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A REMARK ON E. GREEN'S PAPER "COMPLETENESS OF L^p-SPACES OVER FINITELY ADDITIVE SET FUNCTIONS"

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An extended real-valued finitely additive function μ on a field A of subsets of a set X is called a *charge* and (X, A, μ) is called a *charge space.* A charge or a charge space is said to be *positive* (*bounded*) if the charge is positive (bounded). A countably additive charge is called a *measure.*

In [4] Green gave necessary and sufficient conditions for $L_p(X, A, \mu)$ $(1 \leqslant p < \infty)$ to be a complete normed linear space for a positive bounded charge space (X, A, μ). The purpose of this note* is to show that the "necessity" part of Green's result is not correct. The problem of characterizing general positive bounded charge spaces (X, A, μ) for which $L_p(X, A, \mu)$ is complete is still open ([1]).

We assume the reader's knowledge of the theory of integration for charge spaces as in [2] or [1]. For example, $L_p(X, A, \mu)$ stands for $L_p^0(X, A, \mu)/N$, where

$$L_p^0(X, A, \mu) = \{f\colon |f|^p \text{ is integrable}\}$$

and

$$N = \{f \in L_p^0(X, A, \mu)\colon \textstyle\int |f|^p d\mu = 0\}.$$

If μ is a bounded measure on a field A, then $\tilde{\mu}$ stands for the measure which is the extension of μ to the σ-field $\sigma(A)$ generated by A. We shall give our result for $L_1(X, A, \mu)$, where μ is a positive bounded charge only.

THEOREM. *Let μ be a positive bounded measure on a field A of subsets of a set X. Assume that*

(*) *for any $\varepsilon > 0$ and for any $A \in \sigma(A)$ there exist $B, C \in A$ such that $B \subset A \subset C$ and $\mu(C-B) < \varepsilon$.*

Then $L_1(X, A, \mu)$ is complete.

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([1]) After the paper had been submitted, a solution to the problem appeared in G. H. Greco, *Completezza degli spazi L^p per misure finitamente additive*, Annali di Matematica pura ed applicata (4) 32 (1982), p. 243–255. [Note of the Editors]

Proof. We shall show that $L_1^0(X, A, \mu) = L_1^0(X, \sigma(A), \bar{\mu})$. This, of course, would imply that $L_1(X, A, \mu) = L_1(X, \sigma(A), \bar{\mu})$, and so $L_1(X, A, \mu)$ would be complete. For $f \in L_1^0(X, \sigma(A), \bar{\mu})$, let

$$f_n = \sum_{i=1}^{k_n} c_i^{(n)} I_{A_i^{(n)}} \quad \text{with } A_i^{(n)} \in \sigma(A),\ n \geqslant 1,$$

be a sequence of simple functions such that f_n converges to f in measure ($\bar{\mu}$) and $\int |f_n - f_m|\, d\bar{\mu} \to 0$ as $n, m \to \infty$.

For each n and $i \leqslant k_n$ let $B_i^{(n)}$ and $C_i^{(n)}$ be sets belonging to A such that

$$B_i^{(n)} \subset A_i^{(n)} \subset C_i^{(n)} \quad \text{and} \quad \mu(C_i^{(n)} - B_i^{(n)}) < \min\left\{\frac{1}{nk_n |c_i^{(n)}|}, \frac{1}{nk_n}\right\}.$$

If we put

$$g_n = \sum_{i=1}^{k_n} c_i^{(n)} I_{B_i^{(n)}},$$

then

$$\{x\colon |f_n - g_n|(x) > \varepsilon\} \subset \bigcup_{i=1}^{k_n} (C_i^{(n)} - B_i^{(n)}) \quad \text{for any } \varepsilon > 0$$

and, by hypothesis, g_n converges to f in measure (μ) and $\int |g_n - g_m|\, d\mu \to 0$ as $n, m \to \infty$.

Remark 1. As a converse, we can show that if $L_1^0(X, A, \mu) = L_1^0(X, \sigma(A), \bar{\mu})$, then (*) is satisfied. However, if $L_1(X, A, \mu)$ is complete, then (*) need not be satisfied, as $L_1(X, A, \mu)$ is complete for any 0-1 valued charge μ. This can be seen from [1] or [2] and [3].

Now we give some applications of the Theorem.

Example 1. Let X be a countable set and let A be the field of finite and cofinite subsets of X. Then $L_1(X, A, \mu)$ is complete for any positive bounded measure μ on (X, A).

Example 2. Let A be the field generated by all the open subsets of the real line R. Then $L_1(R, A, \mu)$ is complete for any positive bounded measure μ on A, because any positive bounded measure τ on the Borel σ-field of R is regular in the sense that

$$\tau(A) = \sup\{\tau(B)\colon B \subset A,\ B \text{ closed}\} = \inf\{\tau(C)\colon C \supset A,\ C \text{ open}\}.$$

Remark 2. For any charge space (X, A, μ), Green [4] considers the Stone space S' of the quotient Boolean algebra of A with respect to μ and claims that if $L_1(X, A, \mu)$ is complete, then S' is extremally disconnected and every open subset of S' is equivalent to its closure. Our Example 1, where $\mu(\{x\}) > 0$ for all $x \in X$, X countable, and Example 2, where $\mu(U) > 0$ for all open U, are counterexamples to this assertion of Green.

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