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SEPARABLE TRANSLATION-INVARIANT SUBSPACES OF $M(G)$ AND OTHER DUAL SPACES ON LOCALLY COMPACT GROUPS

BY

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DEDICATED TO PROFESSOR STANISŁAW HARTMAN ON THE OCCASION OF HIS SEVENTIETH BIRTHDAY, WITH RESPECT AND THANKS

0. How big can a separable ideal of the algebra of (regular Borel) measures on the locally compact group G be? The question can be put more generally. Let Φ be a normed space on which the locally compact group G has a semi-continuous representation (as isometries). Suppose that Ψ is a G-invariant subspace of Φ. How big is Ψ? A fairly elementary arguments shows that if Ψ is separable, then Ψ consists only of elements of Φ on which the operation of G is norm-continuous. A more complicated argument shows that if Ψ contains an element on which the operation of G is not norm-continuous, then Ψ has dimension at least $\mathfrak{c}$. In the case of abelian G and Ψ an ideal in the algebra of regular Borel measures on G, the analogous conclusion holds with "measure norm" replaced by "spectral radius norm": if Ψ is not contained in the radical of the absolutely continuous measures, then Ψ has dimension at least $\mathfrak{c}$ in spectral radius norm. Related results for the Fourier–Stieltjes algebra $B(G)$ of a non-abelian group G are given.

1. Introduction and discussion of results. Let G be a locally compact group and let T be a lower semi-continuous representation of G on the normed space Φ, that is, the mapping $x \mapsto T(x)$ is a representation of G as

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isometric linear mappings from Φ into Φ such that, for each $\mu \in \Phi$ and $\varepsilon > 0$, the set

$$\{x \in G\colon \|T(x)\mu - \mu\| > \varepsilon\}$$

is open. We denote by Φ_c the set of $\mu \in \Phi$ such that $x \mapsto T(x)\mu$ is a continuous mapping from G to Φ, where Φ has the norm topology. Then Φ_c is a closed invariant subspace of Φ. We prove in this note that Φ_c contains all elements $\mu \in \Phi$ such that the orbit $\{T(x)\mu\colon x \in G\}$ is separable. Furthermore, for each $\mu \in \Phi$, either $\mu \in \Phi_c$ or the orbit generates a subspace of topological dimension at least $\mathfrak{c}$. Those two results are established in Section 2.

Our method is applied to the space $M(G)$ of regular Borel measures on G with the total variation norm in Section 3, where we improve upon a result of Larsen [14] and Tam [18] (see also Glicksberg [8]). Our proof is different (in that we do not use measure theory). In Section 3 we also give two analogous results for the abelian case, using the norms of (i) the supremum of the Fourier–Stieltjes transform as norm and (ii) the spectral radius norm.

In Section 4, we consider translation-invariant subspaces of $L^\infty(G)$. In Section 5, we show how our main Lemma 2.1 can be sharpened in the case of $\Phi = M(G)$ and apply that sharper version to prove the spectral radius result mentioned above. In Section 6 we consider the algebra $B(G)$, when G is a non-abelian group, and give related results.

The result of Larsen and Tam implies that separable ideals of $M(G)$ are contained in $L^1(G)$. That suggests the following version of the problem: what subalgebras of $M(G)$ have separable ideals that are not contained in $L^1(G)$? The answer for some classes of algebras is "none". Examples of such classes are (1) algebras generated by all Riesz products based on a fixed dissociate set and (2) algebras generated by measures on a compact independent set. For background information relating to those two situations see, e.g., [10], Chapter 7 (for Riesz products), and [10], Chapter 6 (for measures on an independent set). We do not know what the situation is for algebras generated by appropriate sets of infinite convolutions.

The question "Must a translation-invariant subspace of $M(G)$ have dimension $\mathfrak{c}$?" was raised by Professor Ryll-Nardzewski and communicated to us by Professor Hartman. We are grateful to them both. We are also grateful to Professor E. E. Granirer for bringing to our attention what we have called Lemma 6.2 and to Professor Hartman again for his careful reading of our manuscript and his helpful comments.

Since the groups involved in this note are not necessarily abelian, group operations will generally be written multiplicatively.

2. Lower semi-continuous representations. The principal tool in this section is the following observation:

LEMMA 2.1. *Let G be a locally compact group and let T be a lower semi-continuous representation of G on the normed space Φ. Let $\mu \in \Phi$ be such that $x \mapsto T(x)\mu$ is not norm-continuous. Then there exists $\varepsilon > 0$ such that*

$$H = \{x\colon \|T(x)\mu - \mu\| > \varepsilon\}$$

is an open dense subset of G.

Proof. That H is an open set follows from the definition of semi-continuity. Since $x \mapsto T(x)\mu$ is not continuous, there exists $\delta > 0$ such that

$$\limsup_{x \to 0} \|T(x)\mu - \mu\| > \delta.$$

Let $\varepsilon = \delta/3$. Fix $y \in G$. If $\|T(y)\mu - \mu\| > \varepsilon$, there is nothing to prove. Thus suppose that $\|T(y)\mu - \mu\| \leqslant \varepsilon$. Let U be a neighborhood of y. Let V be a neighborhood of the identity such that $yV \subseteq U$. Then there exists $x \in V$ such that $\|T(x)\mu - \mu\| > \delta$. Since T maps G onto isometries of Φ, we have

$$\|T(yx)\mu - T(y)\mu\| = \|T(x)\mu - \mu\|.$$

Therefore

$$\|T(yx)\mu - \mu\| \geqslant \|T(yx)\mu - T(y)\mu\| - \|T(y)\mu - \mu\| \geqslant \delta - \varepsilon > \varepsilon.$$

That H is dense follows at once. That completes the proof of Lemma 2.1.

Remarks 2.2. (i) A stronger version of Lemma 2.1 can be established using measure theory: there exists $\varepsilon > 0$ such that H is open and such that the complement of H is locally null.

(ii) Suppose that S is a continuous anti-representation of G on a Banach space X as linear isometries of X, that is, for each $x \in X$ the mapping $g \mapsto S(g)x$ is a continuous mapping of G into X when X has the norm topology. Then the mapping $g \mapsto T(g) = S^*(g)$ is a semi-continuous representation of G on $\Phi = X^*$. That follows because for each $\mu \in \Phi$ the mapping $g \mapsto \|T(g)\mu - \mu\|$ is a supremum of continuous functions, the functions being

$$g \mapsto |\langle \mu, S(g)x\rangle - \langle \mu, x\rangle|,$$

and the supremum being taken over elements $x \in X$ of norm one.

(iii) In the case where $X = C_0(G)$, the continuous complex-valued functions on G that vanish at infinity with the supremum norm, and $S(x) = t_x$, where $t_x f(y) = f(xy)$, $f \in C_0(G)$, $x, y \in G$, it is then true that $T(x)\mu = \delta_x * \mu$ for each $\mu \in M(G) = C_0(G)^*$, and $M(G)_c = L^1(G)$.

We now give our more general version of the result of Larsen [14] and Tam [18].

THEOREM 2.3. *Let G be a non-discrete locally compact group, T a lower semi-continuous representation of G as linear isometries on the normed space Φ, and Ψ a G-invariant subspace of Φ. If Ψ is separable, then $\Psi \subseteq \Phi_c$.*

Proof. Suppose that $\Psi \nsubseteq \Phi_c$. Then there exists $\mu \in \Psi \backslash \Phi_c$. We claim that there exist $\varepsilon > 0$ and a non-countable set $E \subseteq G$ such that $x, y \in E$, $x \neq y$ imply $\|T(x)\mu - T(y)\mu\| > \varepsilon$. The non-separability of Ψ will follow immediately from that claim.

To prove the claim, let $\varepsilon > 0$ be given by Lemma 2.1. Let C be the set of all subsets E of G such that $x, y \in E$, $x \neq y$ imply $\|T(x)\mu - T(y)\mu\| > \varepsilon$. We order C by inclusion and use Zorn's Lemma to choose a maximal chain in C. Clearly, the union, E, of the sets in that chain is again a (maximal) element of C. Suppose that E were countable. We shall show in that case that E is not maximal, a contradiction. We apply Lemma 2.1 to each of the sets yH, H as in the statement of Lemma 2.1 (as y runs through E) to conclude that the set

$$\{z\colon \|T(z)\mu - T(y)\mu\| \leqslant \varepsilon,\ y \in E\}$$

is of first category in G, and that therefore there exists z such that $\|T(z)\mu - T(y)\mu\| > \varepsilon$ for all $y \in E$. Therefore $E \cup \{z\}$ is a set in C, and the chain was not maximal. That completes the proof of Theorem 2.3.

Theorem 2.3 did not need the full strength of Lemma 2.1. The next result does.

THEOREM 2.4. *Let G be a non-discrete locally compact group, T a lower semi-continuous representation of G as linear isometries on the normed space Φ, and Ψ a G-invariant subspace of Φ. If $\Psi \nsubseteq \Phi_c$, then Ψ has dimension at least $\mathfrak{c}$ in the norm topology.*

Proof. Suppose that $\Psi \nsubseteq \Phi_c$. Let $\mu \in \Psi \backslash \Phi_c$. Let ε and H be given by Lemma 2.1. We shall use the standard Cantor set construction to exhibit a set C of cardinality at least $\mathfrak{c}$ such that $x, y \in C$, $x \neq y$ imply $x^{-1}y \in H$. Then $\|T(x)\mu - T(y)\mu\| > \varepsilon$ for all such x, y, and the conclusion of Theorem 2.4 will follow. It remains to find C.

By Lemma 2.1, there exist distinct elements $x(1, 1)$ and $x(1, 2)$ in G such that $x(1, j)^{-1}x(1, k) \in H$ for $1 \leqslant j \neq k \leqslant 2$. For $1 \leqslant j \leqslant 2$ let $U(1, j)$ be compact disjoint neighborhoods of $x(1, j)$ such that

$$U(1, j)^{-1}U(1, k) \subseteq H \quad \text{for } 1 \leqslant j \neq k \leqslant 2.$$

That starts the induction. At the n-th stage we have elements $x(n, k)$ and compact, pairwise disjoint sets $U(n, k)$ such that $U(n, k)$ is a neighborhood of $x(n, k)$ and such that both

$$U(n, j)^{-1}U(n, k) \subseteq H \quad \text{for } 1 \leqslant j \neq k \leqslant 2^n$$

and

$$U(k+1,\ 2j-1) \cup U(k+1,\ 2j) \subseteq U(k,j) \quad \text{for } 1 \leqslant j \leqslant 2^k \text{ and } 1 \leqslant k \leqslant n$$

hold.

We now use Lemma 2.1 (the density of H) to choose, in the interior of each set $U(n, k)$, distinct elements $x(n+1, 2k)$ and $x(n+1, 2k+1)$ such that

$$x(n+1,j)^{-1}\, x(n+1,\ k) \in H \quad \text{for } 1 \leqslant j \neq k \leqslant 2^{n+1}.$$

Since H is open, for each j there exists a compact neighborhood $U(n+1,j)$ of $x(n+1,j)$ such that

$$U(n+1,j)^{-1}\, U(n+1,\ k) \subseteq H \quad \text{for } 1 \leqslant j \neq k \leqslant 2^{n+1}.$$

Of course, the neighborhood $U(n+1,j)$ can be chosen so that

$$U(n+1,\ 2j-1) \cup U(n+1,\ 2j) \subseteq U(n,j) \quad \text{for } 1 \leqslant j \leqslant 2^n.$$

We now set

$$C_n = \bigcup_j U(n,j) \quad \text{and} \quad C' = \bigcap_n C_n.$$

For each sequence $\varepsilon = \{\varepsilon_n\}$ of 0's and 1's, define a sequence $j(n)$ as follows: $j(0) = 1$ and $j(n+1) = 2j(n) - 1 + \varepsilon_n$. Then the set $U_\varepsilon = \bigcap U(n, j(n))$ is non-empty, and if ε' is a sequence distinct from ε, then $U_{\varepsilon'} \cap U_\varepsilon = \emptyset$. For each ε, choose one $x_\varepsilon \in U(\varepsilon)$ and let C be the resulting set of x_ε's. Then $\|T(x)\mu - T(y)\mu\| > \varepsilon$ if x, y are distinct elements of C. That completes the proof of Theorem 2.4.

3. Applications to $M(G)$.

COROLLARY 3.1. *Let G be a non-discrete locally compact group and Ψ a translation-invariant subspace of $M(G)$. If $\Psi \nsubseteq L^1(G)$, then Ψ has dimension at least $\mathfrak{c}$.*

The proof is immediate from Theorem 2.4 and Remarks 2.2.

Let G be a locally compact abelian group and let $\|\mu\|_0$ denote the supremum of the Fourier–Stieltjes transform of $\mu \in M(G)$. Let $M_0(G)$ denote all $\mu \in M(G)$ whose Fourier–Stieltjes transform vanishes at infinity.

COROLLARY 3.2. *Let G be a non-discrete locally compact abelian group and Ψ a translation-invariant subspace of $M(G)$. If $\Psi \nsubseteq M_0(G)$, then Ψ has dimension at least $\mathfrak{c}$ in the $\|\cdot\|_0$-norm topology.*

Proof. We represent G as isometries on $(M(G), \|\cdot\|_0)$ by $T(x)\mu = \delta(x) * \mu$, $\mu \in M(G)$, $x \in G$. Then for each $\mu \in M(G)$, $x \in G$,

$$\begin{aligned} \|\delta(x) * \mu - \mu\|_0 &= \sup \{|\langle \delta(x) * \mu - \mu,\ \chi\rangle| \colon \chi \in \hat{G}\} \\ &= \sup \{|(\chi(x) - 1)\langle \mu, \chi\rangle| \colon \chi \in \hat{G}\}, \end{aligned}$$

which is lower semi-continuous, being the supremum of continuous functions.

A result of Goldberg and Simon [9], Theorem B, asserts that $x \mapsto \delta(x) * \mu$ is continuous in the $\|\cdot\|_0$-norm topology if and only if $\mu \in M_0(G)$. The corollary now follows from Theorem 2.4.

Remark 3.3. Here is a sketch of a proof of the direction we use of the cited assertion [9].

We reduce to the case where G is compact. [We first find a discrete closed subgroup H of G such that G/H has a compact-open subgroup. Then the image $\pi\mu$ of μ in $M(G/H)$ is not in $M_0(G/H)$ if μ is not. We may assume that that image is concentrated on that compact-open subgroup of G/H. It is easy to see that if $x \mapsto \delta(x) * \pi\mu$ is not continuous in the $\|\cdot\|_0$-norm topology on $M(G/H)$, then $x \mapsto \delta(x) * \mu$ is not continuous in the $\|\cdot\|_0$-norm topology on $M(G)$.]

Now choose any sequence $\gamma_n \in \hat{G}$ going to infinity such that $|\hat{\mu}(\gamma_n)| > \varepsilon > 0$ for all n. By passing to a subsequence, we may assume that $\{\gamma_n\}$ is a (1/3)-Kronecker set, so that for any neighborhood U of the identity of G and any function $f\colon (\gamma_n) \to \{\pm 1\}$ there is $x \in U$ such that, except for a finite number of n,

$$|\langle x, \gamma_n \rangle - f(\gamma_n)| < 1/3.$$

It now follows that $x \mapsto \delta(x) * \mu$ is not continuous in the $\|\cdot\|_0$-norm topology.

Before we can state the last result of this section, we need a definition. By $\operatorname{Rad} L^1(G)$ is meant the intersection of all maximal ideals of $M(G)$ that are not contained in the (set of ideals identified with the) dual group $\hat{G}$ of G. That is, a measure μ belongs to $\operatorname{Rad} L^1(G)$ if and only if its Gelfand transform vanishes off $\hat{G}$. (That is, $\operatorname{Rad} L^1(G)$ is the inverse image of the radical of $M(G)/L^1(G)$ under the quotient map.)

THEOREM 3.4. *Let G be a non-discrete locally compact abelian group, and Ψ a translation-invariant subspace of $M(G)$. If $\Psi \nsubseteq \operatorname{Rad} L^1(G)$, then Ψ has dimension at least $\mathfrak{c}$ in the spectral radius norm topology.*

Remark. We could apply Theorem 2.4 if we but knew that $x \mapsto \delta(x) * \mu$ were lower semi-continuous in the spectral radius norm topology. [First, let μ be a measure in Ψ that is not in $\operatorname{Rad} L^1(G)$. The set of $\chi \in \Delta M(G) \setminus \hat{G}$ such that

$$x \mapsto (\delta(x) * \mu)\hat{\ }(\chi) = \delta(x)\hat{\ }(\chi)\,\hat{\mu}(\chi) \tag{1}$$

is discontinuous is dense in $\Delta M(G) \setminus (\hat{G} \cup \hat{\mu}^{-1}(0))$ (see [10], 8.3.5). By the definition of $\operatorname{Rad} L^1(G)$, there exists $\chi \in \Delta M(G) \setminus \hat{G}$ such that $\hat{\mu}(\chi) \neq 0$. By the density, there exists $\chi \in \Delta M(G) \setminus \hat{G}$ such that $x \mapsto \delta(x)\hat{\ }(\chi)\,\hat{\mu}(\chi)$ is discontinuous. Therefore (1) is discontinuous in the spectral radius norm topology. An application of Theorem 2.4 would complete the proof.] Since we do not know that the mapping in question is lower semi-continuous, we put off the proof of Theorem 3.4 until Section 5, where we give a stronger version of

Lemma 2.1, a version valid only for $M(G)$, and the use of which enables us to avoid having to show the semi-continuity of $x \mapsto \delta(x) * \mu$.

4. Applications to subspaces of $L^\infty(G)$. Let $\mathrm{LUC}(G)$ denote the space of bounded left uniformly continuous functions on the locally compact group G, that is, all $f \in \mathrm{CB}(G)$ such that the mapping $x \mapsto t_x f$ (left translation by x) from G into $(\mathrm{CB}(G), \|\cdot\|_\infty)$ is continuous, where $\|\cdot\|_\infty$ is the supremum norm on $\mathrm{CB}(G)$. We shall write $\|\cdot\|_u$ for the essential supremum norm on $L^\infty(G)$. Note that $\|\cdot\|_\infty$ and $\|\cdot\|_u$ agree on $\mathrm{CB}(G)$. Also, if $x \mapsto t_x f$ is continuous in the essential supremum norm topology, then there exists $g \in \mathrm{LUC}(G)$ such that $g = f$ almost everywhere, since f can be approximated in $\|\cdot\|_u$-norm by convolutions $k * f$ (which are necessarily in $\mathrm{LUC}(G)$ for integrable k) and $\mathrm{LUC}(G)$ is complete.

COROLLARY 4.1. *Let G be a non-discrete locally compact group, and Ψ a left translation-invariant subspace of $L^\infty(G)$. If $\Psi \nsubseteq \mathrm{LUC}(G)$, then Ψ has dimension at least $\mathfrak{c}$.*

Proof. If $\Psi \nsubseteq \mathrm{LUC}(G)$, then Ψ contains a function for which translation is discontinuous in the essential supremum norm. The result follows then from Theorem 2.4.

Remark 4.2. In many situations, measurability is related to continuity. For example, Theorem 3.4 of [6] (see also [5], Corollary 2.2) implies that if G is a locally compact group and f is a bounded Haar measurable function on G such that the left orbit $\{t_x f: x \in G\}$ is sup norm separable, then f is in $\mathrm{LUC}(G)$. That also follows from Corollary 4.1 above.

5. Stronger versions of Lemma 2.1 when $\Phi = M(G)$. The following Lemma 5.1 is a variant of the well-known result of [10], 8.3.2 and 8.3.3, the novelty here being in the assertion of "a countable union of compact sets" in place of "Borel". Lemma 5.1 was also what suggested Lemma 2.1 to us.

LEMMA 5.1. *Let ν be a singular measure on the locally compact group G. Then $H = \{x \in G: \delta(x) * \nu \not\perp \nu\}$ is a countable union of compact sets of zero Haar measure.*

Proof. That H is a Borel set of zero Haar measure can be proved like the assertion of [10], 8.3.3. We show that H is a countable union of compact sets of zero Haar measure. We may assume that ν is a probability measure. We note that $\delta(x) * \nu \not\perp \nu$ if and only if $\|\delta(x) * \nu - \nu\| < 2$. The set H is the union of the sets

$$H_j = \{x \in G: \|\delta(x) * \nu - \nu\| \leqslant 2 - 1/j\}, \quad 1 \leqslant j < \infty.$$

It will suffice to show that each H_j is compact. Fix $j \geqslant 1$. Of course, the set H_j is closed, since its complement is open. It remains to show that H_j is compact. We shall show that H_j is contained in a compact set. We choose a compact set F of G such that $\nu(F) > 1-1/(8j)$. Then the set $C = FF^{-1}$ is such that if $x \notin C$, then $F \cap xF = \emptyset$. It follows that for such x

$$\|\delta(x) * \nu - \nu\| > 2(1-1/(8j)) - 2 \cdot 1/(8j) > 2 - 1/j.$$

Therefore $H_j \subseteq C$, so H_j is compact. That completes the proof of Lemma 5.1.

We can establish variants of the idea used in the proof of Theorem 2.4 also in this context.

LEMMA 5.2. *Let ν be a singular measure on the locally compact group G and let $H = \{x \in G\colon \delta(x) * \nu \not\perp \nu\}$. Then there exists a set C of cardinality at least $\mathfrak{c}$ such that $x^{-1} y \notin H$ whenever x and y are distinct elements of C.*

Proof. By Lemma 5.1, $H = \bigcup H_j$, where the H_j's are compact sets of zero Haar measure. Let $x(1, 1)$ and $x(1, 2)$ be distinct elements in G such that $x(1, j)^{-1} x(1, k) \notin H$ for $1 \leqslant j \neq k \leqslant 2$. For $j = 1, 2$, let $U(1, j)$ be a compact neighborhood of $x(1, j)$ such that

$$U(1, j)^{-1} U(1, k) \cap H_1 = \emptyset \quad \text{for } 1 \leqslant j \neq k \leqslant 2.$$

Use the compactness of H_n and an inductive argument similar to that of the proof of Theorem 2.4 to find elements $x(n, k)$ and compact pairwise disjoint sets $U(n, j)$ such that $U(n, j)$ is a neighborhood of $x(n, j)$ and such that both

$$U(n, j)^{-1} U(n, k) \cap H_n = \emptyset$$

holds for $1 \leqslant j \neq k \leqslant 2^n$ and

$$U(k+1, 2j-1) \cup U(k+1, 2j) \subseteq U(k, j)$$

holds for $1 \leqslant j \leqslant 2^k$ and $1 \leqslant k \leqslant n$. Let

$$C_n = \bigcup_j U(n, j) \quad \text{and} \quad C' = \bigcap C_n.$$

Choose C as in the proof of 2.4. That completes the proof of Lemma 5.2.

Proof of Theorem 3.4. First, let μ be a measure in Ψ that is not in $\operatorname{Rad} L^1(G)$. Let $\chi \notin \hat{G}$ be a multiplicative linear functional such that $\hat{\mu}(\chi) \neq 0$. Since the function of the complex variable z defined by

$$h(z) = \hat{\mu}(\chi |\chi|^z / |\chi|)$$

is analytic, there is a purely imaginary value of z such that $h(z) \neq 0$. We may therefore assume that $|\chi_\mu| = 1$ a.e. $d\mu$. A singular measure ν is defined by $\nu = \exp(|\chi\mu|)$. Then $|\chi_\nu| = 1$ a.e. $d\nu$. It is easy to see that H, the set given by Lemma 5.1 (see [10], 8.3.5, for details) is a group; that uses the fact that ν^2 and ν are mutually absolutely continuous. Lemma 5.2 shows that G/H has cardinality at least $\mathfrak{c}$. Therefore, there exists a set $C \subseteq G$ such that $\{x+H\colon x \in C\}$ is an independent subset of G/H having cardinality $\mathfrak{c}$. We will use that fact shortly.

Let ϱ be a character of the (discrete) group G/H, thought also as a character of G whose kernel includes H. We define a multiplicative linear functional $f^{(\varrho)}$ on the L-subalgebra B of $M(G)$ that ν and the discrete measures generate as follows. (For undefined notation and terms and for results used here, see [10].) For $x \in G$, set

$$\langle f^{(\varrho)}, \delta(x) * \omega \rangle = \varrho(x) \int \chi_\omega(z)\, d\omega(z) \qquad \text{for } \omega \in M(G),\ \omega \ll \nu.$$

It is easy to see that that does define a multiplicative linear functional on B (see [10], 8.3.5, for a related argument). Since χ_ν has modulus one a.e. $d\nu$, $|f_\sigma^{(\varrho)}| = 1$ a.e. $d\sigma$ for all measures σ, and therefore $f^{(\varrho)}$ is an element of the Šilov boundary of B (see [10], 5.1.3), so $f^{(\varrho)}$ extends to a multiplicative linear functional on all of $M(G)$. For a multiplicative linear functional λ on $M(G)$, we denote by $\lambda(x)$ the character defined by

$$x \mapsto \int \lambda_{\delta(x)}(y)\, d\delta(x)\, y$$

and apply that notation to χ and $f^{(\varrho)}$. For each pair of distinct elements $x, y \in C$, there exists (by the independence cited at the end of the preceding paragraph) ϱ such that $\varrho(x) = 1$, ϱ is one on H, and the real part of $\varrho(y)$ is at most $2^{-1/2}$. Then

$$\left|(\delta(x) * \mu)\hat{}\,(f^{(\varrho)}) - (\delta(y) * \mu)\hat{}\,(f^{(\varrho)})\right| \geqslant |\hat{\mu}(\chi)|\,(1 - 2^{-1/2}).$$

It follows at once that Ψ' has dimension at least $\mathfrak{c}$ in the spectral radius norm. That completes the proof of Theorem 3.4.

6. On $B(G)$ when G is non-abelian. Let G be a locally compact group. By $A(G)$ and $B(G)$ we will mean the Fourier algebra and Fourier–Stieltjes algebra of G, respectively (see [7]). A locally compact group is an *[IN] group* if its identity has a compact neighborhood that is invariant under all inner automorphisms. A locally compact group G is *almost connected* if G/G_0 is compact, where G_0 is the connected component of the identity ([11], p. 52). A locally compact group G is an *[AU] group* if every unitary representation is atomic, that is, the direct sum of irreducible unitary representations. By $\hat{G}$ we denote the space of equivalence classes of irreducible unitary representations of the locally compact group G. We collect in the next theorem some observations about $B(G)$; they are restatements of results of Taylor [19].

THEOREM 6.1. (i) *Let G be either an [IN] group or almost connected. If $B(G)$ is separable in the norm topology, then G is compact.*

(ii) *There exist non-compact locally compact groups G such that $B(G)$ is separable.*

(iii) *If $B(G)$ is separable, then G contains a compact open subgroup.*

Before proving Theorem 6.1, we state and prove a lemma.

LEMMA 6.2. *Let G be a separable locally compact group. Then the following are equivalent*:

(i) *$\hat{G}$ is countable.*

(ii) *$B(G)$ is separable.*

(iii) *$B(G)$ has the Radon–Nikodým property.*

(iv) *G is $[AU]$.*

Proof. (ii)$\Leftrightarrow$(iii). Let X be any separable Banach space. By [5], Corollary, X^* has the Radon–Nikodým property if and only if X^* is separable. Since $B(G)$ is the dual space of a separable space, (ii)$\Leftrightarrow$(iii) follows.

(i)$\Leftrightarrow$(iv) is 4.5 of [19].

(iii)$\Leftrightarrow$(iv) is 4.2 of [19].

Proof of Theorem 6.1. (i) By Lemma 6.1, G is $[AU]$. Suppose that G is $[IN]$. The assertion of [19], 4.7, is that a group is compact if and only if it is both $[AU]$ and $[IN]$. Suppose that G is almost connected. The assertion of [19], 4.11, is that an almost connected $[AU]$ group is compact. That proves (i).

(ii) The Fell group ([1], 4.5) is non-compact and has a countable dual. By Lemma 6.1, it has the required properties.

(iii) By Lemma 6.2, G is $[AU]$. The assertion of [19], 4.9, is that separable $[AU]$ groups have open subgroups. That proves (iii).

Let E denote the weak* closure of the extreme points of $P_0(G)$, where $P_0(G)$ consists of all positive definite $\varphi \in B(G)$ such that $\varphi(e) \leqslant 1$. Then, when G is abelian, $E \setminus \{0\}$ corresponds exactly to the characters on G. A subset Φ of $B(G)$ is *invariant* if $\varphi f \in \Phi$ for all $\varphi \in E$ and all $f \in \Phi$. Clearly, every ideal is invariant.

Let $H = \{u \in B(G)\colon |u| \leqslant 1, \tilde{u} = u\}$, where $\tilde{u}(x) = \overline{u(x^{-1})}$. Then H is weak* compact and convex. One easily shows that

$$\operatorname{extr} H = (\operatorname{extr} P_0 \cup -\operatorname{extr} P_0) \setminus \{0\}.$$

Let E_H be the weak* closure of $\operatorname{extr} H$. We have

$$E_H \cup \{0\} = E \cup -E.$$

We say that the group G has *property* (Q_C) if there is a net $\{u_\alpha\}$ in $A(G)$ with $\|u_\alpha\|_{MA(G)} \leqslant C$ and $u_\alpha \to 1$ weak* (in $\sigma(L^\infty, L^1)$). We may assume $\tilde{u}_\alpha = u_\alpha$ (consider $(u_\alpha + \tilde{u}_\alpha)/2$). Here $\|\cdot\|_{MA(G)}$ denotes the operator norm of a multiplier on $A(G)$. All amenable groups have (Q_1) as well as many others too (e.g., $\mathrm{SL}(2, \boldsymbol{R})$ and all free groups; see [2]). If G has (Q_C) but not necessarily (Q_1), the following theorem holds with E replaced by CE, and similarly for Theorem 6.5.

THEOREM 6.3. *Let G be a locally compact group with property* (Q_1). *Let Φ be an invariant separable subspace of $B(G)$. If $\varphi \in \Phi$ and $\varphi \neq 0$, then there* *exists $f \in E$ such that $\varphi f \in A(G)$ and $\varphi f \neq 0$. In particular, if G is abelian, then $\Phi \subseteq A(G)$. In general, however, $\Phi \nsubseteq A(G)$* (see Theorem 6.11).

Since $E_H \subset E \cup -E$, it suffices to prove the assertion with E_H instead of E.

Proof. For $\varphi \in \Phi$, $\varphi \neq 0$, let

$$K_\varphi = \{f\varphi : f \in H\} \quad \text{and} \quad E_\varphi = \{f\varphi : f \in E_H\}.$$

Then K_φ is weak* compact and convex and E_φ is a weak* compact subset of K_φ whose weak* closed convex hull is K_φ. By the converse of the Kreĭn–Milman Theorem, E_φ contains the extreme points $\operatorname{extr}(K_\varphi)$ of K_φ. By Lemma 6.4 below (with $\Psi^* = B(G)$), $Z \cap \operatorname{extr}(K_\varphi)$ is a weak* dense subset of $\operatorname{extr}(K_\varphi)$, where Z is the set of all continuity points of the identity map

$$\iota_\varphi : (K_\varphi, \text{weak*}) \to (K_\varphi, \|\cdot\|).$$

Consequently, there exists $\gamma \in E_H$ such that $u = \gamma\varphi \neq 0$, and the identity map ι_φ is continuous at u. By assumption, there exists a net $\{\varphi_\alpha\} \subseteq A(G)$ such that, for all α, $\|\varphi_\alpha\|_{MA(G)} \leqslant 1$ and $\tilde{\varphi}_\alpha = \varphi_\alpha$ and such that $\{\varphi_\alpha\}$ converges to the constant function 1 in the weak* topology. In particular, the net $\{\varphi_\alpha u\}$ converges to u in the weak* topology and lies in K_φ. Consequently, the net $\{\varphi_\alpha u\}$ in $A(G)$ must converge to u in norm. Therefore, $u \in A(G)$.

If G is abelian, $E \setminus \{0\} = \hat{G}$. Hence, if $u = \gamma\varphi \in A(G)$ and $\gamma \in E$, $\gamma \neq 0$, then $\bar{\gamma}\gamma\varphi = \varphi \in A(G)$; that is, $\Phi \subseteq A(G)$. That completes the proof of Theorem 6.3.

Lemma 6.4. *Let Ψ be a Banach space and K a weak* compact convex subset of Ψ^*. Let D denote the weak* closure of* $\operatorname{ext}(K)$ *and let Z be the set of all points of continuity of the identity map* $(K, \text{weak*}) \to (K, \|\cdot\|)$. *If D is norm-separable, then $Z \cap \operatorname{ext}(K)$ is a dense G_δ-subset of* $(\operatorname{ext}(K), \text{weak*})$.

Proof. The assertion is a version of a result of Namioka [16], Theorem 2.2. The proof is the same as Namioka's.

Theorem 6.5. *Let Φ be a separable invariant subspace of* $(B(G), \|\cdot\|_\infty)$, *where $\|\cdot\|_\infty$ denotes the supremum norm on $B(G)$. If G has property* (Q_1), *then for each $\varphi \in \Phi$, $\varphi \neq 0$, there exists $\gamma \in E$ such that $\gamma\varphi \neq 0$ and $\gamma\varphi \in C_0(G)$. In case that G is abelian, that implies that $\Phi \subseteq C_0(G)$. In general, $\Phi \nsubseteq C_0(G)$, however* (see Theorem 6.7 below).

Proof. As in Theorem 6.3 it suffices to prove the assertion with E_H instead of E. Let $\varphi \in \Phi$ and $K_\varphi = \{f\varphi : f \in E_H\}$. Note that a $\|\cdot\|_{B(G)}$-bounded weak* closed subset of $B(G)$ is $\sigma(L^\infty(G), L^1(G))$-closed in $L^\infty(G)$ and that the topologies $\sigma(B(G), C^*(G))$ and $\sigma(L^\infty(G), L^1(G))$ coincide on such a set. Therefore, as in the proof of Theorem 6.3 (using Lemma 6.4 with $\Psi^* = L^\infty(G)$), there is $\gamma \in E_H$ such that $u = \gamma\varphi \neq 0$ and a net $\{\varphi_\alpha\}$ with $\varphi_\alpha u \in A(G)$ and such that $\|\varphi_\alpha u - u\|_\infty \to 0$. Hence $u \in C_0(G)$.

If G is abelian, then each non-zero element $\gamma \in E \setminus \{0\} = \hat{G}$ has a multiplicative inverse, so $\varphi \in C_0(G)$. That completes the proof of Theorem 6.5.

A locally compact group is [*Moore*] if each of its irreducible unitary representations is finite dimensional. All [*Moore*] groups are amenable ([17], p. 698).

THEOREM 6.6. *Suppose that G is a [Moore] group and that $(B(G), \|\cdot\|_\infty)$ is separable. Then G is compact.*

Proof. Let $K = P_0(G)$. An argument similar to that used to prove Theorem 6.3 shows that if Z is the set of all continuity points of the identity map

$$(K, \text{weak*}) \to (K, \|\cdot\|_\infty),$$

then $Z \cap \text{ext}(K)$ is a dense G_δ-subset of $(\text{ext}(K), \text{weak*})$. As in the proof of Theorem 6.5, each element of Z is an element of $C_0(G)$, since [*Moore*] groups are amenable. Since each irreducible representation of G is finite dimensional, functions in $\text{ext}(K)$ are in $\text{AP}(G)$. But $\text{AP}(G) \cap C_0(G) = \{0\}$, unless G is compact. That completes the proof of Theorem 6.6.

THEOREM 6.7. *If G is either the Euclidean motion group or* $\text{SL}(2, \boldsymbol{R})$, *then $(B(G), \|\cdot\|_\infty)$ is separable.*

Proof. If G is the motion group, then

$$\text{WAP}(G) = \text{AP}(G) \oplus C_0(G) \quad \text{and} \quad \text{AP}(G) = C(\boldsymbol{T}),$$

where $\boldsymbol{T}$ is the circle group (see [4] and [15]). Hence $\text{WAP}(G)$ is separable. Therefore, $(B(G), \|\cdot\|_\infty)$ is separable.

If G is $\text{SL}(2, \boldsymbol{R})$, then $\text{WAP}(G) = C \oplus C_0(G)$ by [4]. Hence $\text{WAP}(G)$ is separable. Therefore, $(B(G), \|\cdot\|_\infty)$ is separable.

LEMMA 6.8. *Let G be a locally compact group. If $(A(G), \|\cdot\|_\infty)$ is separable, then G is separable.*

Proof. Let $\{f_n\}$ be a dense sequence in $A(G)$. Let

$$V_n = \{x \colon |f_n(x)| > 1/2\}.$$

Each V_n is an open subset of G. Let x be an element of G and U be a neighborhood of x. Let f be a compactly supported element of $A(G)$ such that $f(x) = 1$ and $\text{supp} f \subseteq U$. Since $\{f_n\}$ is $\|\cdot\|_\infty$-dense, there exists f_m such that $\|f - f_m\|_\infty < 1/4$. Then $x \in V_m$ and $V_m \subseteq U$. Hence $\{V_n\}$ is a basis for the neighborhood system of x. The lemma is proved.

COROLLARY 6.9. *The following are equivalent*:

(i) *$A(G)$ is separable*;

(ii) *$(A(G), \|\cdot\|_\infty)$ is separable*;

(iii) *G is separable.*

THEOREM 6.10. *Suppose that G is discrete and G contains an infinite abelian subgroup H. Then $B(G)$ is not separable.*

Proof. Since H is an infinite discrete abelian group, there exists an uncountable subset $\{f_\beta\}$ of elements of $B(H)$, all of norm 1, such that if $\alpha \neq \beta$, then $\|f_\alpha - f_\beta\| = 2$. Let T: $B_\varrho(H) \to B_\varrho(G)$ be defined by

$$Tf(x) = \begin{cases} f(x) & \text{if } x \in H, \\ 0 & \text{otherwise.} \end{cases}$$

We now show that T is well-defined and isometric. Let g be a finitely supported function on G with $\|g\|_{\mathrm{Cv}(G)} \leqslant 1$, where $\|g\|_{\mathrm{Cv}(G)}$ denotes the operator norm of g acting on $l^2(G)$ by convolution. Then

$$\langle Tf, g \rangle = \sum_{x \in G} f(x) g(x) = \langle f, g|_H \rangle.$$

But

$$\|g|_H\|_{\mathrm{Cv}(H)} \leqslant \|g\|_{\mathrm{Cv}(G)} \leqslant 1.$$

Therefore $\|Tf\|_{B_\varrho(G)} \leqslant \|f\|_{B_\varrho(H)}$.

In particular, T is well-defined and $\|T\| \leqslant 1$. Of course, T is an isometry, because, by choosing g to be supported on H, we infer that

$$\|Tf\|_{B_\varrho(G)} \geqslant \|f\|_{B_\varrho(H)}.$$

It follows that the set $\{Tf_\beta\}$ is a non-separable subset of $B_\varrho(G)$.

THEOREM 6.11. *The "$ax+b$" group contains a separable ideal $\Psi \subseteq B(G)$ such that $\Psi \nsubseteq A(G)$.*

Proof. By [13],

$$B(G) = A(G) \oplus B_s(G),$$

where $B_s(G) = B(\mathbf{R}_+^*) \circ j$, and j is defined by $j(a, b) = a$. Let $K = B(\mathbf{R}_+^*)|_{[1,2]} \circ j$, that is the elements that come from those supported in the interval $[1, 2]$. Since $B(\mathbf{R}_+^*)|_{[1,2]} = A(\mathbf{R}_+^*)|_{[1,2]}$ is separable, K is separable. Therefore $\Psi = A(G) \oplus K$ is separable. But

$$B(G) \cdot \Psi \subseteq A(G) + B(G) \cdot K \subset A(G) + B_s(G) \cdot K \subseteq A(G) + K = \Psi.$$

Hence Ψ is an ideal. Obviously, $\Psi \nsubseteq A(G)$. That completes the proof of Theorem 6.11.

We say that a group G has *property* S if every separable ideal of $B(G)$ is contained in $A(G)$. Theorem 6.11 shows that the "$ax+b$" group does not have property S.

THEOREM 6.12. *If the locally compact group G has an open subgroup H of finite index that has property* S, *then G has property* S.

Proof. Let Ψ be a separable ideal of $B(G)$. Let χ_H denote the characteristic function of H. Since

$$\chi_H \in B(G) \quad \text{and} \quad \sum_{x \in G} \delta(x) * \chi_H = 1$$

(only a finite sum is involved), we have

$$\Psi = \sum_{x \in G} (\delta(x) * \chi_H) \Psi .$$

Since the restriction to (cosets of) H reduces B-norms, the space $T = \chi_H \Psi = \Psi|_H$ is norm-separable in $B(H)$. Since any element of $B(H)$ extends to an element of $B(G)$ (see [12], 32.43), T is an ideal in $B(H)$. Since H has property S,

$$T \subseteq A(H) \quad \text{or} \quad \chi_H \Psi \subseteq A(G).$$

To show that $(\delta(x) * \chi_H) \Psi \subseteq A(G)$, $x \in G$, it will suffice (by the invariance of $A(G)$ under translation) to show that

$$\chi_H (\delta(x^{-1}) * \Psi) \subseteq A(G). \tag{2}$$

But $\delta(x^{-1}) * \Psi$ is again a separable ideal (because translation is an isometric automorphism of $B(G)$). Hence (2) holds. That completes the proof of Theorem 6.12.

Remark 6.13. Theorem 3.4 shows that abelian groups have property S. Hence finite extensions of abelian groups have property S.

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