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MONOTONICITY, CONTINUITY AND LEVELS OF DARBOUX FUNCTIONS

BY

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Introduction. In the present paper* we investigate conditions under which a Darboux function is monotone and establish some properties of nowhere monotone Darboux functions.

Let f be a real-valued Darboux function defined on the real line. It is well known that f is monotone if it is one-to-one. We prove here that f is monotone if it assumes a dense set of its values only once. The function f is shown to be always monotone and continuous relative to the closure of the union of connected levels of f. The function f is thus again monotone whenever its connected levels are dense in its domain. In case f is nowhere monotone, we prove that $f^{-1}(y)$ is dense-in-itself for a comeagre set (see the next page) of real values of y and that there exists another comeagre set of real numbers x such that x is a limit point of $f^{-1}\{f(x)\}$. Several interesting corollaries follow from both these results. We also investigate the Borel measurability of certain sets associated with f, improving thereby two earlier results of Sierpiński [22].

We shall assume throughout the paper R to be the set of real numbers and, unless otherwise stated, f to be a real-valued function defined on R or on a subinterval $[a, b]$ of R. For every $y \in R$, $f^{-1}(y) = \{x: f(x) = y\}$ is called a *level* of f. For each of $a = 1, c, k, d$ and p, $Y_a(f)$ will denote the set of $y \in R$ for which the level $f^{-1}(y)$ is, respectively, a singleton, connected, closed, dense-in-itself or a perfect set and $S_a(f) = f^{-1}\{Y_a(f)\}$.

A function f is called *nowhere monotone* [*nowhere constant*] if it is not monotone [constant] in any subinterval of the domain of f. A nowhere monotone function f is of the *first species* if there exists $r \in R$ such that the function $f(x)+rx$ becomes monotone, and it is of the *second species* if $f(x)+rx$ remains nowhere monotone for every $r \in R$ [8].

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A set $A \subset R$ is a *boundary* set if its interior A^0 is empty, it is *dense-in-itself* if it has no isolated points, and it is *meagre* if it is a countable union of nowhere dense sets. A is further *comeagre* [*cocountable*] if its complement A' is meagre [countable]. $\bar{A}$ and $\mathrm{Fr}\,A$ denote the closure and the frontier of A, respectively.

Section 1 is devoted to conditions under which a Darboux function is monotone or continuous. In Section 2 we investigate the properties of nowhere monotone Darboux functions, generalizing thereby some of the author's earlier results on continuous nowhere monotone functions (see [6]-[9]). In Section 3 we study the Borel measurability of the sets $Y_a(f)$ ($a = 1, c, k, d, p$) for an arbitrary Darboux function f.

1. Monotonicity and continuity of Darboux functions. If a, β, γ are three real numbers, we say that a *is between* β *and* γ if $\beta \leqslant a \leqslant \gamma$ or if $\gamma \leqslant a \leqslant \beta$. We start with the following

LEMMA 1. *A real-valued function f defined on a set $A \subset R$ is monotone if and only if, for every $x_1, x_2, x_3 \in A$, $f(x_2)$ is between $f(x_1)$ and $f(x_3)$ whenever x_2 is between x_1 and x_3.*

Proof. Necessity is trivial. To prove the sufficiency, let f satisfy the given condition. If f is not monotone, there exist $a, b \in A$ such that $a < b$ and $f(a) < f(b)$ and there equally exist $c, d \in A$ such that $c < d$ and $f(c) > f(d)$. If $a = c$, we have $b \neq d$ and we arrive at a contradiction on putting $x_1 = a$, $x_2 = \min\{b, d\}$ and $x_3 = \max\{b, d\}$. If $a \neq c$, to be specific let $a < c$. In case $b = d$, put $x_1 = a$, $x_2 = c$ and $x_3 = b$, and in case $b < d$, put $x_1 = a$, $x_3 = d$ and $x_2 = b$ or c wherever f is larger. In case $b > d$, we have $a < c < d < b$; if $f(c) > f(a)$, put $x_1 = a$, $x_2 = c$ and $x_3 = d$, and if $f(c) \leqslant f(a)$, put $x_1 = a$, $x_2 = d$ and $x_3 = b$.

THEOREM 1. *A Darboux function f is monotone if and only if its level $f^{-1}(y)$ is connected for a set of values of y dense in the range of f.*

Proof. If f is monotone, then, in fact, each of its levels is connected. To prove the sufficiency, let f satisfy the given condition. If f is not monotone, then, by Lemma 1, there exist three points x_1, x_2, x_3 in the domain of f such that $x_1 < x_2 < x_3$ and $f(x_2)$ is not between $f(x_1)$ and $f(x_3)$. To be specific, let $f(x_2)$ be greater than $f(x_1)$ and $f(x_3)$. If $a = \max\{f(x_1), f(x_3)\}$, we have $a < f(x_2)$ and, for every $\beta \in (a, f(x_2))$, f assumes β in (x_1, x_2) as well as in (x_2, x_3) but not at x_2, i.e. $f^{-1}(\beta)$ is disconnected for every $\beta \in (a, f(x_2))$. Since $(a, f(x_2))$ is contained in the range of f, this contradicts our hypothesis, and so f is monotone.

Following is perhaps a little more useful version of Theorem 1:

THEOREM 1'. *A Darboux function f assuming more than one value is monotone if and only if the set of values assumed only once is dense in the range of f.*

Proof. The sufficiency part follows trivially from Theorem 1. To prove the necessity part, let f be monotone and let it assume more than one value. Since f has connected levels, each value that f assumes more than once is assumed by f on a non-degenerate interval. Since there can exist only countably many disjoint non-degenerate intervals, it follows that f assumes all but a countable set of its values only once. The range of f being a non-degenerate interval, the complement of this countable set is clearly dense in the range.

As is evident form this proof, the necessity part of Theorem 1′ can be further strengthened as follows:

COROLLARY 1.1. *A Darboux function f is monotone if and only if, except for a countable set of its values, each is assumed by f only once.*

Since a monotone function is strictly monotone if and only if it is one-to-one, we have

COROLLARY 1.2. *A Darboux function f is strictly monotone if and only if it is one-to-one.*

Remark 1. Corollary 1.2 is well known and was proved independently by Gillespie [13], Jacobsthal [14] and Tricomi [23]. Gillespie did prove the continuity of a Darboux function satisfying the condition of Theorem 1′ but failed to notice its monotonicity. Diaz [4] recently proved that f is monotone if it assumes all of its values but for its supremum and infimum only once. For another proof of the sufficiency part of Theorem 1′ see author's [12].

We next prove that a Darboux function f is always monotone on the closure of $S_c(f)$.

LEMMA 2. *If f is Darboux and $x_1, x_2 \in S_c(f)$, then, for every x between x_1 and x_2, $f(x)$ is between $f(x_1)$ and $f(x_2)$.*

Proof. Let $x_1, x_2 \in S_c(f)$ and let $x_1 < x_2$. If x_1, x_2 are in the same level of f, then f is constant in $[x_1, x_2]$ and the result holds trivially. Otherwise, $f(x)_1 \neq f(x_2)$, say $f(x_1) < f(x_2)$, and let $x_1 < x < x_2$. If $f(x) < f(x_1)$, then $f^{-1}\{f(x_1)\}$ contains x_1 and a point between x and x_2 but not x. As $x_1 \in S_c(f)$, this is not possible, and so we have $f(x_1) \leqslant f(x)$. Since $x_2 \in S_c(f)$, we similarly have $f(x) \leqslant f(x_2)$.

LEMMA 3. *If f is Darboux and x is a limit point of $S_c(f)$ from some side, then f is continuous at x from that side.*

Proof. According to Lemmas 1 and 2, the function f is monotone relative to $S_c(f)$, say non-decreasing. Let x be a limit point of $S_c(f)$ from the right. Since f cannot have a discontinuity of the first kind at any side of a point, each non-degenerate connected level of f contains both of its end points. Hence, if x is a left end point or an interior point of some non-degenerate level of f, f is trivially continuous at x from the right.

Otherwise, there exists a sequence $\{x_n\}$ of points in $S_c(f)$ all belonging to different levels of f such that x_n decreases to x. Then $\{f(x_n)\}$ is decreasing, and so has a limit (possibly $-\infty$), say a. But then it follows from Lemma 2 that a is equally the limit of f from the right at x, and since f cannot have a discontinuity of the first kind, f is continuous at x from the right.

The proof is analogous when x is a limit point of $S_c(f)$ from the left, or when f is non-increasing on $S_c(f)$.

As an immediate consequence of Lemma 3 we have

LEMMA 3′. *A Darboux function f is continuous at every bilateral limit point of $S_c(f)$.*

THEOREM 2. *A Darboux function f is continuous and monotone relative to the closure of $S_c(f)$.*

Proof. Let $g = f/\overline{S_c(f)}$ and $x \in \overline{S_c(f)}$. If x is not a limit point of $\overline{S_c(f)}$ from some side, then g is trivially continuous at x from that side. In case x is a limit point of $\overline{S_c(f)}$ from some side, it is equally a limit point of $S_c(f)$ from that side, and so, by Lemma 3, g is then continuous at x from that side.

According to Lemmas 1 and 2, g is already monotone on $S_c(f)$, say non-decreasing. Let $x, y \in \overline{S_c(f)}$ and $x < y$. There then exist two sequences $\{x_i\}$ and $\{y_i\}$ of elements of $S_c(f)$ such that $x_i \to x$, $y_i \to y$ and $x_i < y_j$ for every pair of natural numbers i, j. Since g is continuous, we have

$$g(x) = \lim_{i\to\infty} g(x_i) \quad \text{and} \quad g(y) = \lim_{i\to\infty} g(y_i).$$

But, for every pair of natural numbers i, j, we have $g(x_i) \leqslant g(y_j)$, and so

$$\lim_{i\to\infty} g(x_i) \leqslant \lim_{i\to\infty} g(y_i),$$

i.e. $g(x) \leqslant g(y)$.

COROLLARY 2.1. *If f is Darboux and $S_c(f)$ is dense in the domain of f, then f is monotone and continuous.*

Remark 2. A continuous function f is known to be monotone if it is so relative to a dense set in its domain. This statement, however, does not hold for a general Darboux function, e.g. for a function that assumes every real value at a dense set of points (see [3], p. 97, Example 3.2). The denseness of $S_c(f)$, on which f is already monotone, makes a Darboux function f monotone only because it also forces f to be continuous.

Since a monotone function has all of its levels connected, we further have

COROLLARY 2.2. *A Darboux function f is monotone if and only if $S_c(f)$ is dense in the domain of f, and is strictly monotone if and only if $S_1(f)$ is dense in the domain of f.*

As seen in Lemma 3′, a Darboux function f is continuous at every bilateral limit point of $S_c(f)$. The same holds at every point $x \in f^{-1}(y)$ if y is a bilateral limit of $Y_c(f)$, or even of $Y_k(f)$ which contains $Y_c(f)$.

THEOREM 3. *If f is Darboux and y is a bilateral limit of $Y_k(f)$, then f is continuous at every point of $f^{-1}(y)$.*

It suffices to prove the following

LEMMA 4. *If f is Darboux and y is a limit point of $Y_k(f)$ from above, then f is upper semicontinuous at every point of $f^{-1}(y)$.*

Proof. Let $x \in f^{-1}(y)$. Given $\alpha > y$, there exists $\beta \in Y_k(f)$ such that $y < \beta < \alpha$. As x is outside the closed set $f^{-1}(\beta)$, there exists an open interval U containing x which does not intersect with $f^{-1}(\beta)$. Clearly, $f(U)$ is then an interval that contains y but not β, and so $f(x') < \beta < \alpha$ for every $x' \in U$.

As $Y_k(f)$ always contains the complement of the range of f, Theorem 3 yields (cf. Lipiński [16])

COROLLARY 3.1. *If f is Darboux and $Y_k(f)$ is dense in the range of f, then f is continuous.*

A function is said to *fulfil the Banach condition* (T_1) if it assumes almost each of its values only a finite number of times. Hence

COROLLARY 3.2. *If a Darboux function f fulfils the Banach condition* (T_1), *then f is continuous.*

The result holds in particular if f is of bounded variation, or monotone, or is a nowhere monotone function of the first species (see [8], p. 83). Conversely, if a Darboux function f is discontinuous, then it is of unbounded variation, and if f is discontinuous at a dense set of points, then it is of unbounded variation in every interval. Hence (see [8], p. 83, f.n. 3),

COROLLARY 3.3. *If a Darboux function f is discontinuous at a dense set of points, then f is a nowhere monotone function of the second species.*

2. Nowhere monotone Darboux functions.

LEMMA 5. *If a Darboux function f is nowhere monotone, then $Y_c(f)$ is nowhere dense.*

Proof. Firstly, $Y_c(f) = Y_1(f)$, for f being nowhere monotone is nowhere constant. If $Y_1(f)$ is not nowhere dense, there exists a non-degenerate interval $[\alpha, \beta]$ with $\alpha, \beta \in Y_1(f)$ such that $Y_1(f)$ is dense in the interval. Then $f^{-1}(\alpha)$ and $f^{-1}(\beta)$ are singletons, say x_1 and x_2, respectively, and $x_1 \neq x_2$. To be specific, let $x_1 < x_2$ and g be the restriction of f to $[x_1, x_2]$. Then the range of g is $[\alpha, \beta]$; for g being Darboux, $g([x_1, x_2]) \supset [\alpha, \beta]$, and if there exists $x \in (x_1, x_2)$ for which $g(x) < \alpha$ $[> \beta]$, then g assumes α $[\beta]$ once again in (x, x_2) $[(x_1, x)]$ which is not possible since α $[\beta] \in Y_1(f)$. Moreover, $Y_1(f) \cap [\alpha, \beta] \subset Y_1(g)$, for if $y \in Y_1(f) \cap [\alpha, \beta]$, y is clearly assumed by g at least once, and it is assumed by g only once since it is assumed by f only once. Thus $Y_1(g)$ is dense in the range $[\alpha, \beta]$ of g, and so, by Theorem 1, g is monotone, i.e. f is monotone in $[x_1, x_2]$. f being nowhere monotone, this is not possible, and so $Y_1(f)$ is nowhere dense.

THEOREM 4. *If f is a nowhere monotone Darboux function, then $f^{-1}(y)$ is a boundary set for every $y \in R$ and is dense-in-itself for a comeagre set of values of y in R.*

Proof. As f is then nowhere constant, for every $y \in R$, $f^{-1}(y)$ cannot contain a non-degenerate interval, and so is a boundary set.

If $y \notin Y_d(f)$, then $f^{-1}(y)$ contains an isolated point, and so there exists an interval I in the domain of f with rational end points (or a or b if $[a, b]$ is the domain of f) such that f assumes y only once in I, or such that $y \in Y_1(f/I)$. Thus if $\{I_n\}$ is an enumeration of the subintervals of the domain of f with rational end points (or a or b), then

$$(*) \qquad R - Y_d(f) = \bigcup_{n=1}^{\infty} Y_1(f/I_n).$$

But, for every n, f is equally Darboux and nowhere monotone in I_n, and so $Y_1(f/I_n)$ is nowhere dense by Lemma 5. Hence $\bigcup_{n=1}^{\infty} Y_1(f/I_n)$ is meagre and $Y_d(f)$ is comeagre.

Theorem 4 holds, in particular, for every Darboux function that is discontinuous at a dense set of points (see Corollary 3.3). It also yields at once

COROLLARY 4.1. *If f is Darboux and $f^{-1}(y)$ is scattered for a set of values of y that is non-meagre in every subinterval of the range of f, then there exists a family of intervals dense in the domain of f in each of which f is monotone.*

This corollary generalizes Corollary 1 of [9], p. 65, and a result of Marcus [18], p. 103.

As proved earlier (see [10], Theorem 2), for every Darboux function f we have

$$m\{R(f) \cap Y_d(f)\} = m\{R(f) - f(D)\},$$

where m denotes the Lebesgue measure, $R(f)$ denotes the range of f and D is the set of points where f has a finite or infinite derivative. Since a nowhere derivable function is always nowhere monotone, Theorem 4 yields further

COROLLARY 4.2. *If a Darboux function f is nowhere derivable, then $f^{-1}(y)$ is dense-in-itself for all but a meagre set of values of y in R that is of measure zero.*

In case f is a nowhere monotone function having everywhere a finite derivative (for examples see [6], p. 176, f.n. 7), then the Darboux function f' is discontinuous at a dense set of points (see [7], p. 666, Corollary 1), and so is nowhere monotone (see Corollary 3.3). Moreover, the only points where f' may have a derivative are the points where f' vanishes (see [6], p. 176, Corollary 2), i.e., $m\{f'(D)\} = m\{0\} = 0$. Hence Theorem 4 yields

COROLLARY 4.3. *If a nowhere monotone function f has everywhere a finite derivative, then the set $\{x: f'(x) = c\}$ is dense-in-itself for all but a meagre set of values of c in R that is of measure zero.*

In case a Darboux function f is everywhere discontinuous, then, by Theorem 3, $Y_k(f)$ cannot have any bilateral limit, and so is countable. The set $Y_1(f)$ is then equally countable, and so is $Y_1(f/I_n)$ in (*) for every natural number n. Clearly, the set $Y_d(f)$ is then cocountable.

COROLLARY 4.4. *If a Darboux function f is everywhere discontinuous, then $f^{-1}(y)$ is dense-in-itself for all but a countable set of values of y in R.*

There exist Darboux functions of which every level is dense-in-itself, e.g., a function that assumes every real value at a dense set of points. This example also shows that a nowhere monotone Darboux function need not have any relative maximum or minimum. A continuous nowhere monotone function, on the other hand, has always dense sets of maxima and minima (see [11], p. 1442, Lemma 2).

Since a continuous function has closed levels, Theorem 4 yields in that case

COROLLARY 4.5. *If f is a continuous nowhere monotone function, then $f^{-1}(y)$ is nowhere dense for every $y \in R$ and is perfect for a comeagre set of values of y in R.*

Remark 3. Corollary 4.5 is the necessity part of author's [9], Theorem 1, and Theorem 4 is a generalization of that result to Darboux functions. The method of proof of [9] was quite different and, in fact, we proved there a little more, viz. that there exists a set of reals y comeagre in the range of f such that f is oscillating at least on one side at every point $x \in f^{-1}(y)$. Lemma 5 was proved by Padmavally [20] for continuous functions but her proof is not applicable to Darboux functions. The method of proof of Theorem 4 was first employed by Sierpiński [22] to prove that, for every function f with a closed graph, the set $Y_p(f)$ is $F_{\sigma\delta}$, and has been lately used by Khanh [15] to prove Corollary 4.5.

The converse of Theorem 4 holds equally. In fact,

THEOREM 4'. *A Darboux function f is nowhere monotone if and only if each of its levels is a boundary set and $f^{-1}(y)$ is dense-in-itself for a set of values of y dense in the range of f.*

Proof. The necessity follows from Theorem 4. To prove the sufficiency, let f be a Darboux function satisfying the given conditions and let, if possible, f be monotone in some non-degenerate interval $[\alpha, \beta]$. As f has boundary levels, it is nowhere constant, and so f is strictly monotone in $[\alpha, \beta]$, i.e. f is one-to-one in $[\alpha, \beta]$. Clearly, $f(\alpha) \neq f(\beta)$ and each value $y \in f((\alpha, \beta))$ is assumed by f once and only once in (α, β), i.e. $f^{-1}(y)$ has an isolated point for every $y \in f((\alpha, \beta))$. This being contrary to the hypothesis, f is nowhere monotone.

Theorem 4′ generalizes Lemma 2 of [9] to Darboux functions.

Next, we need the following

LEMMA 6. *If a Darboux function f is nowhere monotone, then $S_c(f)$ is nowhere dense.*

Proof. If $S_c(f)$ is dense in some interval, then, according to Theorem 2, f is monotone in that interval, which is not possible since f is nowhere monotone.

THEOREM 5. *If f is a nowhere monotone Darboux function, then there exists a comeagre set of points x in the domain of f such that x is a limit point of the level $f^{-1}\{f(x)\}$.*

Proof. Let E be the set of points x in the domain of f such that x is an isolated point of $f^{-1}\{f(x)\}$. For every $x \in E$ there exists an interval I in the domain of f with rational end points (or a or b if $[a, b]$ is the domain of f) such that $I \cap f^{-1}\{f(x)\} = \{x\}$, i.e. such that $x \in S_1(f/I)$. Hence, if $\{I_n\}$ is an enumeration of the subintervals of the domain of f with rational end points (or a or b), then

$$E = \bigcup_{n=1}^{\infty} S_1(f/I_n).$$

But f being equally Darboux and nowhere monotone in each interval I_n, by Lemma 6, $S_1(f/I_n)$ is nowhere dense for every n, and so E is meagre.

COROLLARY 5.1. *If f is a nowhere monotone Darboux function, then it has a median or extreme derivate zero at a comeagre set of points in the domain of f.*

In case f is measurable (in the sense of Lebesgue) and symmetric, then, according to Neugebauer [19], Theorem 10, f has symmetric derivates at a comeagre set of points. As the intersection of two comeagre sets is again comeagre, we have

COROLLARY 5.2. *If a nowhere monotone Darboux function f is measurable and symmetric, then there exists a comeagre set of points in the domain of f where*

$$D_+f = D_-f \leqslant 0 \leqslant D^+f = D^-f.$$

Remark 4. In case f is continuous, then, in Theorem 5, x becomes even a bilateral limit point of $f^{-1}\{f(x)\}$ (see [6], Theorem 2). Recently, Manna [17] (see p. 67, Theorem 4 and Corollary 1) proved Theorem 5 and its Corollary 5.1 under the extra hypothesis of semicontinuity on f. Corollary 5.2 is an extension of Theorem 3 of [6].

It may be still remarked that in Theorems 1, 2, 4 and 5 the connectedness, or at least the local connectedness of the domain of the function is indispensable. For, as shown by Filipczak [5], p. 86, there exists a real-valued continuous nowhere monotone function defined on the Cantor set that is one-to-one.

Let, next, f be a Darboux nowhere monotone function of the second species. For every $r \in R$, the function $f(x)+rx$ is then nowhere monotone but need not be Darboux in general (see Sen [21], p. 21, Example 2). It is known to be Darboux in case it is further of Baire class one (see [21], p. 21, Theorem III). Following an argument parallel to that of [9], Section 2, Theorem 4 leads to the following

THEOREM 6. *If a Darboux function f of Baire class one is a nowhere monotone function of the second species, then, for every countable set $E \subset R$, there exists a comeagre set H in R such that, for every $m \in E$ and for every $c \in H$, the line $y = mx+c$ intersects the curve $y = f(x)$ in a dense-in-itself boundary set.*

Theorem 5 yields, on the other hand,

THEOREM 7. *If a Darboux function f of Baire class one is a nowhere monotone function of the second species, then there exists a comeagre set of points in the domain of f where the bilateral upper and lower derivates of f are $+\infty$ and $-\infty$, respectively.*

Proof. For every natural number n, the functions $f(x) \pm nx$ are both nowhere monotone Darboux functions, and so, by Corollary 5.1, there exist sets E_{+n} and E_{-n} comeagre in the domain of f such that, for every $x \in E_{+n}$ $[E_{-n}]$, we have

$$\bar{D}f(x) \geqslant n \quad [\underline{D}f(x) \leqslant -n].$$

The set

$$E = \bigcap_{n=-\infty}^{\infty} E_n$$

is again comeagre in the domain of f and at every point $x \in E$ we have $\bar{D}f(x) = +\infty$ and $\underline{D}f(x) = -\infty$.

In case f is further symmetric, then it has symmetric derivates at a comeagre set of points in its domain, and so we have

COROLLARY 7.1. *If a Darboux function f of Baire class one is symmetric and is a nowhere monotone function of the second species, then there exists a comeagre set of points in the domain of f, where $D^+f = D^-f = +\infty$ and $D_+f = D_-f = -\infty$.*

This corollary is an extension of Theorem 5 of [8].

COROLLARY 7.2. *If a Darboux function f of Baire class one is derivable at a comeagre set of points, then f is of bounded variation in a family of intervals that is dense in the domain of f.*

For if f is of unbounded variation in every subinterval of some interval

I, then f is a nowhere monotone function of the second species in I, and so, by Theorem 7, f is non-derivable at a non-meagre set of points.

In case a Darboux function f has a non-zero derivative at a comeagre set of points, then, by Corollary 5.1, f is even monotone in a family of intervals dense in the domain of f.

Theorems 6 and 7 hold, in particular, for every finite derivative, or for every finite approximate derivative, for such a function is always Darboux and of Baire class one. Since the derivative of a nowhere monotone function, having everywhere a finite derivative, is discontinuous at a dense set of points (see [7], p. 666, Corollary 1), by Corollary 3.3 it is a nowhere monotone function of the second species, and so Theorems 6 and 7 yield, respectively,

COROLLARY 6.1. *If a nowhere monotone function f has everywhere a finite derivative, then, for every countable set $E \subset R$, there exists a comeagre set H in R such that, for every $m \in E$ and for every $c \in H$, the line $y = mx + c$ intersects the curve $y = f'(x)$ in a dense-in-itself boundary set.*

COROLLARY 7.3. *If a nowhere monotone function f has everywhere a finite derivative, then there exists a comeagre set of points in the domain of f where the derived function of f has its bilateral upper and lower derivates equal to $+\infty$ and $-\infty$, respectively.*

PROBLEM. None of Theorems 1, 4 and 5 of [7] on continuous nowhere monotone functions can be extended to general nowhere monotone Darboux functions. For if f assumes every real value at a dense set of points, it is clearly a nowhere monotone Darboux function and has everywhere a knot-point, i.e. $D^+f = D^-f = +\infty$ and $D_+f = D_-f = -\infty$. It would be interesting to investigate if some of these theorems can be extended to nowhere monotone Darboux functions of Baire class one (**P 853**).

3. Borel measurability of the sets $Y_a(f)$ $(a = 1, c, k, d, p)$.

LEMMA 7. *If f is Darboux, then each of the sets $Y_c(f)$ and $Y_k(f)$ contains all of its bilateral limits.*

Proof. Let, first, y be a bilateral limit of $Y_c(f)$. To prove that $f^{-1}(y)$ is connected, let $x_1, x_2 \in f^{-1}(y)$ and $x_1 < x < x_2$. It will suffice to show that $f(x) = y$. If $f(x) > y$, there exists $\beta \in Y_c(f)$ such that $y < \beta < f(x)$. Then $f^{-1}(\beta)$ contains at least one point in each of the intervals (x_1, x) and (x, x_2) without containing x, which is not possible since $\beta \in Y_c(f)$. Hence $f(x) \leqslant y$ and, similarly, $f(x) \geqslant y$, implying thereby that $f(x) = y$.

Let, next, y be a bilateral limit of $Y_k(f)$ and let x be a limit point of $f^{-1}(y)$. If $x \notin f^{-1}(y)$, we have $f(x) \neq y$, say $f(x) > y$. Then there exists $\beta \in Y_k(f)$ such that $y < \beta < f(x)$. Since $f^{-1}(\beta)$ is closed and $x \notin f^{-1}(\beta)$, there exists an open interval U containing x that does not intersect with $f^{-1}(\beta)$. Clearly, $f(U)$ is then an interval containing $f(x)$ but not β, i.e. $f(U) \subset (\beta, \infty)$, i.e. $U \cap f^{-1}(y) = \emptyset$. Since this contradicts the hypothesis that x is a limit point of $f^{-1}(y)$, we have $x \in f^{-1}(y)$.

Theorem 1 could also be deduced from Lemma 7.

THEOREM 8. *If f is Darboux, then $Y_c(f)$ and $Y_k(f)$ are G_δ-sets of the form $F-C$, where F is closed and C is a countable subset of the frontier of F.*

Proof. If a set E of real numbers contains all of its bilateral limits, then $C = \bar{E}-E$ contains only unilateral limits of E, and so is countable. Clearly, $E = \bar{E}-C$, and as E contains the interior of $\bar{E}$, we have

$$C = \bar{E}-E \subset \bar{E}-\bar{E}^0 = \mathrm{Fr}(\bar{E}).$$

As in Theorem 8 we have $C \subset \mathrm{Fr}(F)$, the set $F-C = F^0 \cup \{\mathrm{Fr}(F)-C\}$, where F^0 is open and $\mathrm{Fr}(F)-C$ is G_δ, and as $\mathrm{Fr}(F)$ is nowhere dense, so is $\mathrm{Fr}(F)-C$. Hence,

COROLLARY 8.1. *If f is Darboux, then $Y_c(f)$ and $Y_k(f)$ are G_δ-sets of the form $G \cup G_\delta$, where G is open and the G_δ-set is nowhere dense.*

The following corollary also follows directly from Theorem 8:

COROLLARY 8.2. *If f is Darboux, then each of the sets $Y_c(f)$ and $Y_k(f)$ contains every open interval in which it is dense.*

THEOREM 9. *If f is Darboux, then $Y_1(f)$ is a G_δ-set of the form $F-C$, where F is closed and C is countable. Moreover, if f is further nowhere constant, then $C \subset \mathrm{Fr}(F)$.*

Proof. In case f is nowhere constant, then $Y_1(f) = Y_c(f)$ and the result follows from Theorem 8. In general, $Y_1(f) \subset Y_c(f)$ and $C_1 = Y_c(f)-Y_1(f)$ is countable since f can have only countably many lines of invariability. Thus $Y_1(f) = Y_c(f)-C_1 = (F-C)-C_1 = F-(C \cup C_1)$.

COROLLARY 9.1. *If f is Darboux, then $Y_1(f)$ is cocountable in every interval in which it is dense. In case f is further nowhere constant, then $Y_1(f)$ contains every open interval in which it is dense.*

THEOREM 10. *If f is Darboux, then $Y_d(f)$ and $Y_p(f)$ are $F_{\sigma\delta}$-sets of the form $G_\delta \cup C$, where C is countable. In case f is further nowhere constant, then $Y_d(f)$ $[Y_p(f)]$ is also of the form $F-G_{\delta\sigma}$ $[G_\delta - G_{\delta\sigma}]$, where the $G_{\delta\sigma}$-set is meagre.*

Proof. As seen in the proof of Theorem 4,

$$R-Y_d(f) = \bigcup_{n=1}^{\infty} Y_1(f/I_n),$$

where $\{I_n\}$ is an enumeration of the subintervals in the domain of f with rational end points (or a or b). For every n, f is equally Darboux in I_n and so, by Theorem 9, $Y_1(f/I_n) = F_n - C_n$, where F_n is closed and C_n is countable. Hence

$$R-Y_d(f) = \bigcup_{n=1}^{\infty}(F_n - C_n) = \bigcup_{n=1}^{\infty} F_n - C,$$

where C is a subset of $\bigcup_{n=1}^{\infty} C_n$, and so is countable. Thus $R - Y_d(f) \epsilon F_\sigma - C$ and, in turn, $Y_d(f) \epsilon G_\delta \cup C$. Moreover, with the help of Theorem 8 we have

$$Y_p(f) = Y_k(f) \cap Y_d(f) = G_\delta \cap (G_\delta \cup C) = G_\delta \cup C_0,$$

where $C_0 \subset C$, and so is countable.

In case f is nowhere constant, then, for every n, $Y_1(f/I_n) = Y_c(f/I_n)$, and so, by Corollary 8.1, $Y_1(f/I_n) = G_n \cup N_n$, where G_n is open and N_n is a nowhere dense G_δ-set. Thus

$$R - Y_d(f) = \bigcup_{n=1}^{\infty} (G_n \cup N_n) = \Big(\bigcup_{n=1}^{\infty} G_n\Big) \cup \Big(\bigcup_{n=1}^{\infty} N_n\Big) = G \cup M,$$

where G is open and M is a meagre $G_{\delta\sigma}$-set, whence $Y_d(f) = G' - M = F - G_{\delta\sigma}$, where the $G_{\delta\sigma}$-set is meagre. Also, then

$$Y_p(f) = Y_k(f) \cap Y_d(f) = G_\delta \cap (F - G_{\delta\sigma}) = G_\delta \cap F - G_\delta \cap G_{\delta\sigma} \epsilon G_\delta - G_{\delta\sigma},$$

where the $G_{\delta\sigma}$-set is still meagre.

COROLLARY 10.1. *If f is Darboux, then each of the sets $Y_d(f)$ and $Y_p(f)$ is comeagre in every interval in which it is $\aleph$-dense. In case f is further nowhere constant, then each of these sets is comeagre in an interval in which it is dense.*

Remark 5. Sierpiński [22] proved that if f has a closed graph, then $Y_1(f)$ is a G_δ-set and $Y_p(f)$ is an $F_{\sigma\delta}$-set. Theorems 9 and 10 not only extend Sierpiński's results to Darboux functions but also provide a more precise form of the sets under consideration, for Corollaries 9.1 and 10.1 are not deducible from the general forms of the sets.

What is known regarding $Y_d(f)$ is that it is an analytic set for every Borel measurable function f [2]. As for Theorem 10, it may be noted that the set of values assumed by a Darboux function infinitely many times is also of the form $G_\delta \cup C$ (see Borsuk [1]). Corollary 10.1 generalizes Theorem 3 of [9].

Some of the results of the present paper can be extended to real-valued functions with locally connected domain. However, as the methods involved are quite different, we postpone the discussion to a subsequent paper.

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