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CONCERNING A PROBLEM DUE TO SAM B. NADLER, JR.

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A function $f\colon X \to X$ is called a *contraction map* if there is a positive number $a < 1$ such that

$$\varrho(f(x), f(y)) \leqslant a\varrho(x, y) \quad \text{for all } x, y \in X.$$

Nadler, Jr., asked ([1]) the following question:

Is it true that, for every compactum X for which the identity map $i_X\colon X \to X$ is a pointwise limit of contraction maps, all Čech cohomology groups of X (over integers) are trivial?

In order to give an affirmative answer to this question, let us recall first some notions belonging to the homology theory.

By a *sequence of chains* in a compactum X we understand a sequence $\varkappa = \{\varkappa_i\}$ with

$$\varkappa_i = a_{i,1}\sigma_{i,1} + a_{i,2}\sigma_{i,2} + \ldots + a_{i,m_i}\sigma_{i,m_i},$$

where $a_{i,j}$ are elements of an abelian group $\mathfrak{A}_i$ (depending, in general, on i) and $\sigma_{i,j}$ are *oriented simplexes* (i.e., finite systems of points (vertices) of X).

Let $\operatorname{mesh}(\varkappa_i)$ denote the maximal diameter of the simplexes $\sigma_{i,1}, \ldots, \sigma_{i,m_i}$. Let us prove the following

LEMMA 1. *If X is a compactum and $f\colon X \to X$ is a map satisfying the condition $\varrho(f(x), f(y)) < \varrho(x, y)$ for every $x, y \in X$ with $x \neq y$, and if $\{\varkappa_i\}$ is a sequence of chains in X with*

$$\overline{\lim_{i\to\infty}} \operatorname{mesh}(\varkappa_i) \leqslant \varepsilon, \quad \textit{where } \varepsilon > 0,$$

then

$$\overline{\lim_{i\to\infty}} \operatorname{mesh}(f(\varkappa_i)) < \varepsilon.$$

Proof. Otherwise there would exist a sequence of indices $i_1 < i_2 < \ldots$ such that in $\varkappa_{i_n}$ there is a simplex σ_{i_n,j_n} containing two vertices x_n, y_n

([1]) S. B. Nadler, Jr., *Some problems concerning stability of fixed points*, Colloquium Mathematicum 27 (1973), p. 263-268; see Problem 2.11 on p. 268.

with

$$\varrho\big(f(x_n\ ), f(y_n)\big) \geqslant \varepsilon - \frac{1}{n}.$$

Since X is compact, we may assume $x_n \to x$ and $y_n \to y$, where $x, y \in X$. Then $\varrho\big(f(x), f(y)\big) \geqslant \varepsilon$ and, consequently, $\varrho(x, y) > \varepsilon$. It follows that there exists a number $\eta > \varepsilon$ such that the inequality $\varrho(x_n, y_n) \geqslant \eta$ is satisfied for almost all n. Hence $\text{mesh}(\varkappa_{i_n}) \geqslant \eta > \varepsilon$ for almost all n, which contradicts our hypothesis that

$$\overline{\lim_{i\to\infty}} \text{mesh}(\varkappa_i) \leqslant \varepsilon.$$

Thus the proof of Lemma 1 is complete.

If $\boldsymbol{\kappa} = \{\varkappa_i\}$ is a sequence of chains in X satisfying the condition

$$\lim_{i\to\infty} \text{mesh}(\varkappa_i) = 0,$$

then we say that $\boldsymbol{\kappa}$ is an *infinite chain in* X. An infinite chain $\boldsymbol{\gamma} = \{\gamma_i\}$ in X is said to be an *infinite cycle in* X if all chains γ_i are cycles, i.e. if their boundaries $\partial\gamma_i$ vanish. If there is a sequence of chains $\boldsymbol{\kappa} = \{\varkappa_i\}$ in X such that

$$\overline{\lim_{i\to\infty}} \text{mesh}(\varkappa_i) \leqslant \varepsilon \quad \text{and} \quad \gamma_i = \partial\varkappa_i \text{ for } i = 1, 2, \ldots,$$

then the infinite cycle $\boldsymbol{\gamma}$ is said to be *ε-homologous to zero in X* and we write $\boldsymbol{\gamma} \underset{\varepsilon}{\sim} 0$ in X.

If there is an infinite chain $\boldsymbol{\kappa} = \{\varkappa_i\}$ in X such that $\gamma_i = \partial\varkappa_i$ for $i = 1, 2, \ldots$, then we write $\boldsymbol{\gamma} = \partial\boldsymbol{\kappa}$ and we say that the infinite cycle $\boldsymbol{\gamma}$ is *homologous to zero in X* (notation: $\boldsymbol{\gamma} \sim 0$ in X). A compactum X is said to be *acyclic* if every infinite cycle in X is homologous to zero in X.

LEMMA 2. *An infinite cycle $\boldsymbol{\gamma}$ in X is homologous to zero in X if and only if $\boldsymbol{\gamma} \underset{\varepsilon}{\sim} 0$ in X for every $\varepsilon > 0$.*

Proof. It is evident that the relation $\boldsymbol{\gamma} \sim 0$ in X implies $\boldsymbol{\gamma} \underset{\varepsilon}{\sim} 0$ in X for every $\varepsilon > 0$. On the other hand, if $\boldsymbol{\gamma} \underset{\varepsilon}{\sim} 0$ in X for every $\varepsilon > 0$, then for every $n = 1, 2, \ldots$ there is in X a sequence $\{\varkappa_i^{(n)}\}$ of chains such that

$$\overline{\lim_{i\to\infty}} \text{mesh}(\varkappa_i^{(n)}) \leqslant \frac{1}{n} \quad \text{and} \quad \partial\varkappa_i^{(n)} = \gamma_i \text{ for } i = 1, 2, \ldots$$

Then for every $n = 1, 2, \ldots$ there is an index i_n such that $\text{mesh}(\varkappa_i^{(n)}) \leqslant 2/n$ for $i \geqslant i_n$. We may assume that $i_{n+1} > i_n$ for $n = 1, 2, \ldots$ Setting

$$\varkappa_i = \begin{cases} \varkappa_i^{(1)} & \text{for } i = 1, 2, \ldots, i_1, \\ \varkappa_i^{(n)} & \text{for } i_n \leqslant i < i_{n+1},\ n = 1, 2, \ldots, \end{cases}$$

we get an infinite chain $\varkappa = \{\varkappa_i\}$ satisfying the condition $\partial\varkappa = \gamma$. Thus the proof of Lemma 2 is complete.

Now let us prove the following

THEOREM. *Let X be a compactum satisfying the following condition: For every $\varepsilon > 0$ there exists a map $f\colon X \to X$ such that $\varrho(f(x), x) < \varepsilon$ for every $x \in X$ and that*

$$\varrho(f(x), f(y)) < \varrho(x, y) \quad \textit{if } x, y \in X \textit{ and } x \neq y. \tag{1}$$

Then X is acyclic.

Proof. If X is not acyclic, then there is an infinite cycle $\gamma = \{\gamma_i\}$ in X such that $\gamma \nsim 0$ in X. We infer, by Lemma 2, that there exist positive numbers ε such that

(2) the relation $\gamma \underset{\varepsilon}{\sim} 0$ in X fails.

Let ε_0 denote the least upper bound of the set of all such numbers ε. We see easily that then there exists a sequence of chains $\varkappa = \{\varkappa_i\}$ in X such that

$$\partial\varkappa_i = \gamma_i \text{ for } i = 1, 2, \ldots \quad \text{and} \quad \overline{\lim_{i\to\infty}} \operatorname{mesh}(\varkappa_i) \leqslant \varepsilon_0.$$

By our hypothesis there is a map $f\colon X \to X$ satisfying condition (1) and such that $\varrho(f(x), x) < \varepsilon_0$ for every $x \in X$. Then there is an infinite chain $\lambda = \{\lambda_i\}$ in X such that $\partial\lambda_i = \gamma_i - f(\gamma_i)$ for $i = 1, 2, \ldots$ Moreover, $\{f(\varkappa_i)\}$ is a sequence of chains in X satisfying, by Lemma 1, the condition

$$\overline{\lim_{i\to\infty}} \operatorname{mesh} f(\varkappa_i) < \varepsilon_0.$$

Setting $\varkappa_i' = \lambda_i + f(\varkappa_i)$ for $i = 1, 2, \ldots$, we see that

$$\overline{\lim_{i\to\infty}} \operatorname{mesh}(\varkappa_i') < \varepsilon_0.$$

Since $\partial(f(\varkappa_i)) = f(\partial\varkappa_i) = f(\gamma_i)$, we infer that

$$\partial\varkappa_i' = \gamma_i - f(\gamma_i) + \partial(f(\varkappa_i)) = \gamma_i \quad \text{for } i = 1, 2, \ldots$$

But this contradicts (2). Thus the proof of the Theorem is complete.

Since every contractive map $f\colon X \to X$ satisfies the condition $\varrho(f(x), f(y)) < \varrho(x, y)$ for $x, y \in X$ and $x \neq y$, and since for an acyclic compactum X all Čech cohomology groups are trivial, the just proved theorem gives an affirmative answer to the problem of Sam B. Nadler, Jr.

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