COLLOQUIUM MATHEMATICUM

VOL. XXX 1974 FASC. 1

SUBALGEBRA LATTICES OF UNARY ALGEBRAS AND AN AXIOM OF CHOICE

BY

WILLIAM A. LAMPE (BOULDER, COLORADO)

0. Introduction. An algebraic lattice is one that can be represented as the lattice of subalgebras of some algebra. The surprising result that every algebraic lattice is isomorphic to the lattice of subalgebras of an algebra of the form $\mathfrak{A} \times \mathfrak{A}$ was first proved by A. A. Iskander. Iskander proved a stronger theorem (see Theorem A), and it is easy to modify the proof and further strengthen this theorem. Many algebraic lattices cannot be isomorphic to the lattice of subalgebras of a unary algebra. We show in this paper* that the analogues for unary algebras of these theorems are true. Also, one of the analogues is shown to be equivalent to an axiom of choice for collections of two-element sets, while another is independent of this axiom.

1. Results. An element c of a complete lattice is *compact* if whenever $c \leqslant \bigvee(a_i \mid i \in I)$, then there is a finite $J \subseteq I$ with $c \leqslant \bigvee(a_i \mid i \in J)$. A lattice $\mathfrak{L}$ is *algebraic* if $\mathfrak{L}$ is complete and every element is the join (supremum) of compact elements.

Let $\mathfrak{A} = \langle A; F\rangle$ be an algebra (sometimes called a *universal algebra* or a *general algebra*). We put $\mathfrak{A}^2 = \mathfrak{A} \times \mathfrak{A}$ and $A^2 = A \times A$. If $B \subseteq A^2$, then we let B^* be the *converse* of B; i.e., $B^* = \{\langle x, y\rangle \mid \langle y, x\rangle \in B\}$. $\mathscr{S}(\mathfrak{A})$ denotes the set of subalgebras and $\mathfrak{S}(\mathfrak{A})$ the lattice of subalgebras of $\mathfrak{A}$.

Birkhoff and Frink [1] characterized the lattice of subalgebras of an algebra as an algebraic lattice. For any algebra $\mathfrak{A}$, the mapping which sends D to D^* is an automorphism of order two of $\mathfrak{S}(\mathfrak{A}^2)$. One might at first suspect that this would put some restriction on the class of lattices that could be represented as an $\mathfrak{S}(\mathfrak{A}^2)$. But, on the contrary, Iskander has shown that, indeed, every algebraic lattice is isomorphic to some

* The research for this paper was done when the author was a lecturer at the University of Manitoba. The results formed a portion of the author's dissertation at the Pennsylvania State University and were announced in abstract #653-295 of the Notices of the American Mathematical Society 15 (1968).

$\mathfrak{S}(\mathfrak{A}^2)$ (see [8] or [4]). On the other hand, not every algebraic lattice is isomorphic to an $\mathfrak{S}(\mathfrak{A}^n)$ if $n \geqslant 3$.

Iskander [8] actually proved the following stronger theorem:

THEOREM A. *If $\mathfrak{L}$ is an algebraic lattice and a is an automorphism of $\mathfrak{L}$ of order two, then there is an algebra $\mathfrak{A}$ with $\mathfrak{L}$ isomorphic to $\mathfrak{S}(\mathfrak{A}^2)$ in such a way that a corresponds to the mapping $D \rightarrow D^*$ for $D \in \mathscr{S}(\mathfrak{A}^2)$.*

One sees from this theorem that one can always have $D = D^*$ for any subalgebra D of $\mathfrak{A}^2$.

It is easy to modify the proof of this theorem so that $|A| \geqslant \mathfrak{m}$ for any preassigned cardinal $\mathfrak{m}$. Also, a proof can be given for this modified theorem that does not require any choice axiom (see [4]).

A *unary algebra* is one in which every operation is unary or nullary. Not every algebraic lattice can be represented as the subalgebra lattice of a unary algebra. If $\mathfrak{A}$ is unary, then the join in $\mathfrak{S}(\mathfrak{A})$ of two subalgebras is just their set union. Thus $\mathfrak{S}(\mathfrak{A})$ is a sublattice of the lattice of all subsets of A; in particular, $\mathfrak{S}(\mathfrak{A})$ is distributive.

Throughout $\boldsymbol{K}$ denotes the class of all lattices isomorphic to $\mathfrak{S}(\mathfrak{A})$ for some unary algebra $\mathfrak{A}$.

One of the theorems of this paper is the following analogue of the modified version of Theorem A:

THEOREM 1. *If $\mathfrak{L} \in \boldsymbol{K}$, a is an automorphism of $\mathfrak{L}$ of order two and $\mathfrak{m}$ is a cardinal number, then there exist a unary algebra $\mathfrak{A}$ and an isomorphism ϱ from $\mathfrak{L}$ onto $\mathfrak{S}(\mathfrak{A}^2)$ with $(xa)\varrho = (x\varrho)^*$ for all $x \in L$ and $|A| \geqslant \mathfrak{m}$.*

Iskander's theorem does not give any information about the kind of operations in the algebra $\mathfrak{A}$. Hence, his theorem does not entail the above-given result.

The following weak axiom of choice will be of special interest to us in the case $n = 2$:

C_n. *If $\mathscr{C}$ is any collection of n-element sets, then there is a function χ with domain $\mathscr{C}$ such that $\chi(A) \in A$ for all $A \in \mathscr{C}$.*

Such a χ is called a *choice function*. Suppose $\mathfrak{L} \in \boldsymbol{K}$, $|L| \geqslant 2$, and a is an automorphism of $\mathfrak{L}$ of order two. We will say that the *unary algebra* $\mathfrak{A} = \langle A; F\rangle$ *represents the pair* $\langle \mathfrak{L}, a\rangle$ if there is an isomorphism ϱ from $\mathfrak{L}$ onto $\mathfrak{S}(\mathfrak{A}^2)$ with $(xa)\varrho = (x\varrho)^*$.

LEMMA 1. *If $\mathfrak{A} = \langle A; F\rangle$ represents $\langle \mathfrak{L}, a\rangle$ and if there is an $a \in L$ such that aa is the complement of a, then there is a choice function χ on the collection of two-element subsets of A.*

Let $\mathfrak{B}$ be the four-element Boolean lattice having elements $\{0, a, b, 1\}$. Let a be the automorphism of $\mathfrak{B}$ with $0a = 0$, $aa = b$, $ba = a$ and $1a = 1$. Set $\boldsymbol{M}$ equal to the class of all unary algebras that represent $\langle \mathfrak{B}, a\rangle$.

The proof we will give for Theorem 1 will assume C_2 (but no stronger choice axiom). Thus, from Lemma 1, we get

COROLLARY. *The following are equivalent:*

(i) C_2;

(ii) *Theorem* 1;

(iii) *for every cardinal* $\mathfrak{m}$, *there is a unary algebra* $\langle A; F\rangle \in \boldsymbol{M}$ *such that* $|A| \geqslant \mathfrak{m}$.

In contrast, as previously mentioned, even the modified form of Theorem A needs no form of the axiom of choice.

Let T be the statement one gets by deleting "$\mathfrak{m}$ is a cardinal" and "$|A| \geqslant \mathfrak{m}$" from Theorem 1. That is, let T be the analogue for unary algebras of Theorem A.

Let W2C be the statement: For every cardinal $\mathfrak{m}$, there is a set A such that $2^{|A|^2} \geqslant \mathfrak{m}$ and there is a choice function on the collection of two-element subsets of A.

THEOREM 2. T *implies* W2C.

Clearly, C_2 implies T. Nothing more than this and Theorem 2 is known about the relationship of T to the axiom of choice and to the other axioms of set theory. The possibility that there is a "real" mathematical statement (namely, T) strictly weaker than C_2 and still independent of the other axioms seems intriguing.

The next theorem (Theorem 3) is clearly a corollary to Theorem 1. But we shall prove Theorem 4 by giving a proof of Theorem 3 that does not use C_2. That this is possible is because we shall assume in Theorem 3 that the automorphism a has a special kind of fixed point.

The element r of the complete lattice $\mathfrak{L}$ is *complete-join irreducible* if, given any representation $r = \bigvee(x_i \mid i \in I)$, where $x_i \in L$, then $r = x_j$ for some $j \in I$.

THEOREM 3. *Suppose that* $\mathfrak{L} \in \boldsymbol{K}$, a *is an automorphism of* $\mathfrak{L}$ *of order two, and* $\mathfrak{m}$ *is any cardinal number. If there is an* $r \in L$ *such that* $r \neq 0$, $ra = r$, *and* r *is complete-join irreducible, then there exist a unary algebra* $\mathfrak{A}$ *with* $|A| \geqslant \mathfrak{m}$ *and an isomorphism* ϱ *from* $\mathfrak{L}$ *onto* $\mathfrak{S}(\mathfrak{A}^2)$ *such that* $(xa)\varrho = (x\varrho)^*$ *for all* $x \in L$.

If a is the identity map, then a, clearly, satisfies the hypothesis in Theorem 3, and we get the following corollary:

COROLLARY. *If* $\mathfrak{L} \in \boldsymbol{K}$, *then there is a unary algebra* $\mathfrak{A}$ *with* $\mathfrak{L}$ *isomorphic to* $\mathfrak{S}(\mathfrak{A}^2)$.

THEOREM 4. *Theorem* 3 *is independent of* C_2.

In a forthcoming paper, R. J. Gauntt has solved the problem concerning mutual interdependences of the axioms of choice for collections of n-element sets (for all $n < \omega$). We quote the precise expression of this statement from his paper:

"If $z = \{n_1, \ldots, n_r\}$ is a finite set of natural numbers, we let C_z mean

C_{n_1} and ... and C_{n_r}. We let $D_{n,z}$ mean, for every subgroup G of S_n without fixed points, there is a subgroup H of G and a finite sequence $H_1, \ldots, H_k$ of proper subgroups of H such that

$$\frac{|H|}{|H_1|} + \ldots + \frac{|H|}{|H_k|} \in z \ldots$$

Mostowski proved that $D_{n,z}$ is sufficient for $(C_z \to C_n)$. We use Cohen type forcing to prove the converse."

In particular, Gauntt's result shows that C_2 does not imply the usual axiom of choice.

Before we can prove our theorems, we will need to know more about members of the class $\boldsymbol{K}$. Theorem B gives us this information which belongs to the folklore of algebra. A reference is [3].

THEOREM B. *For every lattice $\mathfrak{L}$, the following statements are equivalent:*

(i) $\mathfrak{L} \in \boldsymbol{K}$;

(ii) *$\mathfrak{L}$ is a complete sublattice of a complete, atomic Boolean lattice;*

(iii) *$\mathfrak{L}$ is a complete lattice in which every element is the join of complete-join irreducible elements and in which the following distributive law holds for any set I and $x, y_i \in L$:*

$$x \wedge \bigvee(y_i \mid i \in I) = \bigvee(x \wedge y_i \mid i \in I);$$

(iv) *there is a partially ordered set $\langle P; \leqslant\rangle$ with zero such that if $\mathscr{M}$ is the system of hereditary subsets of $\langle P; \leqslant\rangle$, then $\mathfrak{L}$ is isomorphic to $\langle \mathscr{M}; \cup, \cap\rangle$.*

With regard to (iii), in $\mathfrak{S}(\mathfrak{A})$ there is a 1-1 correspondence between complete-join irreducible elements and subalgebras generated by a single element. This is a property peculiar to unary algebras.

One should notice the similarity of representation (iv) with the representation of an algebraic lattice as the lattice of all ideals of a semilattice with zero. In proving (iii) implies (iv), one may let P be the set of complete-join irreducible elements. For each $a \in L$, one then defines $M_a = \{x \mid x \in P \text{ and } x \leqslant a\}$. $a \to M_a$ is an isomorphism from $\mathfrak{L}$ to $\langle \mathscr{M}; \cup, \cap\rangle$.

Let $\mathfrak{L}$ be a lattice, and α an automorphism of $\mathfrak{L}$ of order two. Suppose $\mathfrak{L}$ is an algebraic lattice. In proving Iskander's theorem, one starts with the representation of $\mathfrak{L}$ as the lattice of ideals of the semilattice $\mathfrak{C}$, where $\mathfrak{C}$ *is the semilattice of all compact elements of* $\mathfrak{L}$.

Let $\mathfrak{L} \in \boldsymbol{K}$. The proofs for Theorems 1 and 3 of this paper start with (iv) of Theorem B where *P can be assumed to be the set of complete-join irreducible elements of $\mathfrak{L}$.*

There is always a compact element $c \neq 0$ with $c\alpha = c$. (In particular, take $c = (a \vee a\alpha)$ for some compact $a \neq 0$.) But it is easy to construct examples in which $r\alpha \neq r$ for every complete-join irreducible element $r \neq 0$. It is this difference that brings choice axioms into the picture for unary algebras.

The particular ideas used in the construction of the present paper are based on [4]. The general technique, however, was introduced by Grätzer and Schmidt in [5]. It was also employed by Iskander in his original proof in [8]. Given [4], large portions of the proofs are somewhat routine (and much of this routine work is not reproduced here), but frequently difficulties arise and special considerations are needed. The addition of Lemma 7 is an example.

There has been further work in this general area, since this paper was first written and circulated in preprint form.

In [9], Iskander modified the techniques in [4] and proved the following generalization of Theorem A:

THEOREM C. *Let $\mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3$ be algebraic lattices such that $|L_1|, |L_2| > 1$. Let a_i be an automorphism of $\mathfrak{L}_i$ such that $a_i \circ a_i$ is the identity map for $i = 1$ and 2. Then there are algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ of the same similarity type, having the following properties:*

(a) *there are lattice isomorphisms β_i of $\mathfrak{L}_i$ onto $\mathfrak{S}(\mathfrak{A}_i \times \mathfrak{A}_i)$ for $i = 1$ or 2, and an isomorphism β_3 of $\mathfrak{L}_3$ onto $\mathfrak{S}(\mathfrak{A}_1 \times \mathfrak{A}_2)$;*

(b) *$(la_i)\beta_i = (l\beta_i)^*$ for any $l \in L_i$ for $i = 1$ or 2.*

Note that if $\mathfrak{A}$ is any algebra and $|A| > 1$, then $|\mathscr{S}(\mathfrak{A} \times \mathfrak{A})| > 1$. So if in Theorem C one removed the restriction that both $|L_1|$ and $|L_2|$ are greater than 1, then one would have answered the unsolved problem about the relationship between $\mathscr{S}(\mathfrak{A})$ and $\mathscr{S}(\mathfrak{A} \times \mathfrak{A})$.

George Piegari has announced that by combining the techniques of this paper with those of [9] he can prove the analogue of Theorem C for unary algebras. Thus, he has improved on Theorem 1. Piegari announced this in Abstract 72T-A108, Notices of the American Mathematical Society 19 (1972), A-434.

2. Preliminary lemmas. The lemmas of Section 3 give the initial construction and the add-on construction. Before we proceed, we need to study the algebra freely generated by a partial unary algebra.

Essentially, Lemmas 2-6 are the statements in Section 3 of [4] with slight modifications brought on by the fact that the lemmas here are concerned only with unary algebras. Their proofs are almost identical with those of the earlier versions and are omitted. Lemma 7 is proved.

LEMMA 2. *The algebra $\mathfrak{B}_1 = \langle B_1; F \rangle$ freely generated by the unary partial algebra $\mathfrak{B} = \langle B; F \rangle$ is characterized by the following properties:*

(i) *there is a subset B^* of B_1 such that B^* generates B_1;*

(ii) *the relative subalgebra $\mathfrak{B}^* = \langle B^*; F \rangle$ of $\mathfrak{B}_1$ is isomorphic to $\mathfrak{B}$;*

(iii) *for $f, g \in F$ and $a, b \in B_1$, $f(a) = g(b) \notin B^*$ implies $f = g$ and $a = b$;*

(iv) *if $f \in F$, $a \in B_1$, then $f(a) \in B^*$ implies $a \in B^*$.*

We may take $B^* = B$.

COROLLARY. *Let* $\mathfrak{B} = \langle B; F\rangle$ *and* $\mathfrak{C} = \langle C; G\rangle$ *be partial unary algebras, and let* $\mathfrak{B}_1 = \langle B_1; F\rangle$ *and* $\mathfrak{C}_1 = \langle C_1; G\rangle$ *be the algebras freely generated by* $\mathfrak{B}$ *and* $\mathfrak{C}$, *respectively. If* $B \subseteq C$ *and* $F \subseteq G$, *and, for every* $f \in F$, $D(f, \mathfrak{B}) \supseteq D(f, \mathfrak{C}) \cap B$, *then, for some* $\mathfrak{B}_1$, $B_1 \subseteq C_1$.

Let $B(F) = \{y \mid y \in B \text{ or } y = f(x) \text{ for some } x \in B \text{ and } f \in F\}$. Let $B = {}^0B$. If ${}^nB$ has been defined, then we put ${}^{n+1}B = {}^nB(F)$.

LEMMA 3. *We have*

(i) $B_1 = \bigcup({}^nB \mid n = 0, 1, 2, \ldots)$;

(ii) *if* $y \in {}^{n+1}B - {}^nB$, *then there exist uniquely* $x \in {}^nB$ *and* $f \in F$ *with* $y = f(x)$.

In Lemmas 4, 5 and 6, $\mathfrak{B}_1$ will be the algebra freely generated by the partial unary algebra $\mathfrak{B}$, D a subalgebra of $\mathfrak{B}$, and $\bar{D} = [D]_{\mathfrak{B}_1}$ the subalgebra generated by D in $\mathfrak{B}_1$ (similarly in the case of $D \subseteq B^2$).

LEMMA 4. *We have*

(i) $\bar{D} \cap B = D$;

(ii) *there is a function* θ *defined on the class of unary partial algebras so that* $\theta(\mathfrak{B}) = \beta$ *is a function,* $\beta\colon B_1 \to B$, *with* $\beta(b) = b$ *for* $b \in B$ *and such that* $a \in \bar{D}$ *iff* $\beta(a) \in D$;

(iii) $D \to \bar{D}$ *is a lattice embedding of* $\mathfrak{S}(\mathfrak{B})$, *the subalgebra lattice of* $\mathfrak{B}$, *into* $\mathfrak{S}(\mathfrak{B}_1)$.

The function β in (ii) is defined inductively based on Lemma 3.

LEMMA 5. *If all the operations of* $\mathfrak{B}$ *are injective* (*i.e.*, 1-1), *then*

(i) *the operations of* $\mathfrak{B}_1$ *and* $\mathfrak{B}_1^2$ *are injective*;

(ii) $\langle \overline{B^2}; F\rangle$ *is freely generated by* $\mathfrak{B}^2$.

To summarize, we state the following lemma:

LEMMA 6. *If all the operations of* $\mathfrak{B}$ *are injective, then*

(i) *the operations of* $\mathfrak{B}_1$ *are injective*;

(ii) *for* $D \in \mathscr{S}(\mathfrak{B}^2)$, *the collection of subalgebras of* $\mathfrak{B}^2$, *we have* $\bar{D} \cap B^2 = D$;

(iii) $D \to \bar{D}$ *is a lattice embedding of* $\mathfrak{S}(\mathfrak{B}^2)$ *into* $\mathfrak{S}(\mathfrak{B}_1^2)$;

(iv) $\overline{D^*} = (\bar{D})^*$;

(v) *there is a function* θ *defined on the class of unary partial algebras with injective operations so that* $\theta(\mathfrak{B}) = \beta$ *is a function,* $\beta\colon B_1^2 \to B_1^2$, *with* $\beta(b) = b$ *for all* $b \in B^2$ *and such that, for any* $a \in B_1^2$, $a \in \bar{D}$ *iff* $\beta(a) \in D$.

The important effect of the map β in (v) is, of course, limited to the subalgebra generated by B^2 in $\mathfrak{B}_1^2$. The map is defined on all of B_1^2 for notational convenience.

LEMMA 7. *There is a function* ν *defined on the class of partial unary algebras having injective operations so that, for every* $\mathfrak{B}$, *if* $\nu(\mathfrak{B}) = R$, *then* $[R] = \bar{R} = B_1^2 - \overline{B^2}$, *and in the relative subalgebra* $\langle R; F\rangle$ *the domain of every operation is empty.*

Proof. Let $\mathfrak{B}$ be a partial unary algebra with 1-1 operations. Since the operations of $\mathfrak{B}$ are injective, Lemmas 2 and 5 imply that $B_1^2 - \overline{B^2}$ is a subalgebra of B_1^2. Let

$$Q_0 = \{\langle a_0, a_1\rangle \mid \{a_0, a_1\} \subseteq {}^1B \text{ and } \langle a_0, a_1\rangle \notin \overline{B^2}\}$$

with ${}^1B$ defined as before in Lemma 3. If Q_n has been defined, set

$$Q_{n+1} = \{\langle a_0, a_1\rangle \mid \{a_0, a_1\} \subseteq {}^{n+2}B \text{ and } \langle a_0, a_1\rangle \notin \overline{B^2} \cup \bigcup_{i=0}^{n} [Q_i]\}.$$

Finally, set $R = \bigcup(Q_i \mid i = 0, 1, 2, \ldots)$ and $\nu(\mathfrak{B}) = R$.

Since $B_1^2 - \overline{B^2}$ is a subalgebra, it is easy to check that $[R] = B_1^2 - \overline{B_2}$.

Let $f \in F$, and $\langle a_0, a_1\rangle \notin R$. There exists uniquely a k with $\langle a_0, a_1\rangle \in Q_k$. By Lemma 2 and the definition of Q_i, it is easy to show that $f(\langle a_0, a_1\rangle) \notin Q_i$ for any i. Thus, $f(\langle a_0, a_1\rangle) \notin R$ and the domain of f in the relative subalgebra $\langle R; F\rangle$ is empty. Since f was arbitrary, this completes the proof of the lemma.

3. Construction lemmas. The proofs of Theorems 1 and 3 are trivial if $|L| = 1$. So, hereafter, we will assume that $|L| > 1$.

Elements with the following property will be employed frequently in the rest of the paper:

(**) $\quad r \in L$ is complete-join irreducible, $r \neq 0$ and $ra = r$.

LEMMA 8. *If $\mathfrak{L} \in \boldsymbol{K}$, $\mathfrak{m}$ is any cardinal number, and a any automorphism of $\mathfrak{L}$ of order two, then there exists a unary partial algebra $\mathfrak{B} = \langle B; F\rangle$ and an isomorphism σ from $\mathfrak{L}$ onto $\mathfrak{S}(\mathfrak{B}^2)$ such that*

(i) $\mathfrak{m} \leqslant |B|$;

(ii) $(xa)\sigma = (x\sigma)^*$ *for all* $x \in \mathfrak{L}$;

(iii) *all operations in F are injective*;

(iv) *there is a distinguished element* $\langle b_0^+, b_1^+\rangle \in B^2$ *with* $b_0^+ \neq b_1^+$;

(v) *if there is an $r \in L$ having property* (**), *then* $[\langle b_0^+, b_1^+\rangle]^* = [\langle b_0^+, b_1^+\rangle]$.

Moreover, if there is an r with property (**), *then the existence of $\mathfrak{B}$ does not depend on any choice axiom, otherwise,* C_2 *implies the existence of* $\mathfrak{B}$.

Proof. As in Theorem B, P is the set of complete-join irreducible elements of $\mathfrak{L}$, $\mathcal{M}$ is the set of hereditary subsets of $\langle P; \leqslant\rangle$, and $a \to M_a$ is an isomorphism of $\mathfrak{L}$ onto $\langle \mathcal{M}; \cap, \cup\rangle$. Observe that $(M_a)a = M_{aa}$. Let C be some set with $\mathfrak{m} \leqslant |C|$ and $C \nsubseteq P$. Set $B = P \cup C$, and fix a $b_0 \in B - P$. If there is an r with property (**), fix one such element and call it r_0. If there is no r with (**), fix an $r \neq 0$ and call it r_0.

Define a mapping ψ from P into the power set of B^2 as follows:

$$0\psi = \{\langle b, b\rangle \mid b \in B\};$$

$$r\psi = \begin{cases} \{\langle r, b_0\rangle, \langle b_0, r\rangle\} & \text{if } 0 \neq r \neq r_0 \text{ and } r = ra, \\ \{\langle r, ra\rangle\} & \text{if } r \neq ra \text{ and } r_0 \neq r \neq r_0 a; \end{cases}$$

$$r_0\psi = B^2 - \bigcup(r\psi | r \in P - \{r_0\}) \quad \text{if } r_0 a = r_0.$$

To complete the definition of ψ if $r_0 a \neq r_0$, we will employ C_2. Set

$$Q = B^2 - \big(\bigcup(r\psi \mid r \in P - \{r_0, r_0 a\}) \cup \{\langle r_0, r_0 a\rangle, \langle r_0 a, r_0\rangle\}\big).$$

Let A be the set of two-element subsets of Q that are of the form $\{\langle x, y\rangle, \langle y, x\rangle\}$. Let χ: $A \to Q$ be a mapping guaranteed by C_2. Now set

$$r_0\psi = \{\langle r_0, r_0 a\rangle\} \cup \chi(A) \quad \text{and} \quad (r_0 a)\psi = B^2 - \bigcup(r\psi \mid r \in P - \{r_0 a\}).$$

It is obvious that

(1) $r\psi \neq \emptyset$ for all $r \in P$;

(2) $r_1 \neq r_2$ implies $r_1\psi \cap r_2\psi = \emptyset$;

(3) $(ra)\psi = (r\psi)^*$ for all $r \in P$;

(4) $\bigcup(r\psi \mid r \in P) = B^2$.

Now define partial operations on B as follows:

(5) every $b \in B$ is the value of a nullary operation f_b;

(6) for $0 \neq r_1 \leqslant r_2$ and $\langle x_i, y_i\rangle \in r_i\psi$, a partial unary operation f is given by $f(x_2) = x_1$, $f(y_2) = y_1$, and $D(f) = \{x_2, y_2\}$.

The operation f is well defined and 1-1 since $r_1 \neq 0 \neq r_2$ implies $x_i \neq y_i$.

Let F be the set of all operations defined by (5) and (6), and let $\mathfrak{B} = \langle B; F\rangle$. Define a mapping φ of $\mathscr{M}$ into $\mathscr{S}(\mathfrak{B}^2)$ as follows:

$$M_a\varphi = \bigcup(r\psi \mid r \in M_a).$$

It is a routine matter to check that φ is an isomorphism from $\langle \mathscr{M}; \cap, \cup\rangle$ onto $\mathfrak{S}(\mathfrak{B}^2)$. Now define σ: $L \to \mathscr{S}(\mathfrak{B}^2)$ by $x\sigma = M_x\varphi$ for $x \in \mathfrak{L}$. Since σ is the composition of two isomorphisms, it is an isomorphism. Since $M_a a = M_{aa}$ and $(ra)\psi = (r\psi)^*$, we infer that $(xa)\sigma = (x\sigma)^*$ for all $x \in L$.

For $r_0 a = r_0$, set $\langle b_0^+, b_1^+\rangle = \langle r_0, b_0\rangle$, and for $r_0 a \neq r_0$, set $\langle b_0^+, b_1^+\rangle = \langle r_0, r_0 a\rangle$. Clearly, $\langle b_0^+, b_1^+\rangle$ has the required properties.

Finally, the only place in the proof that a choice axiom was used was in the definition of ψ in case there was no r with property (**).

The unary partial algebra $\mathfrak{B} = \langle B; G\rangle$ is an *expansion* of the partial algebra $\mathfrak{A} = \langle A; F\rangle$ if $A \subseteq B$, the similarity type of $\mathfrak{A}$ is contained in the similarity type of $\mathfrak{B}$, and if $f(a_0)$ is defined in $\mathfrak{A}$, then $f(a_0)$ is defined in $\mathfrak{B}$, and the two values are equal. $\mathfrak{B}$ is a *singular expansion* of $\mathfrak{A}$ if $\mathfrak{B}$

is an expansion of $\mathfrak{A}$, $\mathfrak{A}$ is a relative subalgebra of $\langle B; F\rangle$, $[A] = B$, and $A \subseteq D(f, \mathfrak{B})$ for $f \in F$.

Let Γ be the class of ordered pairs of the form $\langle \mathfrak{B}, \langle b_0, b_1\rangle\rangle$, where $\mathfrak{B} = \langle B; F\rangle$ is a partial unary algebra with injective operations, $b_0 \neq b_1$, $\langle b_0, b_1\rangle \in B^2$, and $[\langle b_0, b_1\rangle] = [\langle b_1, b_0\rangle]$. Let D be a subalgebra of $\mathfrak{B}^2$.

LEMMA 9. *There is a function* $\gamma\colon \Gamma \to \Gamma$ *such that if* $\langle \mathfrak{B}^+, \langle b_0^+, b_1^+\rangle\rangle = \gamma(\langle \mathfrak{B}, \langle b_0, b_1\rangle\rangle)$, *then*

(i) $\mathfrak{B}^+$ *is a singular expansion of* $\mathfrak{B}$;

(ii) $D \to [D]_{(\mathfrak{B}^+)^2}$ *is an isomorphism of* $\mathfrak{S}(\mathfrak{B}^2)$, *the subalgebra lattice of* $\mathfrak{B}^2$, *onto* $\mathfrak{S}\big((\mathfrak{B}^+)^2\big)$.

Remark. In Lemmas 9 and 10, if D is a subalgebra of $\mathfrak{B}^2$, then $[D]$ means $[D]_{(\mathfrak{B}^+)^2}$ while $\bar{D}$ means $[D]_{\mathfrak{B}_1^2}$.

Proof. Let $\langle \mathfrak{B}, \langle b_0, b_1\rangle\rangle \in \Gamma$, where $\mathfrak{B} = \langle B; F\rangle$. Let $\mathfrak{B}_1 = \langle B_1; F\rangle$ be the algebra freely generated by $\mathfrak{B}$, and $\beta\colon \mathfrak{B}_1^2 \to \mathfrak{B}_1^2$ the mapping given by (v) of Lemma 6. Let $\langle a_0, a_1\rangle \in B_1^2$, and $\beta(\langle a_0, a_1\rangle) = \langle c_0, c_1\rangle$.

For each $\langle a_0, a_1\rangle \in \overline{B^2} - B^2$, define a partial unary operation f on B_1 with $D(f) = \{a_0, a_1\}$, $f(a_0) = c_0$ and $f(a_1) = c_1$. Because of the properties of β and since the diagonal is always a subalgebra, $a_0 = a_1$ iff $c_0 = c_1$. Thus, such an operation f is well defined and 1-1. Let F_1 be the set of all such partial operations; i.e., for each $\langle a_0, a_1\rangle \in \overline{B^2} - B$, there exists an f (defined above) in F_1.

For each $\langle a_0, a_1\rangle \in B_1^2 - \overline{B^2}$, define two partial unary operations f and g on B_1 with $D(f) = \{a_0, a_1\}$, $D(g) = \{b_0, b_1\}$, $f(a_0) = b_0$, $f(a_1) = b_1$, $g(b_0) = a_0$ and $g(b_1) = a_1$. Since the diagonal is a subset of $\overline{B^2}$, we get $a_0 \neq a_1$. Since $b_0 \neq b_1$, such f and g are well defined and 1-1. Let F_2 be the set of all such partial unary operations. (The set of operations F_2 is partly, for dealing with the fact that while B generates B_1, B^2 need not generate B_1^2.)

Let $\mathfrak{B}^+ = \langle B_1; F \cup F_1 \cup F_2\rangle$, set $\langle b_0^+, b_1^+\rangle = \langle b_0, b_1\rangle$, and write $\gamma\big(\langle \mathfrak{B}; \langle b_0, b_1\rangle\rangle\big) = \langle \mathfrak{B}^+, \langle b_0^+, b_1^+\rangle\rangle$. Clearly, the operations of $\mathfrak{B}^+$ are injective, and $b_0^+ \neq b_1^+$.

Since $\langle B_1; F\rangle$ is the algebra freely generated by $\mathfrak{B}$, it is obvious that $\mathfrak{B}^+ = \langle B_1; F \cup F_1 \cup F_2\rangle$ is a singular expansion of $\mathfrak{B}$; i.e., (i) is satisfied.

Let D be a subalgebra of $\mathfrak{B}^2$, and let E be a subalgebra of $(\mathfrak{B}^+)^2$. In order to complete the proof of the lemma, we need only the following five statements:

(a) *Let* $f \in F_1$ *and let* $f(\langle a_0, a_1\rangle)$ *be defined. Then* $\langle a_0, a_1\rangle \in \bar{D}$ *iff* $f(\langle a_0, a_1\rangle) \in D$.

(b) *If* $\langle b_0, b_1\rangle \notin D$, *then* $\bar{D} = [D]$.

(c) *If* $\langle b_0, b_1\rangle \in D$, *then* $[D] = \bar{D} \cup (B_1^2 - \overline{B^2})$.

(d) $[D] \cap B^2 = D$.

(e) $E = [E \cap B^2]$.

Their proofs are routine and are left to the reader.

Now consider the mapping $D \to [D]$ which is from $\mathscr{S}(\mathfrak{B}^2)$ into $\mathscr{S}\big((\mathfrak{B}^+)^2\big)$. Since $\mathfrak{B}^+$ is a singular expansion of $\mathfrak{B}$, if E is a subalgebra of $(\mathfrak{B}^+)^2$, then $E \cap B^2$ is a subalgebra of $\mathfrak{B}^2$. Thus, (e) implies the mapping is onto. Let D_0 and $D_1 \in \mathscr{S}(\mathfrak{B}^2)$. By (d), $D_0 \subseteq D_1$ iff $[D_0] \subseteq [D_1]$. Thus, the mapping is an isomorphism. So (ii) holds.

Now, since $D \to [D]$ is an isomorphism from $\mathfrak{S}(\mathfrak{B}^2)$ onto $\mathfrak{S}\big((\mathfrak{B}^+)^2\big)$ and since $[\langle b_0, b_1\rangle]_{\mathfrak{B}^2} = [\langle b_1, b_0\rangle]_{\mathfrak{B}^2}$, it follows that $[\langle b_0, b_1\rangle]_{(\mathfrak{B}^+)^2} = [\langle b_1, b_0\rangle]_{(\mathfrak{B}^+)^2}$. Thus $\langle \mathfrak{B}^+, \langle b_0, b_1\rangle\rangle \in \Gamma$ and $\gamma\colon \Gamma \to \Gamma$.

It should be observed that γ was defined without the use of any choice axiom, provided, of course, that, without using any choice axiom, one can prove the existence of a function on the class of partial unary algebras so that the image of any partial algebra is the algebra freely generated by that partial algebra. That this can be done is both known and fairly obvious. Moreover, it can be done in such a way that the statement of the Corollary to Lemma 2 is satisfied by the images of $\mathfrak{B}$ and $\mathfrak{C}$ under this function.

Lemma 9 will be used in a proof of Theorem 3 which is independent of any choice axiom. The next lemma will be a similar one for proving Theorem 1.

Let Δ be the class of ordered pairs $\langle \mathfrak{B}, \langle b_0, b_1\rangle\rangle$, where $\mathfrak{B}$ is a partial unary algebra with injective operations, $b_0 \neq b_1$, and $\langle b_0, b_1\rangle \in B^2$. (So $\Gamma \subseteq \Delta$.) Let $\mathfrak{B} = \langle B; F\rangle$, and let $\mathfrak{B}_1 = \langle B_1; F\rangle$ be the algebra freely generated by $\mathfrak{B}$. Let Δ' be the class of ordered triples $\langle \mathfrak{B}, \langle b_0, b_1\rangle, \chi\rangle$, where $\langle \mathfrak{B}, \langle b_0, b_1\rangle\rangle \in \Delta$ and χ is a choice function on the collection of two-element subsets of B_1. Let $K \subseteq F$. Let $\mathfrak{B} - K$ denote the partial algebra derived from $\mathfrak{B}$ by "throwing away" the operations that are in K.

LEMMA 10. *There is a function $\delta\colon \Delta' \to \Delta$ such that if*

$$\delta\big(\langle \mathfrak{B}, \langle b_0, b_1\rangle, \chi\rangle\big) = \langle \mathfrak{B}^+, \langle b_0^+, b_1^+\rangle\rangle,$$

then

(i) *$\mathfrak{B}^+$ is a singular expansion of $\mathfrak{B}$;*

(ii) *$D \to [D]$ is an isomorphism from $\mathfrak{S}(\mathfrak{B}^2)$, the subalgebra lattice of $\mathfrak{B}^2$, onto $\mathfrak{S}\big((\mathfrak{B}^+)^2\big)$;*

(iii) *$\mathfrak{B}^+ = \langle B_1; F \dot\cup G \dot\cup H\rangle$ and $\mathfrak{B}^+ - (G \cup H) = \mathfrak{B}_1$.*

Moreover, there is a function δ' defined on Δ' so that $\delta'\big(\langle \mathfrak{B}, \langle b_0, b_1\rangle, \chi\rangle\big) = \langle \tau, \tau'\rangle$, where τ and τ' are 1-1 indexings of G and H by U and V which are subsets of $B_1^2 - B^2$.

Remark. The proof is very similar to that of Lemma 9. We will examine only those details that are different. Since we have a choice function (and of necessity), we will use only an appropriate "half" of the set F_2 of operations. Half of this half will be the set H. The set G will consist of the set F_1, as in the proof of Lemma 9, together with the other half of the appropriate half of the set F_2. The function δ' is the reason for this rearrangement.

Proof. Let $\langle \mathfrak{B}, \langle b_0, b_1 \rangle, \chi \rangle \in \Delta'$, and let R be the image of $\mathfrak{B}$ under the function of Lemma 7. Set

$$X = \{\langle a_0, a_1 \rangle \mid \langle a_0, a_1 \rangle \in R \text{ and } a_0 = \chi(\{a_0, a_1\})\}.$$

Now $X \cup X^* = R$. Since $R \subseteq B_1^2 - \overline{B^2}$, $X \cap X^* = \emptyset$. Set $Y = \bar{X}$. So $Y^* = \bar{X}^* = \overline{X^*}$. By Lemma 7, we infer that the domain of every operation in the relative subalgebra $\langle R; F \rangle$ is void. So, by Lemma 5 and by (iii) and (iv) of Lemma 2, it follows that $Y \cap Y^* = \emptyset$. Also, since $\bar{R} = B_1^2 - \overline{B^2}$, $X \cup X^* = R$, and $\mathfrak{B}$ is a *unary* partial algebra, we have $Y \cup Y^* = B_1^2 - B^2$.

Define the set F_1 of operations as in the proof of Lemma 9.

For each $\langle a_0, a_1 \rangle \in Y$, define an operation f with $D(f) = \{a_0, a_1\}$, $f(a_0) = b_0$ and $f(a_1) = b_1$.

Let G be the set of all those partial operations defined above including F_1. Set $U = (\overline{B^2} - B^2) \cup Y$. Thus, G has a natural 1-1 indexing by U, call it τ, and $U \subseteq B_1^2 - B^2$.

For each $\langle a_0, a_1 \rangle \in Y$, define a partial unary operation g with $D(g) = \{b_0, b_1\}$, $g(b_0) = a_0$ and $g(b_1) = a_1$. Let H be the set of all these operations, and let $Y = V$. There is a natural 1-1 indexing of H by V, call it τ', and $V \subseteq B_1^2 - B^2$. Set $\delta'(\langle \mathfrak{B}, \langle b_0, b_1 \rangle, \chi \rangle) = \langle \tau, \tau' \rangle$.

Set $F_2' = H \cup (G - F_1)$. (This is the appropriate half of F_2.)

Set $\mathfrak{B}^+ = \langle B_1; F \dot{\cup} G \dot{\cup} H \rangle$, $\langle b_0^+, b_1^+ \rangle = \langle b_0, b_1 \rangle$, and $\delta(\langle \mathfrak{B}, \langle b_0, b_1 \rangle, \chi \rangle) = \langle \mathfrak{B}^+, \langle b_0^+, b_1^+ \rangle \rangle$. As in the proof of Lemma 9, all the additional operations are well defined and 1-1; since $b_0 \neq b_1$, it follows that $\delta\colon \Delta' \to \Delta$.

By construction, (i) holds. Clearly, (ii) holds as in the proof of Lemma 9.

Statements (a), (d) and (e) from the proof of Lemma 9 do not need to be altered, but (b) and (c) do.

(b′) *If* $\{\langle b_0, b_1 \rangle, \langle b_1, b_0 \rangle\} \cap D = \emptyset$, *then* $[D] = \bar{D}$.

(c_0') *If* $\langle b_0, b_1 \rangle \in D$ *and* $\langle b_1, b_0 \rangle \notin D$, *then* $[D] = \bar{D} \cup Y$.

(c_1') *If* $\langle b_0, b_1 \rangle \notin D$ *and* $\langle b_1, b_0 \rangle \in D$, *then* $[D] = \bar{D} \cup Y^*$.

(c_2') *If* $\{\langle b_0, b_1 \rangle, \langle b_1, b_0 \rangle\} \subseteq D$, *then* $[D] = \bar{D} \cup (B_1^2 - \overline{B^2})$.

By (a), if D is any subalgebra of $\mathfrak{B}^2$, then $\bar{D}$ is closed under $F \cup F_1$. By construction, Y and Y^* are closed under F. Also, $B_1^2 - \overline{B^2}$ is closed under F. Since the operations of F_1 are defined only within $\overline{B^2}$, we infer that Y, Y^* and $B_1 - \overline{B^2}$ are closed under F_1. Since we are dealing with unary algebras, $\bar{D} \cup Y$, $\bar{D} \cup Y^*$ and $D \cup (\overline{B_1^2} - B^2)$ are all closed under $F \cup F_1$.

It is easily shown that

(1) $\{\langle b_0, b_1\rangle\}\cup Y$ *is the smallest subset of* B_1^2 *containing* $\langle b_0, b_1\rangle$ *and closed under* F_2';

(2) $\{\langle b_1, b_0\rangle\}\cup Y^*$ *is the smallest subset of* B_1^2 *containing* $\langle b_1, b_0\rangle$ *and closed under* F_2';

(3) $\{\langle b_0, b_1\rangle, \langle b_1, b_0\rangle\}\cup(B_1^2-\overline{B}^2)$ *is the smallest subset of* $\overline{B}_1^2$ *containing both* $\langle b_0, b_1\rangle$ *and* $\langle b_1, b_0\rangle$ *and closed under* F_2';

(4) $\overline{B}^2-\{\langle b_0, b_1\rangle, \langle b_1, b_0\rangle\}$ *is closed under* F_2'.

Now statements (c_i') follow easily from these four statements.

The rest of the proof is identical with that of Lemma 9.

It should be observed that the proof of Lemma 10 requires *no* choice axiom.

Let Γ' be the class of partial unary algebras with injective operations, let $\mathfrak{B} = \langle B; F\rangle \in \Gamma'$, and let $\mathfrak{B}_1 = \langle B_1; F\rangle$ be the algebra freely generated by $\mathfrak{B}$.

LEMMA 11. *There is a function* $\gamma'\colon \Gamma'\to\Gamma'$ *such that if* $\gamma'(\mathfrak{B}) = \langle B'; F'\rangle$, *then* $B' = B_1$, $F' = F\dot{\cup}G'\dot{\cup}H'$ *and* $\mathfrak{B}'-(G'\cup H') = \mathfrak{B}_1$. *Also, there is a function* γ'' *defined on* Γ' *such that* $\gamma''(\mathfrak{B}) = \langle \mu, \mu'\rangle$, *where* μ *and* μ' *are* 1-1 *indexings of* G' *and* H', *respectively, by* $B_1^2-B^2$. *Moreover, for* $f\in F'-F$, *it holds that* $D(f, \mathfrak{B}') = \emptyset$.

The proof is trivial.

4. Proofs of the theorems. Our treatment will be informal. We will not deal with a specific axiom system for set theory. We will assume that we are dealing with a theory that is strong enough so that we can form algebras, the set of finite sequences on a set, and so on. We will, of course, assume no choice axiom is part of our system.

We give a proof for Theorem 1 which employs C_2 but no other choice axiom, and we give a proof of Theorem 3 using no choice axiom. All of this rests on the fact that there exists a function defined on the class of partial algebras so that the image of a partial algebra is the algebra freely generated by it.

We complete the paper by giving a proof of Lemma 1 and Theorem 2.

Proof of Theorem 3. Let $\mathfrak{m}$ be any cardinal number, let $\mathfrak{L}\in \boldsymbol{K}$, let α be an automorphism of $\mathfrak{L}$ of order two, and assume that there is a complete-join irreducible element $r\neq 0$ with $r\alpha = r$. Let ${}^0\mathfrak{B} = \langle {}^0B; {}^0F\rangle = \mathfrak{B}$ as given by Lemma 8, and let $\langle {}^0b_0, {}^0b_1\rangle = \langle b_0^+, b_1^+\rangle$ be given by Lemma 7. By (iii), (iv) and (v) of Lemma 8, $\langle {}^0\mathfrak{B}, \langle {}^0b_0, {}^0b_1\rangle\rangle \in \Gamma$. If $\langle {}^n\mathfrak{B}, \langle {}^nb_0, {}^nb_1\rangle\rangle$ has been defined, set

$$\begin{aligned}\langle {}^{n+1}\mathfrak{B}, \langle {}^{n+1}b_0, {}^{n+1}b_1\rangle\rangle &= \langle\langle {}^{n+1}B; {}^{n+1}F\rangle, \langle {}^{n+1}b_0, {}^{n+1}b_1\rangle\rangle\\ &= \gamma\big(\langle {}^n\mathfrak{B}, \langle {}^nb_0, {}^nb_1\rangle\rangle\big),\end{aligned}$$

where γ is the function given by Lemma 9. Let $A = \bigcup({}^nB \mid n = 0, 1, 2, \ldots)$, $F = \bigcup({}^nF \mid n = 0, 1, 2, \ldots)$, and $\mathfrak{A} = \langle A; F\rangle$.

Let $a \in A$ and $f \in F$. There exists a k with $a \in {}^kB$ and $f \in {}^kF$. Since ${}^{k+1}\mathfrak{B}$ is a singular expansion of ${}^k\mathfrak{B}$, $f(a)$ is defined in ${}^{k+1}\mathfrak{B}$. Also, since ${}^{j+1}\mathfrak{B}$ is a singular expansion of ${}^j\mathfrak{B}$, the value of $f(a)$ in ${}^j\mathfrak{B}$ for $j \geqslant k+1$ is the same as in ${}^{k+1}\mathfrak{B}$. So $\mathfrak{A}$ is a well defined unary algebra.

Let us agree that, for a subalgebra D of $\mathfrak{B}^2$, $[D] = [D]_{\mathfrak{A}^2}$ and $[D]_k = [D]_{({}^k\mathfrak{B})^2}$. It can easily be shown that $[D] = \bigcup([D]_k \mid k = 0, 1, 2, \ldots)$. Using this, it is easy to show that the map $D \to [D]$ is an isomorphism of $\mathfrak{S}(\mathfrak{B}^2)$ *onto* $\mathfrak{S}(\mathfrak{A}^2)$.

Now $\mathfrak{m} \leqslant |B| \leqslant |A|$. By Lemma 8, there is an isomorphism σ from $\mathfrak{L}$ onto $\mathfrak{S}(\mathfrak{B}^2)$. Thus, if $x\varrho = [x\sigma]$, then ϱ is an isomorphism from $\mathfrak{L}$ onto $\mathfrak{S}(\mathfrak{A}^2)$. Since, for any $X \subseteq A^2$, we have $[X^*] = [X]^*$, and since, by (ii) of Lemma 8, $(x\alpha)\sigma = (x\sigma)^*$, we infer that $(x\alpha)\varrho = [(x\alpha)\sigma] = [(x\sigma)^*] = [x\sigma]^* = (x\varrho)^*$.

The proof of Theorem 1 will mimic the proof of Theorem 3. However, we must be careful to avoid making countably many choices of choice functions since we want to assume only C_2.

Proof of Theorem 1. Let $\mathfrak{m}$ be any cardinal number, let $\mathfrak{L} \in \boldsymbol{K}$, and let α be an automorphism of $\mathfrak{L}$ of order two. Let ${}^0\mathfrak{B} = \langle {}^0B; {}^0F\rangle = \mathfrak{B}$ and $\langle {}^0b_0, {}^0b_1\rangle = \langle b_0^+, b_1^+\rangle$, where $\mathfrak{B}$ and $\langle b_0^+, b_1^+\rangle$ are given in Lemma 8.

Set ${}^0\mathfrak{C} = {}^0\mathfrak{B}$. If ${}^n\mathfrak{C}$ has been defined, set $\langle {}^{n+1}C, {}^{n+1}G\rangle = {}^{n+1}\mathfrak{C} = \gamma'({}^n\mathfrak{C})$ and $\langle {}^{n+1}\mu, {}^{n+1}\mu'\rangle = \gamma''({}^n\mathfrak{C})$, where γ' and γ'' are functions given by Lemma 11. Let $D = \bigcup({}^nC \mid n = 0, 1, 2, \ldots)$, assume C_2, and let χ be a choice function on the collection of two-element subsets of D.

Since ${}^0B \subseteq {}^0B_1 = {}^1C \subseteq D$, if ${}^0\chi$ is the restriction of χ to the collection of two-element subsets of ${}^0B_1$, then $\langle {}^0\mathfrak{B}, \langle {}^0b_0, {}^0b_1\rangle, {}^0\chi\rangle \in \Delta'$. Suppose ${}^n\mathfrak{B} = \langle {}^nB; {}^nF\rangle$, and suppose that the element $\langle {}^n\mathfrak{B}, \langle {}^nb_0, {}^nb_1\rangle, {}^n\chi\rangle$ of Δ' has been defined in such a way that ${}^nB \subseteq {}^nC$, ${}^nF \subseteq {}^nG$ and so that $D(f, {}^n\mathfrak{B}) \supseteq D(f, {}^n\mathfrak{C}) \cap {}^nB$. Set

$$\langle {}^{n+1}\mathfrak{B}', \langle {}^{n+1}b_0, {}^{n+1}b_1\rangle\rangle = \delta\big(\langle {}^n\mathfrak{B}, \langle {}^nb_0, {}^nb_1\rangle, {}^n\chi\rangle\big),$$

where δ is given by Lemma 10, and ${}^{n+1}\mathfrak{B}' = \langle {}^{n+1}B'; {}^{n+1}F'\rangle$. By the Corollary to Lemma 2, we obtain ${}^{n+1}B = {}^nB_1 \subseteq {}^nC_1 = {}^{n+1}C$. In accordance with Lemmas 10 and 11, we have

$${}^{n+1}F' = {}^nF \,\dot{\cup}\, G_n \,\dot{\cup}\, H_n \quad \text{and} \quad {}^{n+1}G = {}^nG \,\dot{\cup}\, G_n' \,\dot{\cup}\, H_n'.$$

Now, using the functions γ'' and δ', as given by Lemmas 11 and 10, and their values μ, μ' and τ, τ', respectively, we set $\hat{f} = \mu\big(\tau^{-1}(f)\big)$ for $f \in G_n$ and $\hat{f} = \mu'\big(\tau'^{-1}(f)\big)$ for $f \in H_n$. Set

$${}^{n+1}F = {}^nF \cup \{\hat{f} \mid f \in G_n \cup H_n\}.$$

Note that ${}^{n+1}F \subseteq {}^{n+1}G$. We define the unary partial algebra ${}^{n+1}\mathfrak{B} = \langle {}^{n+1}B; {}^{n+1}F\rangle$ by setting ${}^{n+1}B = {}^{n+1}B'$, by retaining all operations from ${}^{n}F$ as they were defined in ${}^{n+1}\mathfrak{B}'$, and by having the value $\hat{f}(x)$ in ${}^{n+1}\mathfrak{B}$ equal the value $f(x)$ in ${}^{n+1}\mathfrak{B}'$. Note that D is a subalgebra of ${}^{n+1}\mathfrak{B}' \times {}^{n+1}\mathfrak{B}'$ iff D is a subalgebra of ${}^{n+1}\mathfrak{B} \times {}^{n+1}\mathfrak{B}$ and that ${}^{n+1}\mathfrak{B}$ is a singular expansion of ${}^{n}\mathfrak{B}$. Also, we infer that ${}^{n+1}B \subseteq {}^{n+1}C$. From Lemmas 10 and 11 and from the definition of ${}^{n+1}\mathfrak{B}$, it follows that $D(f, {}^{n+1}\mathfrak{B}) \supseteq D(f, {}^{n+1}\mathfrak{C}) \cap B$ for all $f \in {}^{n+1}F$. Using the Corollary to Lemma 2, we obtain ${}^{n+1}B_1 \subseteq {}^{n+1}C_1 = {}^{n+2}C \subseteq D$. So we can define ${}^{n+1}\chi$ to be the restriction of χ to ${}^{n+1}B_1$, and we have $\langle {}^{n+1}\mathfrak{B}, \langle {}^{n+1}b_0, {}^{n+1}b_1\rangle, \chi^{n+1}\rangle \in \Delta'$. Then the rest of the required conditions are satisfied, as noted above.

Now we have the sequence of partial unary algebras ${}^{0}\mathfrak{B}, \ldots, {}^{n}\mathfrak{B}, \ldots$ The rest of the proof is exactly as in the proof of Theorem 3.

Proof of Lemma 1. We suppose that $\mathfrak{A} = \langle A; F\rangle$ represents $\langle \mathfrak{L}, a\rangle$ and that aa is the complement of a for some $a \in L$. We take ϱ as the isomorphism from $\mathfrak{L}$ to $\mathfrak{S}(\mathfrak{A}^2)$ satisfying $(xa)\varrho = (x\varrho)^*$. Set $B = a\varrho$. So, we have $B \vee B^* = B \cup B^* = A^2$ since A^2 is the greatest element of $\mathfrak{S}(\mathfrak{A}^2)$. We also know that $B \wedge B^* = B \cap B^*$ is the zero of $\mathfrak{S}(\mathfrak{A}^2)$. $D = \{\langle a, a\rangle \mid a \in A\}$ is always a subalgebra of $\mathfrak{A}^2$. So $B \cap B^* \subseteq D$. Now let $a_0, a_1 \in A$ with $a_0 \neq a_1$. These facts imply that precisely one of $\langle a_0, a_1\rangle$ and $\langle a_1, a_0\rangle$ is a member of B. So we define the required χ by $\chi(\{a_0, a_1\}) = a_0$ iff $\langle a_0, a_1\rangle \in B = a\varrho$.

We close with a proof of Theorem 2.

Proof of Theorem 2. We need only show that, for each cardinal $\mathfrak{m}$, there is a pair $\langle \mathfrak{L}, a\rangle$ with $|L| \geqslant \mathfrak{m}$ and with an $l \in L$ such that la is the complement of l. Let $|X| \geqslant \mathfrak{m}$, $Y = X \times \{0, 1\}$, and $\mathfrak{L} = \langle 2^Y; \cap, \cup\rangle$. Let a be the unique extension of the mapping which sends $\langle x, 0\rangle$ to $\langle x, 1\rangle$ and $\langle x, 1\rangle$ to $\langle x, 0\rangle$ for all $x \in X$. We are done if we take $l = X \times \{0\}$.

Acknowledgement. The author wishes to thank the referee for his many comments and suggestions.

REFERENCES

[1] G. Birkhoff and O. Frink, *Representations of lattices by sets*, Transactions of the American Mathematical Society 64 (1948), p. 229-316.

[2] G. Grätzer, *Universal algebra*, Princeton, New Jersey, 1968.

[3] — *On the family of certain subalgebras of a universal algebra*, Indagationes Mathematicae 27 (1965), p. 790-802.

[4] — and W. A. Lampe, *On subalgebra lattices of universal algebras*, Journal of Algebra 7 (1967), p. 263-270.

[5] G. Grätzer and E. T. Schmidt, *Characterizations of congruence lattices of abstract algebras*, Acta Scientiarum Mathematicarum 24 (1963), p. 34-59.

[6] J. D. Halpern, *The independence of the axiom of choice from the Boolean prime ideal theorem*, Fundamenta Mathematicae 54 (1964), p. 57-65.

[7] — and A. Lévy, *The Boolean prime ideal theorem does not imply the axiom of choice*, in *Axiomatic set theory*, Proceedings of Symposia in Pure Mathematics 13 (1971), p. 83 - 134.

[8] А. А. Искандер, *Структура соответствий универсальной алгебры*, Известия Академии наук СССР, серия математическая, 29 (1965), p. 1357 - 1372.

[9] — (A. A. Iskander), *On subalgebra lattices of universal algebra*, Proceedings of the American Mathematical Society 32 (1972), p. 32 - 36.

[10] W. A. Lampe, *Notes on related structures of a universal algebra*, Pacific Journal of Mathematics 43 (1972), p. 189 - 205.

[11] — *On related structures of a universal algebra*, Ph. D. Dissertation, Pennsylvania State University, March 1969.

UNIVERSITY OF HAWAII
HONOLULU, HAWAII

Reçu par la Rédaction le 20. 11. 1971;
en version modifiée le 16. 7. 1973