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BOUNDED MONTEL UNIVALENT FUNCTIONS

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1. Preliminary remarks. The class $\mathcal{S}$ of functions $f(z)$ regular and univalent in the open unit disk Δ,

$$\Delta = \{z \in C: |z| < 1\},$$

and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ has been the prime target of analysts working in geometric function theory for over 7 decades. The motivation for much of their work was Bieberbach's Conjecture [2]. Now that Professor Louis de Branges has shown the conjecture to be true, attention of investigators may turn to other classes as well. For example, meromorphic univalent functions, regular univalent functions with other normalizations and multivalent functions. The object of this report* is the class of bounded functions with Montel's normalization.

Let a be fixed, $0 < a < 1$, and let $f(z)$ be regular and univalent in Δ. The function $f(z)$ is in *Montel's class* $\mathcal{M}(a)$ if

$$f(0) = 0 \quad \text{and} \quad f(a) = a, \tag{1.1}$$

i.e., 0 and a are fixed points. (Normalizations such as this were proposed by P. Montel and encouraged by work of R. Nevanlinna and G. Pick; see, e.g., [9].) If $f(z)$ is in $\mathcal{M}(a)$, then there is a function $g(z)$ in $\mathcal{S}$ such that

$$f(z) = \frac{ag(z)}{g(a)}, \tag{1.2}$$

and conversely. This relationship has been exploited by a number of authors (Lewandowski [8], Krzyż [6], Krzyż and Złotkiewicz [7]) to determine distortion and covering properties for $\mathcal{M}(a)$ and some of its subclasses.

The subclass of $\mathcal{M}(a)$ whose members $F(z)$ are bounded by M,

$$|F(z)| \leqslant M \quad \text{for } z \in \Delta,$$

* This work was done while the second-named author was a visiting professor at the University of Delaware.

will be denoted by $\mathscr{M}(a; M)$. [In this notation, $\mathscr{M}(a)$ is sometimes written $\mathscr{M}(a; +\infty)$. However, for brevity, we will usually write $\mathscr{M}$ for $\mathscr{M}(a)$, and $\mathscr{M}(M)$ for $\mathscr{M}(a; M)$.]

It is our purpose to establish some properties of functions in $\mathscr{M}(M)$. For example, if $F(z) = A_1 z + A_2 z^2 + \ldots$, we find sharp upper and lower bounds for $|A_1|$. We give the sharp Koebe constant for the class along with some distortion theorems and establish the correct upper bound for

$$\int_{-\pi}^{\pi} \Phi(\log|F(re^{i\theta})|)\, d\theta.$$

2. Coefficient and distortion theorems. Transformation (1.2) can be used to determine some properties of M from $\mathscr{S}$. However, (1.2) is not sufficiently restrictive to give good information about $\mathscr{M}(M)$. The next statement gives a more effective transformation.

LEMMA 1. *If $F(z)$ is in $\mathscr{M}(M)$ and α is real, then*

$$G(z) = \frac{F(z)(1+e^{i\alpha}a/M)^2}{(1+e^{i\alpha}F(z)/M)^2} \tag{2.1}$$

is in $\mathscr{M}$.

This follows from properties of the Koebe function

$$k_\theta(z) = \frac{z}{(1+e^{i\theta}z)^2}. \tag{2.2}$$

THEOREM 1. *If $F(z) = A_1 z + A_2 z^2 + \ldots$ is in $\mathscr{M}(M)$, then*

$$\left(\frac{1-a}{M-a}M\right)^2 \leqslant |A_1| \leqslant \left(\frac{1+a}{M+a}M\right)^2. \tag{2.3}$$

Furthermore, both bounds are best possible.

From (2.1) we get

$$G'(0) = A_1\left(1+\frac{e^{i\alpha}a}{M}\right)^2 \tag{2.4}$$

The distortion theorem for $\mathscr{S}$ and (1.2) give

$$(1-a)^2 \leqslant |G'(0)| \leqslant (1+a)^2. \tag{2.5}$$

Now, with proper choices of α, (2.4) and (2.5) give the inequalities (2.3). The left-hand side is rendered sharp by

$$L(z) = M\breve{k}_\pi\left(M\left(\frac{1-a}{M-a}\right)^2 k_\pi(z)\right), \tag{2.6}$$

and the right-hand side by

$$K(z) = M\check{k}_0\left(M\left(\frac{1+a}{M+a}\right)^2 k_0(z)\right) \tag{2.7}$$

($\check{k}$ denotes the inverse of k).

Our next theorem provides information about the region of values of A_1 for $F(z)$ in $\mathcal{M}(M)$.

THEOREM 2. *If* $F(z) = A_1 z + A_2 z^2 + \ldots$ *is in* $\mathcal{M}(M)$, *then the region of values of* $\log A_1$ *lies in the simply-connected and convex set given by*

$$\bigcup_{\alpha}\left\{\omega\colon \left|\omega \log\frac{M^2(1-a^2)}{(M+e^{i\alpha}a)^2}\right| \leqslant \log\frac{1+a}{1-a}\right\} \tag{2.8}$$

for $0 \leqslant \alpha < 2\pi$.

Our justification of Theorem 2 follows from (1.2), (2.4) and the inequality

$$\left|\log\frac{f(z)}{z} + \log(1-|z|^2)\right| \leqslant \log\frac{1+|z|}{1-|z|} \tag{2.9}$$

for $f(z)$ in $\mathcal{S}$ due to Grunsky ([3]; see also [4], p. 107). From (1.2) and (2.4) we get

$$\log\frac{a}{g(a)} = \log\left[A_1\left(1+\frac{e^{i\alpha}a}{M}\right)^2\right] \tag{2.10}$$

and, consequently,

$$-\log\frac{g(a)}{a} - \log(1-a^2) = \log\frac{A_1}{(1-a^2)} + 2\log\left(1+\frac{e^{i\alpha}a}{M}\right). \tag{2.11}$$

Let $W = \log(A_1/(1-a^2))$; then (2.9) and (2.11) show that W lies in the disk

$$D_\alpha = \left\{\omega\colon \left|\omega + 2\log\left(1+\frac{e^{i\alpha}a}{M}\right)\right| \leqslant \log\frac{1+a}{1-a}\right\} \tag{2.12}$$

for some α.

The mapping $L(z) = 2\log(1+az/M)$ is univalent and convex for $|z| \leqslant 1$; hence

$$\Gamma = \{\omega\colon \omega = L(e^{i\theta}),\ 0 \leqslant \theta < 2\pi\}$$

is a closed convex Jordan curve which lies in the rectangle given by

$$2\log(1-a/M) \leqslant \operatorname{Re}\{\omega\} \leqslant 2\log(1+a/M)$$

and

$$|\operatorname{Im}\{\omega\}| \leqslant 2\arcsin(a/M).$$

Now, since $2\arcsin(a/M) < \log(1+a)/(1-a)$, we conclude that the set $\bigcup_{\alpha} D_\alpha$ is simply-connected and convex and that all values $\log(A_1/(1-a^2))$ lie in $\bigcup_{\alpha} D_\alpha$. This concludes Theorem 2.

Is Theorem 2 best possible? Functions corresponding to the boundary points of the region of values of $\log(f(a)/a)$ (inequality (2.9)), in $\mathscr{S}$, map Δ onto the plane slit along a ray or a spiral [4]. However, if it is a spiral, then (2.1) is not applicable. Hence the question of whether the region of values of $\log A_1$ is exactly the set given by (2.8) remains open.

The $\frac{1}{4}$-Theorem for $\mathscr{S}$ has an analog in $\mathscr{M}(M)$.

THEOREM 3. *All values omitted by any member of $\mathscr{M}(M)$ lie outside the open disk given by*

$$|\omega| \leqslant M\check{k}_0\left(\frac{M}{4}\left(\frac{1-a}{M-a}\right)^2\right). \tag{2.13}$$

Suppose $d = \min|F(z)|$ for $|z| = 1$. Then (1.1) and (2.1) and the triangle inequality give

$$\frac{d|1+e^{i\alpha}a/M|^2}{(1-d/M)^2} \geqslant \min_{\mathscr{M}}|G(z)| \geqslant \min_{\mathscr{S}}\left|\frac{af(z)}{f(a)}\right| \geqslant \frac{1}{4}(1-a)^2.$$

Now, if we choose $\alpha = \pi$, then

$$\frac{d(1-a/M)^2}{(1-d/M)^2} \geqslant \frac{(1-a)^2}{4},$$

which, in turn, is equivalent to (2.13). This bound is sharp as (2.6) shows.

Theorem 3 gives the Koebe disk about the origin for $\mathscr{M}(M)$: one may ask for the Koebe disk centered at a. Our solution uses symmetrization and properties of hyperbolic distance.

First we recall some known results. If G is a region in the plane, then its circularly symmetric image with respect to the ray $\{x \in \boldsymbol{R}\colon x \leqslant a\}$ will be denoted by G^*. Also, the hyperbolic distance between ω_1 and ω_2 relative to G will be written as $\varrho(\omega_1, \omega_2; G)$. It is known [6] that this metric does not increase under symmetrization; consequently, we may write

$$\varrho(0, a; G) \geqslant \varrho(0, a; G^*). \tag{2.14}$$

Suppose now that d_0 is the radius of the Koebe disk about a for $\mathscr{M}(M)$. This means there is a function $h(z)$ in the class such that

$$d_0 = \operatorname{dist}\{a, \partial h[\Delta]\}, \tag{2.15}$$

where $h[\Delta]$ is the image of Δ under $h(z)$ and $\partial h[\Delta]$ is its boundary.

Let $D = \{|\omega| < M\colon \omega \notin [a+d_0, M]\}$, i.e., D is the open disk of radius M centered at the origin and slit along $[a+d_0, M]$. Then $(h[\Delta])^* \subset D$ and (see [10])

$$\varrho(0, a; (h[\Delta])^*) \geqslant \varrho(0, a; D). \tag{2.16}$$

The hyperbolic distance is a conformal invariant, i.e.,

$$\varrho(0, a; \Delta) = \varrho(0, h(a); h[\Delta])$$

(see [10]) and this with (2.14) and (2.16) implies that

$$\varrho(0, a; \Delta) \geqslant \varrho(0, a; D).$$

The function $K(z)$ of (2.7) maps Δ onto the disk of radius M centered at the origin and slit along $[K(1), M]$. Also, $D \subset K[\Delta]$ and

$$\varrho(0, a; \Delta) \geqslant \varrho(0, a; D) \geqslant \varrho(0, a; K[\Delta]).$$

However, $\varrho(0, a; K[\Delta]) = \varrho(0, a; \Delta)$, which guarantees that $D \equiv K[\Delta]$ and that d_0 is given by $K(z)$. This proves the next statement.

THEOREM 4. *If*

$$d_0 = M\check{k}_0\left(\frac{M}{4}\left(\frac{1+a}{M+a}\right)^2\right) - a,$$

then

$$\{\omega\colon |\omega - a| < d_0\} \subset \bigcap_{\mathcal{M}(M)} F[\Delta].$$

The value d_0 is best possible.

The question of finding the Koebe domain, $\bigcap_{\mathcal{M}(M)} F[\Delta]$, remains open. We have shown it includes the disks of Theorems 3 and 4. Is it disjoint for some a? (**P 1355**) Other methods will have to be devised for a resolution of this question.

3. The integral means. Bounds on the integral means of functions and their derivatives have proved useful, and obtaining them has provided challenges of intrinsic interest. The most significant contribution has been made by Baernstein [1]. His methods have been exploited by a number of authors (see, e.g., [2]).

THEOREM 5. *If $\Phi(x)$ is convex and non-decreasing on $\boldsymbol{R}$, and $K(z)$ is as in* (2.7), *then*

$$\int_{-\pi}^{\pi} \Phi(\log|F(re^{i\theta})|)\, d\theta \leqslant \int_{-\pi}^{\pi} \Phi(\log|K(re^{i\theta})|\, d\theta) \tag{3.1}$$

for all $F(z)$ in $\mathcal{M}(M)$ and $0 < r < 1$.

The proof makes use of the method of Baernstein with some modifications. We state some notation and theorems given by Baernstein [1] (see also [2]).

If g is a measurable, extended real-valued function on $[-\pi, \pi]$, then

$$g^*(\theta) = \sup_E \int_E g(\theta)\,d\theta, \tag{3.2}$$

the supremum being taken over all Lebesgue measurable sets E in $[-\pi, \pi]$ for which $m(E) = 2\theta$.

LEMMA 2 ([1]). *If u is continuous and subharmonic in the annulus $r_1 < |z| < r_2$, then u^* is continuous in the semi-annulus*

$$\{re^{i\theta}\colon r_1 < r < r_2,\ 0 \leqslant \theta \leqslant \pi\}$$

and subharmonic in its interior.

LEMMA 3 ([1]). *The following are equivalent for g and h in $L^1[-\pi, \pi]$:*

(a) *For every convex non-decreasing function Φ on $\boldsymbol{R}$,*

$$\int_{-\pi}^{\pi} \Phi(g(x))\,dx \leqslant \int_{-\pi}^{\pi} \Phi(h(x))\,dx.$$

(b) *For every t in $\boldsymbol{R}$,*

$$\int_{-\pi}^{\pi} [g(x)-t]^+\,dx \leqslant \int_{-\pi}^{\pi} [h(x)-t]^+\,dx.$$

(c) $g^*(\theta) \leqslant h^*(\theta)$ *for* $0 \leqslant \theta \leqslant \pi$.

For the proof of our theorem we let

$$u(\omega) = \begin{cases} -\log|F(\omega)| & \text{for } \omega \text{ in } F[\Delta], \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

$$v(\omega) = \begin{cases} -\log|K(\omega)| & \text{for } \omega \text{ in } K[\Delta], \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Both these functions are continuous and subharmonic for $0 < |\omega|$ and they are harmonic in $F[\Delta]\setminus\{0\}$ and $K[\Delta]\setminus\{0\}$, respectively. The equivalence of (a) and (b) in Lemma 3 shows that to establish (3.1) it suffices to prove that

$$\int_{-\pi}^{\pi} \log^+\left(\frac{|F(re^{i\theta})|}{\varrho}\right)d\theta \leqslant \int_{-\pi}^{\pi} \log^+\left(\frac{|K(re^{i\theta})|}{\varrho}\right)d\theta, \tag{3.5}$$

$\varrho > 0$; and this, in view of the formula of Cartan (see [1]) is equivalent to

$$\int_{-\pi}^{\pi} [u(\varrho e^{i\theta})+\log r]^+\,d\theta \leqslant \int_{-\pi}^{\pi} [v(\varrho e^{i\theta})+\log r]^+\,d\theta \tag{3.6}$$

for $0 < r < 1$ and $0 < \varrho \leqslant M$. Finally, the equivalence of (b) and (c) in Lemma 3 shows that (3.6), and hence Theorem 5, follows from showing that

$$u^*(\varrho e^{i\theta}) \leqslant v^*(\varrho e^{i\theta}) \tag{3.7}$$

for $0 \leqslant \theta \leqslant \pi$ and $0 < \varrho < M$.

$u^*(\varrho e^{i\theta})$ is subharmonic in the open semi-disk

$$D = \{\omega\colon |\omega| < M,\ \operatorname{Im}\omega > 0\},$$

while $v^*(\varrho e^{i\theta})$ is harmonic there (see, e.g., [1], Proposition 5). Consequently, $u^*(\varrho e^{i\theta}) - v^*(\varrho e^{i\theta})$ is subharmonic in D.

For ω in some neighborhood of the origin we may write

$$u(\omega) = -\log\left|\frac{\omega}{F'(0)}\right| + u_1(\omega)$$

and

$$v(\omega) = -\log\left|\frac{\omega}{K'(0)}\right| + v_1(\omega),$$

$u_1(\omega)$ and $v_1(\omega)$ are harmonic and $u_1(0) = v_1(0) = 0$. Then

$$u(\omega) - v(\omega) = \log\left|\frac{F'(0)}{K'(0)}\right| + u_1(\omega) - v_1(\omega)$$

is harmonic about the origin. Therefore $(u^* - v^*)(\omega)$ is bounded and continuous on $\bar{D}$, the closure of D. To guarantee (3.7), an application of the Phragmén–Lindelöf Principle (see [10]) shows that it is enough to prove that $(u^* - v^*)(\omega) \leqslant 0$ on ∂D.

On $|\omega| = M$ and for $0 \leqslant \omega \leqslant M$, $u^*(\omega) = v^*(\omega) = 0$; consequently, $(u^* - v^*)(\omega) \leqslant 0$ there. Hence only the interval $[-M, 0]$ must still be considered.

Let d be the Koebe constant for $\mathcal{M}(M)$; d is the radius of the disk given in (2.13) and $d = \operatorname{dist}\{0, \partial K[\Delta]\}$. If $d_0 = \operatorname{dist}\{0, \partial F[\Delta]\}$, then, unless $F \equiv K$, $d_0 > d$.

First we assume $-d \leqslant \varrho e^{i\pi} \leqslant 0$. In this case both $u(\omega)$ and $v(\omega)$ are harmonic and we may write

$$u^*(\varrho e^{i\pi}) = -2\pi\log\left|\frac{\varrho}{F'(0)}\right|, \quad v^*(\varrho e^{i\pi}) = -2\pi\log\left|\frac{\varrho}{K'(0)}\right|$$

and

$$(u^* - v^*)(\varrho e^{i\pi}) = -2\pi\log\left|\frac{F'(0)}{K'(0)}\right| \leqslant 0.$$

This completes the proof for $-d \leqslant -\varrho < 0$.

For ϱ near zero,

$$(u^* - v^*)(\varrho e^{i\pi}) = -2\theta \log \left|\frac{F'(0)}{K'(0)}\right| + O(\varrho),$$

which implies that

$$\lim_{\omega \to 0} (u^* - v^*)(\omega) \leqslant 0.$$

This covers the case at zero.

For $d < \varrho < M$ and $\varepsilon > 0$ we let

$$P(\varrho e^{i\theta}) = (u^* - v^*)(\varrho e^{i\theta}) - \varepsilon\theta \tag{3.8}$$

for a given $\varepsilon > 0$. The function $P(\omega)$ is subharmonic in D and continuous in $\bar{D} \setminus \{0\}$. Choose ϱ_0 $(-M < \varrho_0 < -d)$; then

$$P(\varrho_0 e^{i\pi}) = \sup \{P(\varrho_0 e^{i\theta})\}.$$

The symmetric non-increasing rearrangement u^* of u is

$$u^*(\varrho e^{i\theta}) = \int_{-\theta}^{\theta} u(\varrho e^{it})\, dt.$$

Hence we may conclude (see [1], Proposition 2) that

$$\lim_{\theta \to \pi} \frac{u^*(\varrho_0 e^{i\pi}) - u^*(\varrho_0 e^{i\theta})}{\pi - \theta} = 2 \min_{0 \leqslant \theta \leqslant \pi} u(\varrho_0 e^{i\theta}) = 0, \tag{3.9}$$

as the circle $|\omega| = \varrho_0$ meets the complement of $F[\Delta]$.

Furthermore, for $\varrho < d_0$ we have

$$v^*(\varrho e^{i\theta}) = -2\theta \ln \frac{\varrho}{|F'(0)|} + \int_{-\theta}^{\theta} v_1(\varrho e^{it})\, dt,$$

which gives

$$\frac{\partial v^*(\varrho e^{i\theta})}{\partial \theta} = -2 \ln \frac{\varrho_0}{|F'(0)|} + 2v_1(\varrho_0 e^{i\theta}) = 2v(\varrho_0 e^{i\theta}) \geqslant 0 \tag{3.10}$$

and, as a result, (3.9) and (3.10) yield

$$\lim_{\theta \to \pi} \frac{\partial P}{\partial \theta}(\varrho_0 e^{i\theta}) = -\varepsilon - 2v(\varrho_0 e^{i\pi}) < 0.$$

(3.9) and (3.10) taken together contradict the condition that

$$P(\varrho_0 e^{i\pi}) - P(\varrho_0 e^{i\theta}) \geqslant 0.$$

From this we infer that

$$u^*(-\varrho) \leqslant v^*(-\varrho) + \varepsilon\pi$$

for $d < \varrho < d_0$; then letting $\varepsilon \to 0$ gives the result sought.

The case where $-M < -\varrho \leqslant d_0$ is handled in a similar manner. This concludes our discussion of the theorem.

COROLLARY. *If $F(z)$ is in $\mathcal{M}(M)$, then*

$$|F(z)| \leqslant \max_{|z|=r} |K(z)| = K(-|z|).$$

This is obtained by choosing $\Phi(t) = \exp t^p$ and letting $p \to +\infty$.

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