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ON SEPARABILITY IN SEMI-STRATIFIABLE SPACES

BY

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1. Introduction. A set Y in a space is said to be *peripherally separable* [6] if $\mathrm{Bd}\, Y$ is separable. A space is said to be *locally peripherally separable* [6] if it has a basis consisting of peripherally separable open sets. A space X is said to satisfy *condition* A if, given any collection of disjoint closed sets $\{F_\lambda \mid \lambda \in \Lambda\}$ in X, for each $\lambda \in \Lambda$ there exists an $x_\lambda \in F_\lambda$ such that

$$\mathrm{Cl}\big(\bigcup_\lambda F_\lambda\big) - \bigcup_\lambda F_\lambda \subset \mathrm{Cl}\{x_\lambda \mid \lambda \in \Lambda\}.$$

This condition guarantees that any pseudo-open mapping f from X onto Y satisfies the following property: for each subset S of Y, there exists a subset T of $f^{-1}(S)$, containing exactly one point of each $f^{-1}(S)$, such that $f(\mathrm{Cl}\, T) \supset \mathrm{Cl}\, S$ (we have $f(\mathrm{Cl}\, T) \subset \mathrm{Cl}\, S$ by continuity). This property is analogous to the defining property of almost open mappings: a mapping $f\colon X \to Y$ is *almost open* [1] if, for each $y \in Y$, there is a point $x \in f^{-1}(y)$ such that whenever U is an open set containing x, then $y \in \mathrm{Int} f(U)$. (Equivalently, $f(x^{\mathrm{Int}}) \subset y^{\mathrm{Int}}$, where $z^{\mathrm{Int}} = \{W \mid z \in \mathrm{Int}\, W\}$. Note that the continuity of f implies $f(x^{\mathrm{Int}}) \supset y^{\mathrm{Int}}$.)

Jones [6] has investigated the property of local peripheral separability in metric spaces and has shown that a connected, locally connected, locally peripherally separable metric space is separable. Treybig [10] and Roy [7] have given examples showing that the hypothesis of local connectedness is required. We show that, in the class of stratifiable spaces, condition A implies that every open set is peripherally separable. We conclude from this that a connected stratifiable space satisfying condition A is separable. In fact, we show that, in the class of Nagata spaces, condition A is equivalent to the separability of a space minus its isolated points. Analogous results are given for developable spaces. All spaces are assumed to be T_1.

2. Semi-stratifiable spaces. The following diagram summarizes the relationships between several generalizations of metric spaces which we shall refer to:

$$\begin{tikzcd}
 & \text{metric} \arrow[dl] \arrow[dr] & \\
\text{Nagata [4]} \arrow[d] & & \text{developable} \arrow[d] \\
\text{stratifiable [2]} \arrow[dr] & & \text{semi-metric [4]} \arrow[dl] \\
 & \text{semi-stratifiable [5]} &
\end{tikzcd}$$

We also recall that the Nagata spaces are precisely the stratifiable first countable spaces (see [4], Theorem 3.1), and the semi-metric spaces are precisely the first countable semi-stratifiable spaces (see [5], Corollary 1.4).

LEMMA. *If X is a stratifiable space, then, given an open set $U \subset X$, there is a sequence $\{U_n\}$ of open sets such that*

$$\bigcup_n U_n = U \quad \textit{and} \quad \mathrm{Cl}\, U_n \subset U_{n+1} \quad \textit{for each } n.$$

Proof. Let $\{U'_n\}$ be the sequence of open sets assigned to U by the stratification of X. Let $U_1 = U'_1$. Since X is normal ([4], Theorem 2.2), there exists an open set V_2 such that $\mathrm{Cl}\, U_1 \subset V_2 \subset \mathrm{Cl}\, V_2 \subset U$. We let $U_2 = U'_2 \cup V_2$ and continue by an obvious induction argument.

THEOREM 1. *If a stratifiable space X satisfies condition* A, *then every open subset of X is peripherally separable.*

Proof. For an open U, let $\{U_n\}$ be the sequence assigned by the Lemma. Let $F_0 = \mathrm{Cl}\, U_1$ and $F_n = \mathrm{Cl}\, U_{n+1} - U_n$ for $n = 1, 2, \ldots$ Then $\{F_n \mid n \text{ is even}\}$ and $\{F_n \mid n \text{ is odd}\}$ are collections of disjoint closed sets, so for each n such that $F_n \neq \emptyset$ there exists an $x_n \in F_n$ such that

$$\mathrm{Cl}\bigcup_n F_{2n} - \bigcup_n F_{2n} \subset \mathrm{Cl}\{x_{2n}\} \quad \text{and} \quad \mathrm{Cl}\bigcup_n F_{2n+1} - \bigcup_n F_{2n+1} \subset \mathrm{Cl}\{x_{2n+1}\}.$$

Thus, $\mathrm{Bd}\, U \subset \mathrm{Cl}\{x_n\}$ which implies that $\mathrm{Bd}\, U$ is separable. (Separability is inherited in stratifiable spaces.)

Example 1. This example shows that Theorem 1 does not hold for Hausdorff developable spaces (nor for semi-metric or semi-stratifiable spaces). Let X be the closed upper half-plane with the following topology: points above the x-axis are open; a basic open neighborhood of a point p on the x-axis is p together with the intersection of an open disk centered at p with the open upper half-plane. To see that X satisfies condition A, let $\{F_\lambda \mid \lambda \in \Lambda\}$ be a collection of disjoint closed subsets of X. The set

$$B = \mathrm{Cl}\bigcup_\lambda F_\lambda - \bigcup_\lambda F_\lambda,$$

considered as a subset of the usual x-axis, has a countable dense subset D. Let $\mathscr{U}(d)$ be a countable local basis in X at d for each $d \in D$, and let $\{U_1, U_2, \ldots\}$ be an enumeration of $\bigcup_d \mathscr{U}(d)$. By an induction argument, for each n we can find an $x_n \in U_n \cap$ (some $F_{\lambda(n)}$) such that $\lambda(n) \neq \lambda(m)$ whenever $n \neq m$. Then $D \subset \mathrm{Cl}\{x_n\}$, so $B \subset \mathrm{Cl}\{x_n\}$. Thus X satisfies condition A, but the open upper half-plane is an open set with non-separable boundary. (In fact, X is not even locally peripherally separable.)

QUESTION. If a Moore space X satisfies condition A, then is every open set in X peripherally separable? (**P 955**)

Example 2. $[0, \Omega[$ is a normal, first countable space satisfying condition A ([9], Example 3.23), but containing non-peripherally separable open sets.

THEOREM 2. *For a semi-stratifiable space X, the following are equivalent*:

(1) *Every open set is peripherally separable.*

(2) *Every set is peripherally separable.*

(3) $\mathrm{Cl}\, Y - Y$ *is separable for every* $Y \subset X$.

(4) $X - D$, *where D is the set of isolated points in X, is hereditarily separable.*

Proof. We clearly have $(4) \to (3) \to (1)$ and $(4) \to (2) \to (1)$. To prove $(1) \to (4)$, first note that $S = X - D$ is a semi-stratifiable space ([5], Theorem 2.2). If S is not hereditarily separable, then there is an uncountable subset B of S which has no cluster points ([5], Theorem 2.8). Then $X - B$ is an open subset of X having a non-separable boundary.

COROLLARY 1. *If X is a stratifiable space satisfying condition* A, *then X satisfies each condition in Theorem* 2.

COROLLARY 2. *A connected stratifiable space satisfying condition* A *is separable.*

THEOREM 3. *Let X be a first countable space with the property that* $\mathrm{Cl}\, Y - Y$ *is separable for each $Y \subset X$. Then X satisfies condition* A.

Proof. Let D be a countable dense subset of

$$\mathrm{Cl} \bigcup_\lambda F_\lambda - \bigcup_\lambda F_\lambda,$$

where $\{F_\lambda \mid \lambda \in \Lambda\}$ is a collection of disjoint closed sets in X; let $\mathscr{U}(d)$ be a countable local basis at each $d \in D$, and let $\{U_1, U_2, \ldots\}$ be an enumeration of $\bigcup \{\mathscr{U}(d) \mid d \in D\}$. Since U_1 is a neighborhood of some $d \in D$, $U_1 \cap F_{\lambda(1)} \neq \emptyset$ for some $\lambda(1)$, so let $x_1 \in U_1 \cap F_{\lambda(1)}$. Assume we have found $\lambda(1), \ldots, \lambda(n)$ in Λ and $x_1, \ldots, x_n$ in X such that $\lambda(i) \neq \lambda(j)$ if $i \neq j$ and with $x_i \in U_i \cap F_{\lambda(i)}$. Then, since U_{n+1} meets infinitely many of the sets F_λ, we have $U_{n+1} \cap F_{\lambda(n+1)} \neq \emptyset$ for some new $\lambda(n+1)$, and we let $x_{n+1} \in U_{n+1} \cap F_{\lambda(n+1)}$.

Now

$$\mathrm{Cl}\bigcup_{\lambda} F_\lambda - \bigcup_{\lambda} F_\lambda \subset \mathrm{Cl}\{x_n\}.$$

COROLLARY 3. *If X is a semi-metric space satisfying any one of the conditions in Theorem 2, then X satisfies condition* A.

This corollary tells us that hereditary separability implies condition A in semi-metric spaces. That separability alone does not imply condition A, even in complete Moore spaces, is shown by the following example:

Example. Let X be the x-axis together with the rational points above the x-axis. Points above the x-axis are open; a basic open neighborhood of a point p on the x-axis is p together with an "open" disk tangent to the x-axis at p. That X does not satisfy condition A is proved in [9], Example 1.25.

COROLLARY 4. *For a Nagata space X, the following are equivalent:*

(1) *X satisfies condition* A.

(2) *Every open subset of X is peripherally separable.*

(3) *Every subset of X is peripherally separable.*

(4) $\mathrm{Cl}\, Y - Y$ *is separable for every $Y \subset X$.*

(5) *$X - D$, where D is the set of isolated points in X, is separable.*

COROLLARY 5. *Every subspace of a Nagata space satisfying condition* A *also satisfies condition* A.

That condition A is not hereditary in general is seen by considering $[0, \Omega]$; see [9], Example 3.24.

3. Developable spaces. We have seen that condition A implies the separability of a Nagata space minus its isolated points. That this is not true for developable spaces (even developable Baire spaces) is shown by Example 1. We do have results for some types of developable spaces, however. For example, in a developable Baire space with condition A, the non-isolated points are at least contained in a separable subspace. We begin with the following theorem, which also gives a result for developable spaces containing a dense set with empty interior.

THEOREM 4. *Let X be a developable space satisfying one of the following properties:*

(1) *X is a Baire space with no isolated points, or*

(2) *X contains a dense set B with empty interior.*

Then if X satisfies condition A, *it is separable.*

Proof. For X satisfying property (1), let $B = X$. Otherwise, let B be the dense set given by property (2). Suppose B has been well ordered, $B = \{b_\gamma \mid \gamma < \Gamma\}$, where Γ is an ordinal.

Let $\mathscr{U}_1 \supset \mathscr{U}_2 \supset \ldots$ be a development for X. Letting n be a fixed positive integer, we first define a set C_n having no cluster points and an empty interior.

Let $c_0 = b_0$. Let c_1 be the first element of B such that $c_1 \notin \mathrm{St}(c_0, \mathscr{U}_n)$. Suppose $\alpha < \Gamma$, and assume c_γ has been defined for all $\gamma < \alpha$. Then we let c_α be the first element of B such that

$$c_\alpha \notin \bigcup_{\gamma<\alpha} \mathrm{St}(c_\gamma, \mathscr{U}_n),$$

if such an element exists. We continue by transfinite induction on $\gamma < \Gamma$ until we reach an α such that no such c_α exists. Let C_n be the set of c_γ's so defined.

We now prove several facts about each C_n.

(a) *C_n has no cluster points.*

For let $x \in X$. If $\mathrm{St}(x, \mathscr{U}_n)$ contains points of C_n, let c_γ be such a point. Then $\mathrm{St}(c_\gamma, \mathscr{U}_n)$ is an open set containing x, but it contains no other points of C_n. Hence x is not a cluster point of C_n.

(b) *C_n has an empty interior.*

For suppose U is an open set in X contained in C_n. Letting $x \in U$, we see that $U \cap \mathrm{St}(x, \mathscr{U}_n)$ contains no points of C_n other than x. Since x is not an isolated point, we have a contradiction.

(c) *$\bigcup_n C_n$ is a dense subset of X.*

For let U be an open set in X. Since B is dense in X, U contains some element b of B. Then, for some integer n, we have $\mathrm{St}(b, \mathscr{U}_n) \subset U$. Now if $b \notin C_n$, then we must have $b \in \mathrm{St}(c_\gamma, \mathscr{U}_n)$ for some c_γ preceding b. Thus $c_\gamma \in \mathrm{St}(b, \mathscr{U}_n) \subset U$.

(d) *$\bigcup_n C_n$ has an empty interior.*

If X is a Baire space, this is true since each C_n has an empty interior. Otherwise, $\bigcup_n C_n \subset B$, which has an empty interior by the hypothesis.

To obtain a collection of disjoint closed sets in X, we let

$$F_n = C_n - (C_1 \cup \ldots \cup C_{n-1})$$

for each positive integer n. Since each C_n has no cluster points, each F_n is closed. The F_n's are also disjoint, and

$$\bigcup_n F_n = \bigcup_n C_n.$$

By condition A, for each n there exists a point $x_n \in F_n$ such that

$$\mathrm{Cl} \bigcup_n F_n - \bigcup_n F_n \subset \mathrm{Cl}\{x_n\}.$$

But since $\bigcup_n F_n$ is a dense set with empty interior, $\mathrm{Cl}\bigcup_n F_n - \bigcup_n F_n$ is a dense subset of X. Thus $X = \mathrm{Cl}\{x_n\}$.

THEOREM 5. *If X is a developable Baire space satisfying condition* A, *then X can be written as $D \cup S$, where D is the set of isolated points of X and S is separable.* (D *and* S *are not necessarily disjoint.*)

Proof. $X - \mathrm{Cl}D$ is a Baire space containing no isolated points. An elementary argument shows that if a space Y contains a dense Baire subspace, then Y is Baire. Thus $Y = \mathrm{Cl}(X - \mathrm{Cl}D)$ is a Baire space containing no isolated points. Since condition A is clearly closed hereditary, Y is separable by Theorem 4.

Now let $B = \mathrm{Bd}D$. (Note that $X - D$ is contained in $Y \cup B$.) For each positive integer n, let

$$F_n = [\mathrm{St}(B, \mathscr{U}_n) - \mathrm{St}(B, \mathscr{U}_{n+1})] \cap D,$$

where $\mathscr{U}_1 \supset \mathscr{U}_2 \supset \ldots$ is a development for X. We have the following facts:

(a) The F_n's are disjoint.

(b) Each F_n is closed in X. (For suppose $x \in \mathrm{Cl}F_n - F_n$. Then $x \in B$, so $\mathrm{St}(x, \mathscr{U}_{n+1})$ is a neighborhood of x missing F_n.)

(c) $B = \mathrm{Cl}\bigcup_n F_n - \bigcup_n F_n$. (For suppose $b \in B$. Then $b \notin \bigcup_n F_n$, but each $\mathrm{St}(b, \mathscr{U}_n)$ meets some F_m, $m \geqslant n$.)

Thus, since X satisfies condition A, for each n there exists a point $x_n \in F_n$ such that $B \subset \mathrm{Cl}\{x_n\}$. Then $S = Y \cup \mathrm{Cl}\{x_n\}$ satisfies our requirements.

Theorem 5 holds for complete Moore spaces since such spaces are Baire. In fact, a slightly larger class of Moore spaces, introduced by Rudin, is contained in the class of Baire spaces: a Moore space is said to satisfy Axiom 1″ [8] if it has a development $\mathscr{U}_1 \supset \mathscr{U}_2 \supset \ldots$ with the property that if $U_1, U_2, \ldots$ is a sequence of sets such that $U_n \in \mathscr{U}_n$ and $\mathrm{Cl}\, U_{n+1} \subset U_n$ for each n, then

$$\bigcap_n U_n \neq \emptyset.$$

(That this axiom is weaker than the completeness axiom of Moore is shown in [8], Theorem 9.) An easy argument shows that a Moore space satisfying Axiom 1″ is a Baire space.

COROLLARY 6. *If X is a locally compact semi-stratifiable Hausdorff space satisfying condition* A, *then X can be expressed as $D \cup S$, where D is the set of isolated points of X and S is separable.*

Proof. Creede has shown that a locally compact semi-stratifiable Hausdorff space is a Moore space ([5], Corollary 4.11). Also, locally compact Hausdorff spaces are Baire.

That the requirement of semi-stratifiability in Corollary 6 cannot be substantially weakened is shown by the spaces $[0, \Omega]$ and $[0, \Omega[$, which are locally compact and satisfy condition A ([9], Examples 3.22 and 3.23).

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