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ON COMPLETIONS OF PROXIMITY AND UNIFORM SPACES

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1. Introduction. The theory of proximity and uniform spaces includes a variety of "completions". Among these we meet the completion [1] of a proximity space (X, δ) taken to be relative to the total (generalized) uniformity $\mathscr{U}_a$ in the proximity class $\pi(\delta)$ of δ, and the completion [6] of (X, δ) by clusters. In addition, realcompletions of proximity spaces are defined in [11], and functionally complete uniform spaces $(X, \mathscr{U})$, with $\mathscr{U} \in \pi(\delta)$, are studied in [2].

Harris has shown [4] that the completion of (X, δ) in the realcompact structure of δ is the Q-closure of X in the Smirnov compactification δX of (X, δ).

The purpose of this paper is to determine relationships among these completions. In particular, we show that, barring spaces of measurable cardinal, each of the following conditions implies the next:

(A) $(X, \mathscr{U})$ is functionally complete for some $\mathscr{U} \in \pi(\delta)$;

(B) (X, δ) is realcomplete;

(C) $(X, \mathscr{U}_a)$ is complete;

(D) (X, δ) is complete in its realcompact structure.

Conditions (A) and (B) are in fact equivalent, and examples are provided to show that the remaining implications cannot be reversed. Functionally complete uniform spaces are also characterized by means of stable clusters and stable families of closed sets. From this a characterization of realcompact spaces is obtained.

2. Completion of $(X, \mathscr{U}_a)$ and functionally complete spaces. Let $\mathscr{G}_a$ be the gauge associated with $\mathscr{U}_a$ according to Leader's Theorem of [5]. The uniform structures in this paper are generalized in the sense of [1], which also provides that $\mathscr{U}_a$ is the largest member of the class $\pi(\delta)$.

Now $\mathscr{U}_a$ is precisely the collection of subsets of $X \times X$ which are "entourage-like" (relative to δ). If σ is any pseudometric for X compatible with δ, the subset $V = \{(x, y): \sigma(x, y) < 1\}$ of $X \times X$ is entourage-like, so that $V \in \mathscr{U}_a$ and $\sigma \in \mathscr{G}_a$. Since every pseudometric in $\mathscr{G}_a$

is compatible with δ, evidently $\mathscr{G}_a$ is the collection of all compatible pseudometrics for (X, δ).

In Theorem 2.1 we make explicit the fact that the completion X^* of (X, δ) by clusters may be regarded as the completion of $(X, \mathscr{U}_a)$. By Theorem 5 of [6], every member σ of $\mathscr{G}_a$ has an extension to a compatible pseudometric σ^* on X^*, where X^* is regarded as a p-subspace of δX. Thus, if $\mathscr{G}_a^*$ is the gauge for the total uniform structure $\mathscr{U}_a^*$ for X^*, then $\mathscr{G}_a^* = \{\sigma^* : \sigma \in \mathscr{G}_a\}$.

THEOREM 2.1. *The completion of* $(X, \mathscr{U}_a)$ *is* $(X^*, \mathscr{U}_a^*)$.

Proof. Clearly, $(X, \mathscr{U}_a)$ is a dense uniform subspace of $(X^*, \mathscr{U}_a^*)$. It remains to show that $(X^*, \mathscr{U}_a^*)$ is complete.

Let $\mathscr{F}^*$ be a Cauchy round filter in $(X^*, \mathscr{U}_a^*)$. Since δX is the Smirnov compactification of (X^*, δ^*), where δ^* is the proximity associated with $\mathscr{U}_a^*, \mathscr{F}^*$ converges to a point x^* of δX. Now x^* is close to small sets relative to $\mathscr{G}_a^*$, hence $x^* \in X^{**}$. But $X^{**} = X^*$ by Theorem 5 of [6], so $x^* \in X^*$ and the proof is completed.

Let $\mathscr{U}_P$ be the weak (generalized) uniform structure determined by the collection $P(X)$ of all real-valued proximity functions on (X, δ), and let $\nu_\delta X$ be the realcompletion of (X, δ). By $\mathscr{V}_P$ we denote the weak uniform structure on $\nu_\delta X$ determined by $P(\nu_\delta X)$. The algebra of bounded imembers of $P(X)$ is indicated by $P^*(X)$.

Given an admissible uniformity $\mathscr{U}$ for X, we denote by $U(X, \mathscr{U})$ the collection of real-valued functions on X which are uniformly continuous with respect to $\mathscr{U}$ and the standard metric uniformity on the real numbers. Let $\mathrm{w}\mathscr{U}$ be the weak uniform structure for X generated by $U(X, \mathscr{U})$. Evidently, $\mathrm{w}\mathscr{U} \subseteq \mathscr{U}$. Following Fenstad [2], we say that $\mathscr{U}$ is *functionally determined* if $\mathscr{U} = \mathrm{w}\mathscr{U}$, and $(X, \mathscr{U})$ is *functionally complete* if $\mathscr{U}$ is functionally determined and $(X, \mathscr{U})$ is complete.

For $\mathscr{U} \in \pi(\delta)$, $U(X, \mathscr{U})$ satisfies

$$P^*(X) \subseteq U(X, \mathscr{U}) \subseteq P(X).$$

Thus the functionally determined uniform structures are precisely the weak uniform structures $\mathscr{U}_S$, where S satisfies $P^*(X) \subseteq S \subseteq P(X)$. Njåstad has shown (see Theorem 3 of [11]) that the completions of the spaces $(X, \mathscr{U}_S)$ are realcompletions of (X, δ). However, not all completions of $(X, \mathscr{U})$, where $\mathscr{U} \in \pi(\delta)$, are realcompletions (cf. Example 2.6 of [9]).

THEOREM 2.2. *The total structure* $\mathscr{U}_a$ *in* $\pi(\delta)$ *is functionally determined if and only if* $(X^*, \mathscr{U}_a^*) = (\nu_\delta X, \mathscr{V}_P)$.

Proof. The necessity follows from Theorem 2.1 and from Corollary 2.5 of [9], which provides that $(\nu_\delta X, \mathscr{V}_P)$ is the completion of $(X, \mathscr{U}_P)$. The sufficiency is obvious, and the proof is complete.

Equality of the subsets X^* and $v_\delta X$ of δX is, however, inadequate to guarantee that $\mathscr{U}_a = \mathscr{U}_P$, as is shown by the following example.

Example 2.1. Let X be any uncountable discrete space of non-measurable cardinal, let $\mathscr{U}_a$ and δ be the uniform structure and proximity associated with the discrete metric, respectively. Here $P(X) = C(X)$, the ring of continuous real-valued functions on X. Since X is realcompact, $v_\delta X = X$. The only small sets in X are single points, so $X^* = X$ also. Now $v_\delta X = X^*$, but $\mathrm{w}\mathscr{U}_P = \mathscr{U}_P$ is properly contained in $\mathscr{U}_a$ (see 15.23B of [3]).

In [2] the question is raised as to the existence of a complete uniform space $(X, \mathscr{U})$ which is not functionally determined and such that $(X, \mathrm{w}\mathscr{U})$ is also complete. Example 2.1 settles this question affirmatively, since both $(X, \mathscr{U}_a)$ and $(X, \mathrm{w}\mathscr{U}_a)$ are complete, but $\mathrm{w}\mathscr{U}_a \neq \mathscr{U}_a$.

By definition, every point of X^* is close to $\mathscr{U}_a$-small sets, and since $\mathscr{U}_P \subseteq \mathscr{U}_a$, we have $X^* \subseteq v_\delta X$. Example 2.6 of [9] shows that this inclusion may be proper. Thus $(X^*, \mathscr{U}_a^*)$ need not be the completion of $(X, \mathscr{U})$ for any functionally determined uniform structure $\mathscr{U} \in \pi(\delta)$.

THEOREM 2.3. *A proximity space (X, δ) is realcomplete if and only if there exists $\mathscr{U} \in \pi(\delta)$ for which $(X, \mathscr{U})$ is functionally complete.*

Proof. Sufficiency. Suppose that $(X, \mathscr{U})$ is functionally complete for some $\mathscr{U} \in \pi(\delta)$, and let $\mathscr{F}$ be a Cauchy round filter in $(X, \mathscr{U}_P)$. Each $f \in U(X, \mathscr{U})$ determines a pseudometric σ_f for X. Now $U(X, \mathscr{U}) \subseteq P(X)$ implies $\sigma_f \in \mathscr{G}_P$, the gauge for $\mathscr{U}_P$. Thus $\mathscr{F}$ contains sets of arbitrarily small σ_f-diameter. Since $\mathscr{U}$ is functionally determined, the pseudometrics σ_f, $f \in U(X, \mathscr{U})$, generate a gauge for $\mathscr{U}$. Thus, $\mathscr{F}$ is a Cauchy round filter in $(X, \mathscr{U})$, so that $\mathscr{F}$ converges to a point x of X. It now follows that $v_\delta X = X$.

Necessity. If (X, δ) is realcomplete, then $(X, \mathscr{U}_P)$ is complete, and the proof is completed.

Example 2.2. Let X be the unit ball in l_2, the space of square-summable real sequences, and let δ be the proximity for X associated with the standard metric for X. As is shown in Example 2.6 of [9], $P(X) = P^*(X)$, so that $\pi(\delta)$ contains no functionally determined uniform structures other than the unique totally bounded structure $\mathscr{U}_\omega$ in $\pi(\delta)$. However, since $\pi(\delta)$ contains at least two distinct uniform structures, it must contain uncountably many by Corollary 2.1.3 of [12]. We note that while (X, δ) is not realcomplete, X is realcompact.

If (X, δ) is any non-compact, realcomplete proximity space, then $(X, \mathscr{U}_P)$ is functionally complete, but $(X, \mathscr{U}_\omega)$ is not. Thus, in case where $\pi(\delta)$ does contain some functionally complete uniform structure, $(X, \mathscr{U})$ need not be functionally complete for all functionally determined $\mathscr{U}$ in $\pi(\delta)$.

3. $\mathscr{U}$-stable clusters and families of closed sets. Let $\mathscr{U} \in \pi(\delta)$.

Definition. A family of sets $\mathscr{M} = \{F_a : a \in A\}$ is *$\mathscr{U}$-stable* if, for each $f \in U(X, \mathscr{U})$, there exists $F_a \in \mathscr{M}$ such that f is bounded on F_a.

This definition follows Mandelker's definition of a stable family in [8], and the definitions coincide in case where $\mathscr{U}$ is the uniformity for X determined by $C(X)$. Clusters in proximity spaces are defined in [7], where it is shown that a necessary and sufficient condition that (X, δ) be compact is that every cluster in (X, δ) contain a point. The following theorem includes a similar characterization for functionally complete spaces.

THEOREM 3.1. *If $\mathscr{U} \in \pi(\delta)$ and $\mathscr{U}$ is functionally determined, then the following conditions are equivalent:*

(A) *$(X, \mathscr{U})$ is functionally complete.*

(B) *Every $\mathscr{U}$-stable cluster in (X, δ) contains a point.*

(C) *Every $\mathscr{U}$-stable family of closed subsets of X having the finite intersection property has a non-empty intersection.*

Proof. (A) $\Rightarrow$ (B). Let $\mathscr{C}$ be a $\mathscr{U}$-stable cluster in (X, δ). From Theorems 2 and 3 of [7] it follows that there exists $p \in \delta X$ satisfying

$$p \in \bigcap \{\mathrm{Cl}_{\delta X} A : A \in \mathscr{C}\}.$$

Let $\mathscr{F}^p$ be the unique maximal round filter in (X, δ) which converges to p. Since $U(X, \mathscr{U}) \subseteq P(X)$, each $f \in U(X, \mathscr{U})$ has an extension f^δ mapping δX into the Smirnov-compactification of the real numbers R (taken with respect to the standard metric proximity for R). If $f^\delta(p)$ is not real for some $f \in U(X, \mathscr{U})$, then by Theorem 2.2 of [9] the sets

$$F_n = \{x \in X : |f(x)| \geqslant n\}$$

belong to $\mathscr{F}^p$ for each positive integer n. Since $\mathscr{F}^p$ is the trace on X of the family of neighborhoods of p in δX, each F_n meets every member A of $\mathscr{C}$. But $|f| \geqslant n$ on $F_n \cap A$, hence f is unbounded on every member of $\mathscr{C}$, which is a contradiction.

Thus, $f^\delta(p)$ is real for each $f \in U(X, \mathscr{U})$, and since $\mathscr{U}$ is functionally determined, $\mathscr{F}^p$ is a Cauchy filter in $(X, \mathscr{U})$. By (A), $p \in X$ so that $\{p\} \in \mathscr{C}$.

(B) $\Rightarrow$ (C). With appropriate modifications, this is similar to the proof that (B) implies (C) in Theorem 4.3 of [10].

(C) $\Rightarrow$ (A). Suppose that $(X, \mathscr{U})$, where $\mathscr{U} \in \pi(\delta)$, is not complete. Then the completion of $(X, \mathscr{U})$ is a subset of δX which contains X properly (see Theorem 8 of [1]). Choose $p \in \delta X - X$ such that p is a point in the completion of $(X, \mathscr{U})$. Let $\mathscr{F}^p$ be the unique maximal round filter in (X, δ) which converges to p.

Now every $f \in U(X, \mathscr{U})$ can be extended to a uniformly continuous function f_1 mapping the completion of $(X, \mathscr{U})$ into R. For each $\varepsilon > 0$, the inverse image under f of the ε-ball about $f_1(p)$ must be a member of $\mathscr{F}^p$. Since $\mathscr{U} = \mathrm{w}\mathscr{U}$, it is clear that $\mathscr{F}^p$ is a $\mathscr{U}$-Cauchy filter. Thus the family $\{\mathrm{Cl}_X F : F \in \mathscr{F}^p\}$ is $\mathscr{U}$-stable and has the finite intersection property. But

$$\bigcap \{\mathrm{Cl}_X F : F \in \mathscr{F}^p\} = \emptyset,$$

which contradicts (C).

This completes the proof.

We note that if any of the conditions of Theorem 3.1 are satisfied for functionally determined $(X, \mathscr{U})$ with $\mathscr{U} \in \pi(\delta)$, then (X, δ) is realcomplete and each of the conditions of Theorem 4.3 of [10] holds. However, for any realcomplete, non-compact proximity space (X, δ) we have $\mathscr{U}_\omega$ functionally determined but not functionally complete. Thus conditions (A)-(C) of Theorem 3.1 are not equivalent to those of Theorem 4.3 of [10].

Setting $\delta = \beta$, where β is the proximity associated with the Stone-Čech compactification of X, we have $\mathscr{U}_P = \mathscr{C}$, the uniform structure generated by $C(X)$. From Theorem 3.1 we obtain

COROLLARY 3.1. *For a completely regular space X, the following conditions are equivalent:*

(A) *X is realcompact.*

(B) *$(X, \mathscr{C})$ is complete.*

(C) *Every stable cluster contains a point.*

(D) *Every stable family of closed sets with the finite intersection property has a non-empty intersection.*

The equivalence of (A) and (D) is Theorem 5.1 of [8].

4. Other relations among completions. The Q-closure of a proximity space (X, δ) is the set $Q_\delta X$ of points in δX such that $p \in Q_\delta X$ if and only if, whenever $f \in P^*(X)$ satisfies $f^\delta(p) = 0$, there exists $x \in X$ for which $f(x) = 0$ (see [4]). Harris has shown that $Q_\delta X$ is the completion of X in the realcompact structure of δ, and that $p \in Q_\delta X$ if and only if $\mathscr{F}^p$ has the countable intersection property (see 7.1 and Theorem F of [4]).

We say that (X, δ) is *Q-closed* if $Q_\delta X = X$. Let N be the positive integers and, for subsets A and B of X, we let $A \ll B$ denote that B is a p-neighborhood of A.

THEOREM 4.1. *Let (X, δ) be a proximity space, where every closed discrete subspace has non-measurable cardinal. Then each of the following conditions implies the next:*

(A) *(X, δ) is realcomplete.*

(B) *$(X, \mathscr{U}_a)$ is complete, where $\mathscr{U}_a$ is the total uniform structure in $\pi(\delta)$.*

(C) *(X, δ) is Q_δ-closed.*

Proof. (A)$\Rightarrow$(B). This follows from Theorem 2.5 of [9], Theorem 2.1 and the fact that $X^* \subseteq \nu_\delta X$.

(B)$\Rightarrow$(C). To facilitate the proof, we adapt some of the techniques of 15.17 and 15.18 of [3] to the present context.

Let $p \in Q_\delta(X)$. By 7.1 (B) of [4], $\mathscr{F}^p$ has the countable intersection property. Let σ be any pseudometric for X compatible with δ, and take $\varepsilon > 0$. From 15.17 of [3] it follows that there exist sets A_{nx} such that

$$X = \bigcup \{A_{nx} : n \in N, x \in X\}$$

and where, for each $n \in N$, the family $\{A_{nx} : x \in X\}$ is σ-discrete of gauge $\varepsilon/4n$ and $\sigma[A_{nx}] < \varepsilon/2$.

For each $n \in N$, set

$$F_{nx} = \{y \in X : \sigma[A_{nx}, y] < \varepsilon/12n\}.$$

Now $A_{nx} \ll F_{nx}$ for all $n \in N$ and $x \in X$. Moreover, the family $\mathscr{M}_n = \{F_{nx} : x \in X\}$ is σ-discrete and $\sigma[F_{nx}] < \varepsilon$. Clearly,

$$X = \bigcup \{F_{nx} : n \in N, x \in X\}.$$

For each $n \in N$, set $M_n = \bigcup \{A_{nx} : x \in X\}$. If, for each $n \in N$, there exists $B_n \in \mathscr{F}^p$ satisfying $M_n \cap B_n = \emptyset$, then

$$\bigcap \{B_n : n \in N\} = \emptyset,$$

which is a contradiction.

Thus, there exists $m \in N$ for which $B \cap M_m \neq \emptyset$ for all $B \in \mathscr{F}^p$. Take

$$K_m = \bigcup \{F_{mx} : x \in X\}.$$

Evidently, $M_m \ll K_m$. Since every member of $\mathscr{F}^p$ meets M_m, the maximality of $\mathscr{F}^p$ provides that $K_m \in \mathscr{F}^p$.

Select a point p_x from each F_{mx} in K_m, and set

$$S = \{p_x : F_{mx} \in K_m\}.$$

Thus, S is σ-discrete and realcompact, since card S is non-measurable.

Construct a filter $\mathscr{G}_S$ on S as follows. For $A \subseteq S$, put $A \in \mathscr{G}_S$ if and only if

$$\bigcup \{F_{mx} : p_x \in A\} \in \mathscr{F}^p.$$

The sets F_{mx}, $x \in X$, are disjoint so that the correspondence

$$A \leftrightarrow \bigcup \{F_{mx} : p_x \in A\}$$

preserves union and intersection. Thus, $\mathscr{G}_S$ has the countable intersection property.

Let $A \subseteq S$ and consider the sets

$$C = \bigcup \{F_{mx} : p_x \in A\} \quad \text{and} \quad D = \bigcup \{F_{mx} : p_x \in S - A\}.$$

If neither C nor D belongs to $\mathscr{F}^p$, then there exist sets U, V in $\mathscr{F}^p$ for which

$$U \cap \left(\bigcup \{A_{mx} : p_x \in A\}\right) = \emptyset \quad \text{and} \quad V \cap \left(\bigcup \{A_{mx} : p_x \in S - A\}\right) = \emptyset.$$

Select $U_1, V_1 \in \mathscr{F}^p$, where $U_1 \ll U$ and $V_1 \ll V$. Then

$$\bigcup \{A_{mx}: p_x \in A\} \ll X - U_1 \quad \text{and} \quad \bigcup \{A_{mx}: p_x \in S - A\} \ll X - V_1,$$

so that

$$M_m \ll [(X - U_1) \cup (X - V_1)].$$

Now both $(X - U_1) \cup (X - V_1)$ and $U_1 \cap V_1$ belong to $\mathscr{F}^p$, which is impossible. Thus $C \in \mathscr{F}^p$ or $D \in \mathscr{F}^p$, which implies that either $A \in \mathscr{G}_S$ or $S - A \in \mathscr{G}_S$. It now follows that $\mathscr{G}_S$ is a z-ultrafilter in S with the countable intersection property. Since S is realcompact, there is a point $p_x \in S$ such that $\{p_x\} \in \mathscr{G}_S$. Thus, the corresponding set $F_{mx} \in \mathscr{F}^p$ and $\sigma[F_{mx}] < \varepsilon$. Since σ is an arbitrary pseudometric compatible with δ, $\mathscr{F}^p$ is a Cauchy filter relative to $\mathscr{U}_a$. Hence $p \in X^*$ and $X \subseteq Q_\delta X \subseteq X^*$. Now (B) implies $X^* = X$, by Theorem 2.1.

This completes the proof.

None of the implications of Theorem 4.1 can be reversed. Example 2.2 demonstrates a proximity space (X, δ) for which $(X, \mathscr{U}_a)$ is complete, but (X, δ) is not realcomplete.

If we let $X = (0, 1)$ with the proximity δ induced by the usual metric on R, then $\delta X = [0, 1]$. Now $Q_\delta X = X$, but the completion of $(X, \mathscr{U}_a)$ is δX

Finally, suppose the existence of a measurable cardinal. Let X be a discrete space of such a cardinal and let δ be the associated proximity. Then $X^* = X$, but X is not realcompact, so $Q_\delta X \neq X$. Thus the restriction concerning discrete subspaces cannot be avoided.

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