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INVARIANT MEASURES ON ABELIAN METRIC GROUPS

BY

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We consider complete countably additive σ-finite measures which vanish on singletons and are non-identically zero. A measure m defined on a σ-algebra $\mathfrak{M}$ of subsets of a group $(G, +)$ is called *invariant* iff $g+A \in \mathfrak{M}$ and $m(g+A) = m(A)$ for any $g \in G$ and $A \in \mathfrak{M}$. It is called *symmetric* if $-A \in \mathfrak{M}$ and $m(-A) = m(A)$ for any $A \in \mathfrak{M}$. If d is a metric on $(G, +)$, then d is called *invariant* iff $d(x, y) = d(g+x, g+y)$ for any $g, x, y \in G$. Given a metric d on $(G, +)$, a measure m is called *d-invariant* iff the following condition holds: for any subsets $A, B \subset G$, whenever there exists a function $f\colon A \xrightarrow{\text{onto}} B$ such that $d(x, y) = d(f(x), f(y))$ for any $x, y \in A$, and the set A is m-measurable, then B is also m-measurable and $m(A) = m(B)$. Since left translations are isometries for any invariant metric, a measure invariant with respect to an invariant metric is clearly invariant. The converse turns out to be sometimes true as well. Bandt [1] proved that every left Haar measure on a locally compact metric group is invariant with respect to every invariant metric. The aim of this paper is to show that this is a rather specific feature of Haar measures in the sense that many other invariant measures on groups do not enjoy it. Our main result is the following

THEOREM 1. *Let $(G, +)$ be an abelian group without elements of order 2. Then every invariant measure on G can be extended to an invariant measure which is not invariant with respect to any invariant metric.*

Since for abelian groups the inverse element operation is an isometry for every invariant metric, Theorem 1 follows immediately from

THEOREM 2. *Let $(G, +)$ be an abelian group without elements of order 2. Then every invariant measure on G can be extended to an invariant non-symmetric measure.*

Proof. We follow the idea from Pelc [2]. The crucial points of the argument are recalled in order to make the present paper self-contained. First we need the following lemma, essentially due to Szpilrajn [3]. We omit the easy proof.

LEMMA A. *Let m be an invariant measure defined on a σ-algebra $\mathfrak{M}$ of subsets of the group $(G, +)$ and let E be a subset of G. If the family*

$$I = \{X \subset G\colon X \subset K+E \text{ for some countable subset } K \text{ of } G\}$$

consists of sets of inner measure zero, then m can be extended to an invariant measure $\tilde{m}$ defined on the σ-algebra generated by $\mathfrak{M} \cup I$ and vanishing on all sets from I.

It is easy to see that the only sets on which the extended measure $\tilde{m}$ assumes value zero are those of the form $X \cup Y$, where $m(X) = 0$ and $Y \in I$.

If the set E from Lemma A is of positive outer measure and for any countable $K \subset G$ we have $m(-E \cap (K+E)) = 0$, then either $m(-E) = 0$ and the measure m itself is non-symmetric or $-E$ has positive outer measure as well, and hence it is not of the form $X \cup Y$, where $m(X) = 0$ and $Y \in I$. This proves the following

LEMMA B. *Let m be an invariant measure defined on a σ-algebra $\mathfrak{M}$ of subsets of the group $(G, +)$ and let E be a subset of G satisfying the following conditions:*

(a) *E is a set of positive outer measure,*

(b) *for any countable $K \subset G$ the set $K+E$ has inner measure zero,*

(c) *for any countable $K \subset G$ the set $-E \cap (K+E)$ has measure zero.*

Then m can be extended to an invariant non-symmetric measure.

In order to prove our theorem it is enough to construct a set $E \subset G$ satisfying conditions (a), (b) and (c) of Lemma B. We shall call such a set a *(G, m)-extender.*

Case 1. Additive group of a linear space over a countable field.

Let V be a linear space over a countable field F and m any measure on V invariant with respect to addition. Fix a linear basis $B = \{v_\alpha\colon \alpha < \varkappa\}$ of V over F. For any positive integer n, let V_n denote the set of those elements of V which have n summands in the basis B representation. For any sequence $s = (f_1, \ldots, f_r)$ of elements of F let V_s denote the set of all elements of the form $f_1 v_{\alpha_1} + \ldots + f_r v_{\alpha_r}$ for some $\alpha_1 > \ldots > \alpha_r$. Let n_0 be the least natural number for which V_{n_0} has positive outer measure and let s_0 be such a sequence $(f_1, \ldots, f_{n_0})$ for which V_{s_0} has positive outer measure. We claim that $E = V_{s_0}$ is a (V, m)-extender. Condition (a) of Lemma B is trivially satisfied. In order to prove condition (b) let K be any countable subset of V, and H any uncountable subset of B none of whose elements appears in the representation of any $k \in K$. It is easy to see that for distinct $h_1, h_2 \in H$ we have

$$(h_1 + K + E) \cap (h_2 + K + E) \subset K + h_2 + F \cdot h_1 + V_{n_0 - 1}.$$

In view of the definition of n_0 and of the invariance of measure m it follows that the set $K+E$ has inner measure zero.

Finally, we prove that E satisfies condition (c) of Lemma B. Indeed, assume that for some countable set $K \subset G$ we have

$$x \in -E \cap (K+E).$$

Hence

$$(*) \qquad x = -f_1 v_{\alpha_1} - \ldots - f_{n_0} v_{\alpha_{n_0}} = k + f_1 v_{\beta_1} + \ldots + f_{n_0} v_{\beta_{n_0}}.$$

Let D be the countable set consisting of those elements of B which appear in the representation of some $k \in K$. It follows from $(*)$ that, for some i $(1 \leqslant i \leqslant n_0)$, $v_{\beta_i} \in D$ (otherwise, $k = 0$, which contradicts the fact that V does not have elements of order 2). Hence

$$-E \cap (K+E) \subset K + F \cdot D + V_{n_0 - 1}.$$

However, the latter set is a union of countably many translates of V_{n_0-1} (which has measure zero by definition) and, consequently, has measure zero. This completes the proof in case 1.

Case 2. Torsion free abelian group.

Let G be a torsion free abelian group. There exists a homomorphic embedding of G into the additive group of a linear space V over the field Q of rationals, such that a certain basis $B = \{v_\alpha : \alpha < \varkappa\}$ of V consists of elements of G. Let m be any invariant measure on G and denote by $\tilde{m}$ its trivial extension to V (putting 0 outside of G). Clearly, the measure $\tilde{m}$ is invariant with respect to translations from G.

Let V_n and V_s have the same meaning as in case 1. Denote by n_0 the least index n for which $b + V_n$ has positive outer $\tilde{m}$-measure for some $b \in V$. Let s_0 be such a sequence $(q_1, \ldots, q_{n_0})$ of rationals for which $b + V_{s_0}$ has positive outer $\tilde{m}$-measure. Finally, put $E = (b + V_{s_0}) \cap G$. We claim that the set E is a (G, m)-extender. Condition (a) of Lemma B is trivially satisfied. In order to prove condition (b) let K be any countable subset of G. Take any uncountable set H of elements of the basis B which do not appear in the representation of any $k \in K$ or b. Let c be a natural number distinct from all $\pm q_i$, $\pm q_i \pm q_j$ $(i, j \leqslant n_0)$. Clearly,

$$(ch_1 + K + E) \cap (ch_2 + K + E) = \emptyset$$

for distinct $h_1, h_2 \in H$ and $ch \in G$ for any $h \in H$. This proves that the set $K + E$ has inner measure zero.

It remains to prove condition (c) of Lemma B. Take any countable subset K of the group G and consider any element $x \in -E \cap (K+E)$. Hence

$$x = -b - q_1 v_{\alpha_1} - \ldots - q_{n_0} v_{\alpha_{n_0}} = k + b + q_1 v_{\beta_1} + \ldots + q_{n_0} v_{\beta_{n_0}}$$

for some $k \in K$. This implies

$$-k - 2b = q_1 v_{\alpha_1} + \ldots + q_{n_0} v_{\alpha_{n_0}} + q_1 v_{\beta_1} + \ldots + q_{n_0} v_{\beta_{n_0}}.$$

Denote by D the countable subset of the basis B consisting of all elements appearing in the representation of b or of some $k \in K$. Hence $v_{\beta_i} \in D$ for some i $(1 \leqslant i \leqslant n_0)$. It follows that

$$-E \cap (K+E) \subset K+b+Q \cdot D+V_{n_0-1}.$$

By the definition of n_0 we get $\tilde{m}(K+b+Q \cdot D+V_{n_0-1}) = 0$, which implies $m(-E \cap (K+E)) = 0$ and completes the proof in case 2.

Case 3. The general case.

Let $(G, +)$ be an arbitrary abelian group without elements of order 2. Denote by H its torsion subgroup.

If $m(H) = 0$, then we define a measure m_1 on the quotient group G/H by putting

$$m_1(\{a+H\colon a \in A\}) = m(A+H)$$

for all sets A such that $A+H$ is m-measurable.

The group G/H is torsion free and the measure m_1 is invariant. Hence, in view of case 2, there exists a $(G/H, m_1)$-extender. Call it E_1. It is easy to see that the set $E = \bigcup E_1$ is a (G, m)-extender.

If the subgroup H has positive outer measure, consider its subgroups H_i consisting of elements whose orders divide i. Since

$$H = \bigcup_{i=1}^{\infty} H_i,$$

one of the groups H_i must have positive outer measure. Let i_0 be the least index of such a group. We proceed by induction on the number k of prime divisors of i_0 (counting multiple divisors many times). We may assume that i_0 is odd.

If $k = 1$, then i_0 is prime, and hence H_{i_0} is the additive group of a linear space over the field Z_{i_0}. Let E have the same meaning as in case 1 (for the linear space H_{i_0} and the measure m). It is easy to see that E satisfies condition (b) of Lemma B for G and m. In order to show that E is a (G, m)-extender it is enough to prove that

CLAIM. *For any countable set* $K \subset G$, $m(-E \cap (K+E)) = 0$.

Let K be any countable subset of G and let $K' = K \cap H_{i_0}$. In view of the argument in case 1 we get $m(-E \cap (K'+E)) = 0$. On the other hand, $-E \cap ((K \setminus K')+E) = \emptyset$. This proves the claim and completes the proof for $k = 1$.

Suppose that for i_0 having k prime divisors there exists a (G, m)-extender $E \subset H_{i_0}$. Let now $i_0 = p_1 \ldots p_{k+1}$ (p_i are primes, $k \geqslant 1$) and H' be the subgroup of H_{i_0} consisting of elements of order p_1.

Since $m(H') = 0$, we can define an invariant measure m' on G/H' just as before. H_{i_0}/H' is a subgroup of G/H' all of whose elements have orders dividing the number $p_2 \ldots p_{k+1}$, and hence different from 2. By definition, H_{i_0}/H' has positive outer m'-measure. Hence by the inductive hypothesis there exists a $(G/H', m')$-extender $E' \subset H_{i_0}/H'$. It is easy to see that the set $E = \bigcup E'$ is a (G, m)-extender. This completes the proof in the general case.

Our next result shows that Theorem 1 fails to be true for arbitrary abelian groups. We give an example of an abelian group $(G, +)$ and an invariant metric d on G such that, whenever subsets A and B are isometric, one must be a translation of the other. Hence every invariant measure on G must be d-invariant. In our example every element of G has order 2.

PROPOSITION 1. *There exist an abelian group $(G, +)$ and an invariant metric d on G such that every invariant measure on G is d-invariant.*

Proof. Take as $(G, +)$ the group of all sets of natural numbers with symmetric difference as the group operation. The metric d is defined as follows: if $a, b \subset \omega$, let $f\colon \omega \to \{0, 1\}$ be the characteristic function of $a \Delta b$. Define

$$d(a, b) = \sum_{i=0}^{\infty} \frac{f(i)}{3^i}.$$

It is easy to show that d is actually an invariant metric on $(G, +)$.

CLAIM. *For any $a \in G$ and $r \in R$ there exists at most one $b \in G$ such that $d(a, b) = r$.*

Indeed, suppose that $d(a, b') = d(a, b'')$ and let f and g be the characteristic functions of $a\Delta b'$ and $a\Delta b''$, respectively. Since

$$\sum_{i=0}^{\infty} \frac{f(i)}{3^i} = \sum_{i=0}^{\infty} \frac{g(i)}{3^i},$$

it follows that $f(i) = g(i)$ for every natural i, and hence $a\Delta b' = a\Delta b''$, which gives $b' = b''$ and proves the claim.

Suppose that for some subsets A and B of our group G there is an isometry $f\colon A \xrightarrow{\text{onto}} B$ with respect to the metric d. Let x be any element of A. Put $g = f(x) - x$. Then for any $a \in A$ we have

$$d(x, a) = d(f(x), f(a)) \quad \text{and} \quad d(x, a) = d(g+x, g+a).$$

Since $f(x) = g+x$, this implies $d(f(x), f(a)) = d(f(x), g+a)$, and hence, by the claim, we get $f(a) = g+a$ for any $a \in A$. This shows that B is a translation of A. Hence, whenever m is an invariant measure on G, it must be d-invariant as well.

We close the paper with some remarks on the existence of invariant non-symmetric measures on groups which are not abelian. Clearly, we have to assume that uncountably many elements do not have order 2; otherwise, every measure is symmetric. We do not know if Theorem 2 holds without the assumption about commutativity of the group. We are even not able to decide if every group without elements of order 2 carries an invariant non-symmetric measure. Our last proposition shows that this is true under a stronger assumption.

PROPOSITION 2. *Let G be an uncountable group for which the function $f\colon G \to G$ given by $f(g) = 2g$ is one-to-one. Then G carries an invariant non-symmetric measure.*

Proof. Let $(G, +)$ be as above. In order to prove the proposition it is enough to construct a set $A \subset G$ such that $-A \not\subset K+A$ for any countable $K \subset G$. Then the desired measure m can be defined as follows: $m(X) = 0$ for $X \subset K+A$, where K is any countable subset of G, and $m(Y) = 1$ for Y being a complement of X as above.

Let $\{g_\alpha\colon \alpha < \varkappa\}$ be a one-to-one enumeration of G. We define the required set $A = \{a_\alpha\colon \alpha < \varkappa\}$ by induction. If $\{a_\alpha\colon \alpha < \beta\}$ are already defined, let a_β be an element outside of the group generated by $\{g_\alpha\colon \alpha < \beta\} \cup \{a_\alpha\colon \alpha < \beta\}$ such that $2a_\beta$ is outside of this group as well. The existence of an element with this property follows from the assumptions about G.

First assume that the cardinal $\varkappa$ has uncountable cofinality.

Suppose that $K = \{g_{\alpha_n}\colon n \in \omega\}$ is a countable subset of G and let $\alpha = \sup\{\alpha_n\colon n \in \omega\}$. It is enough to show that $-a_\alpha \notin K+A$. Assume the contrary. Then $-a_\alpha = g_{\alpha_n} + a_\beta$ for some natural number n and $\beta < \varkappa$. If $\beta > \alpha$, this implies that a_β belongs to the group generated by a_α and g_{α_n}. If $\beta < \alpha$, this implies that a_α belongs to the group generated by a_β and g_{α_n}. If $\beta = \alpha$, this implies the equality $2a_\alpha = -g_{\alpha_n}$. In each case we get a contradiction with the definition of A. If $\mathrm{cf}(\varkappa) = \omega$, let G_1 be any subgroup of G of cardinality ω_1. We construct a set A for G_1 as above. For any countable $M \subset G$ we have

$$M+A = [(M \cap G_1)+A] \cup [(M \setminus G_1)+A].$$

Since $-A$ is not contained in the first summand (by the above argument) and the second summand is disjoint from G_1, we get $-A \not\subset M+A$. This completes the proof.

Let us finally remark that using the set A constructed above it is possible to define an invariant measure enjoying a stronger property. If we put

$$m_1(X) = 0$$

for $X \subset (K+A) \cup G \setminus (L-A)$, where K and L are countable subsets of G, and

$$m_1(Y) = 1$$

for Y being a complement of X as above, we see that m_1 is an invariant non-symmetric measure defined on a symmetric σ-algebra. Hence not only is m_1 non-symmetric but also it does not have any symmetric extension. We do not know if Theorem 2 can be improved in a similar way.

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