COLLOQUIUM MATHEMATICUM

VOL. LVIII 1989 FASC. 1

BLOWUP IN NONLINEAR PARABOLIC EQUATIONS

BY

PIOTR BILER (WROCŁAW)

1. Introduction. Our aim in this paper is to compare the properties of temporally nonglobal positive solutions to some semilinear heat equations with sources with those to the porous medium equation.

The problem of studying the noncontinuable solutions is not only of pure mathematical interest (as, e.g., the problem of characterization of the optimal conditions for global existence) but also – and for some equations primarily – of practical one. For instance, for the first kind of equations mentioned above these solutions may describe ignition, combustion phenomena, propagation of flames, and finally deflagration in an active nonlinear medium.

The equations studied here are special cases of the general quasilinear heat diffusion equation

$$u_t = \nabla\cdot(K(u)\nabla u)+Q(u), \tag{1.1}$$

where $u \geqslant 0$ is (generally) interpreted as the temperature, $K(u) \geqslant 0$ is the conductivity coefficient, $Q(u) \geqslant 0$ describes the nonlinear sources (the response of thermally active medium). Several serious mathematical difficulties may appear in a study of such equations. We mention only the possible degeneracy when $K(u)$ becomes zero and the critical growth of $Q(u)$ preventing from even local existence.

Let us remark that the physically relevant and the most interesting models are those with

$$\int_0^1 K(u)u^{-1}\,du < \infty, \tag{1.2}$$

$$\int_1^\infty (Q(u))^{-1}\,du < \infty. \tag{1.3}$$

The first condition guarantees (when $Q \equiv 0$) the finite velocity of propagation of the initial disturbances. The second condition, when diffusion is lacking: $K \equiv 0$, implies the finite time of explosion of any nontrivial solution (u tends to infinity for some x and $t \to T < \infty$).

One usually chooses power functions as K and Q not only in order to simplify the equations (since they introduce some homogeneity) but also to exhibit a rich symmetry structure (the equation is invariant under a large group of transformations). Besides universality it is a rather frequent situation that the most symmetric solution describes the common asymptotics of (nearly) all the solutions. Of course, one should be careful in order not to overestimate the universal role of power functions K and Q (see Example (3.4)).

A typical effect observed in the asymptotic behavior of solutions to a semilinear heat equation with sources is the spatial localization of solutions near the blowup time, and simultaneously the self-similar character of the blowup phenomena. On the contrary, for the porous medium and linear heat equation the blowup is not localized in the space and, in some sense, comes from infinity.

We will consider here only the Cauchy problem on the whole space $\boldsymbol{R}^N$ or incidentally the initial-boundary value problem on balls in $\boldsymbol{R}^N$. For some special boundary value problems with blowing up solutions see [28]. These problems are in general more complicated bacause of delicate questions of uniqueness of solutions (cf. Remark (2.4) and Example (4.15) below).

The new results in this paper are presented in Sections 4, 6 and 7.

Acknowledgment. The author wishes to thank Michel Pierre for helpful remarks during his visit in Wrocław in October 1986 and for his kind hospitality and stimulating discussions during the author's stay at University of Nancy 1 in May 1987.

2. Parabolic equations with sources. Here we recall some general results concerning the equation

$$u_t = \Delta u + u^p, \tag{2.1}$$

where $p > 1$. Equation (2.1) considered on a bounded domain in $\boldsymbol{R}^N$ and supplemented with homogeneous Dirichlet boundary conditions admits, under some natural restrictions on p and N, positive solutions which blow up in finite time. The blowup phenomenon occurs for a large class of initial data only at one point x_0 of the domain:

$$\lim_{t\to T} u(x, t) = 0 \text{ for all } x \neq x_0 \quad \text{and} \quad \lim_{t\to T} \|u(\cdot, t)\|_\infty = \infty. \tag{2.2}$$

We refer the reader to [31] and [15] for precise statements and generalizations for other nonlinear source terms.

We leave apart some interesting mathematical questions concerning the existence and the regularity of solutions to equations of type (2.1), pointing out only the remarks (2.3)–(2.5).

(2.3) The only reason of noncontinuability of a solution to (2.1) is (like for ordinary differential equations) the explosion of its norm (in some L^r-space, in particular L^∞ as above); cf. [31] and [3].

(2.4) Some delicate problems are connected with the definition of assuming the initial values. If one takes too weak one, the nonuniqueness of solutions would appear in a rather dramatic way. For instance, there would be infinitely many weak solutions with prescribed blowup time (less than the maximal time of existence of the "proper" solution selected according to a physically motivated variational principle) and the same initial data; cf. [22] and [3].

(2.5) The solution to (2.1) cannot be extended beyond $T_{\max}$ and blows up everywhere on $\Omega \times [T_{\max}, \infty)$ (more precisely: any reasonable approximation process leading to the weakest solution (2.1) — the so-called integral solution — with $T_{\max} < \infty$ diverges to infinity after $T_{\max}$); cf. [4].

Some similar results hold for the model equation

$$u_t = \Delta(u^m) + u^p, \tag{2.6}$$

where $m > 1$, $p > 1$, which is a (possibly) degenerate quasilinear parabolic equation. Here the blowup is confined to a set of measure zero if $p > m$ or to a bounded (but not too "thin") set if $p = m$.

For extensive reviews of results on the solvability of the initial value problem on $\boldsymbol{R}^N$ or on a ball in the class of radial functions, on the symmetry groups, self-similar solutions and blowing up solutions to (2.6) we refer to [18], [37], [16] and [17]. Moreover, a profound discussion is given there of the physical significance of equations like (1.1); the spatial localization of singularities of radial solutions to such equations is interpreted from the point of view of synergetics, finally the applicability of these equations in the description of dissipative structures is considered. Numerical results suggesting the symmetry breaking phenomena are also presented. The main tools in proofs are various generalizations of the strong maximum principle for parabolic equations.

Besides these reviews with abundant list of references of numerical, physical and mathematical character we mention also [32], [36], [30] where, in addition to the existence and nonexistence, the regularity questions of solutions to (2.6) for $p \leqslant m$ are studied using a generalization of the Aronson–Bénilan inequality for the porous medium equation (cf. especially Proposition 2.3 in [36]); see (6.5) below.

For the occurrence of the blowup for the finite difference approximation of the one-dimensional equation (2.1) see [8].

3. Self-similar blowup for equation (2.1). For equation (2.1) simpler than (2.6) there are more rigorous results concerning behavior of blowing up solutions near the critical moment of time. Namely, Giga and Kohn proved in a series of papers [19]–[21] that any blowing up solution resembles asymptotically a self-similar solution to (2.1). This statement requires a comment because in this situation (unbounded solutions near their explosion points) the familiar notions of stability and proximity of solutions have no sense. In fact, Giga and Kohn proved, modulo some technical details concerning u_0, p, N and described in [19]–[21] and [41], that for any local solution to (2.1) defined on $\{(x, t)$:

$|x| < 1,\ -1 < t < 0\}$ the limit of rescaled solutions (by the Boltzmann scaling)

$$u_\lambda(x, t) = \lambda^{2/(p-1)} u(\lambda x, \lambda^2 t), \quad \lambda > 0, \tag{3.1}$$

$$\lim_{\lambda \to 0} (-t)^{1/(p-1)} u_\lambda(x, t) \tag{3.2}$$

is equal to $\pm(p-1)^{1/(1-p)}$ or 0. Here $p < (N+2)/(N-2)$ and the blowup time T is shifted to 0. The latter possibility corresponds in fact to a regular solution without any singularity for $T = 0$ (see [21]). In other words (but in the previous notation),

$$\lim_{t \to T} (T-t)^{1/(p-1)} u(y_0 + y(T-t)^{1/2}, t) = \pm(p-1)^{1/(1-p)} \tag{3.3}$$

uniformly for $|y| \leqslant$ const.

Their description shows that any blowing up solution to (2.1) behaves asymptotically for $t \to T$ like the spatially homogeneous solution $(T-t)^{1/(1-p)} \times (p-1)^{1/(1-p)}$. This statement does not exclude of course the possibility of localized blowup as mentioned in Section 2 (cf. [31]). In such a situation the detection of the explosion point requires a study of translates of u in x. Remark that the asymptotic behavior of u in a parabolic region in (3.3) may be very different from the profile at a given time t near T.

Finally, let us note that the restrictions imposed above on p and N are important as the examples in [39] show.

(3.4) The example below due to M. Pierre shows that the asymptotically self-similar character of the blowing up solutions is not universal for nonautonomous versions of (2.1), that is, a perturbation of equation (2.1) may have a solution which blows up in L^∞ but is not a blowing up weak solution. Namely,

$$u(x, t) = (|x|^2 + t^2)^{-1}, \quad x \in \mathbf{R}^N,\ N > 4,\ -1 < t < a,\ a > 0,$$

satisfies a nonlinear heat equation of the form

$$u_t = \Delta u + g(x, t) u^2 \tag{3.5}$$

with a positive function g bounded from below and from above. Rescaling this solution as in (3.1) we get

$$\lim_{\lambda \to 0} u_\lambda(x, t) = 1/|x|^2,$$

which shows that the stability results for the blowing up solutions seem to be invalid for the perturbed equation, i.e., there is no structural stability with respect to a change of the coefficients of the equation. The same example shows the sharpness of the result in [4] concerning the strong noncontinuability of unbounded solutions to (2.1).

4. The linear heat equation and its various blowing up solutions. The simplest diffusion equation

$$u_t = \Delta u \tag{4.1}$$

is a very special case of the diffusion equations

$$u_t = \Delta(u^m), \quad \text{where } m > 0. \tag{4.2}$$

As we remarked in the Introduction, the case $m > 1$ corresponds to finite velocity of propagation of thermal waves, $m < 1$ gives the fast diffusion (the equation is used, e.g., in plasma modeling). The nonglobal solutions of the Cauchy problem in the former case will be studied in the next sections. In the latter case there are no restrictions on the growth of initial data in order to have global solvability of (4.2). They should be only not too concentrated measures, i.e., they should not charge too small sets. For the precise meaning of this statement we refer to [23] and [35].

As concerns the blowup properties, the linear heat equation has quite different properties than the semilinear equation (2.1) or the porous medium equation (4.2).

Here we consider nonnegative solutions to (4.1) blowing up at time T, invariant under Boltzmann scaling

$$u(x, t) = \lambda^{2\gamma} u(\lambda x, T-\lambda^2(T-t)), \quad \lambda > 0, \tag{4.3}$$

therefore of the form

$$u(x, t) = (T-t)^{-\gamma} w(y), \tag{4.4}$$

where $x = (T-t)^{1/2} y$ and $\gamma > 0$ is not determined yet. The functions $w = w(y)$ satisfy the following equation in $\mathbf{R}^N$:

$$\tfrac{1}{2} y \cdot \nabla w + \gamma w = \Delta w \tag{4.5}$$

or, equivalently,

$$\nabla \cdot \left(\nabla w - \frac{y}{2} w \right) = \left(\gamma - \frac{N}{2} \right) w. \tag{4.5'}$$

Remark that, in the similarity space variables y and $s = -\log(T-t)$, (4.1) transforms into

$$w_s + \tfrac{1}{2} y \cdot \nabla w + \gamma w = \Delta w, \tag{4.6}$$

where $u(x, t) = (T-t)^{-\gamma} w(y, s)$. Of course, studying the solutions to (4.1) near the blowup time T is equivalent to analyzing the large time asymptotics, $s \to +\infty$, of (4.6).

Evidently, $w(y) = \exp(|y|^2/4)$ is a solution to (4.5) with $\gamma = N/2$. Returning to the old variables we obtain

$$u(x, t) = (T-t)^{-N/2} \exp\left(|x|^2/4(T-t)\right). \tag{4.7}$$

In regard to the special form of (4.7), which resembles the backward fundamental solution, let us notice that the nontrivial solutions to (4.5) should have exponential growth in y.

(4.8) PROPOSITION. *Let* $\varrho(y) = \exp(-|y|^2/4)$ *and let a solution* w *to* (4.5) *satisfy*

$$\varrho w \in L^1(\mathbf{R}^N) \quad \text{and} \quad \lim_{R\to\infty} \int_{|y|=R} \varrho(y)\nabla w(y)\cdot \bar{n}(y)dS(y) = 0.$$

Then w *is identically zero.*

Proof. Multiplying (4.5) by ϱ and integrating over the ball of radius R, and then passing with R to infinity we get

$$\gamma\int\varrho w + \tfrac{1}{2}\int\varrho y\cdot\nabla w = \int\varrho\Delta w, \quad \gamma\int\varrho w - \int\nabla\varrho\cdot\nabla w = \int\varrho\Delta w,$$

$$\gamma\int\varrho w + \int\varrho\Delta w = \int\varrho\Delta w, \quad \gamma\int\varrho w = 0,$$

so $w = 0$ almost everywhere. The integration by parts and taking the limits above can be easily justified.

Note that there exist positive radial solutions to (4.5) for arbitrary $\gamma > 0$, $g(r) = w(y)$, $r = |y|$, satisfying

$$g'' + g'\left(\frac{N-1}{r} - \frac{r}{2}\right) = \gamma g,$$

e.g.,

$$g(r) = C\left(1 + \sum_{k=1}^{\infty} r^{2k} \frac{\gamma(\gamma+1)\dots(\gamma+k-1)}{k!\ 4^k(N/2)(N/2+1)\dots(N/2+k-1)}\right). \tag{4.9}$$

All these functions satisfy the estimate

$$w(y) \sim C\exp(|y|^2/4) \quad \text{for } |y|\to\infty.$$

The last remark and a simple observation below permit us to construct also nonradial solutions to (4.5) with a given $\gamma > 0$ in arbitrary dimension. Namely, we have

(4.10) LEMMA. *If* $w_1 = w_1(y_1)$ *and* $w_2 = w_2(y_2)$ *satisfy* (4.5) *for* $y_1 \in \mathbf{R}^{N_1}$, $y_2 \in \mathbf{R}^{N_2}$ *with* γ_1, γ_2, *respectively, then*

$$w(y) = w_1(y_1)\cdot w_2(y_2), \quad y = (y_1, y_2) \in \mathbf{R}^N,\ N = N_1 + N_2,$$

is a nonradial solution to (4.5) *with* $\gamma = \gamma_1 + \gamma_2$ *and* w *has the same exponential growth* $\exp(|y|^2/4)$ *as before.*

The proof follows from a straightforward calculation.

The solutions of such a rapid growth may be treated classically using the Weierstrass kernel. The well-known Widder representation theorem (cf. e.g., [1]) gives, for any positive solution u of the heat equation, the formula

$$u(x, t) = \int\exp(-|x-y|^2/4t)u_0(y)dy \tag{4.11}$$

valid for all sufficiently small $t > 0$ if and only if u_0 is a measure of not too rapid spatial growth:

(4.12) $$\int u_0(y)\exp(-c|y|^2)dy$$

should be finite for some positive c.

The self-similar solutions to (4.5), as expected, have the critical rate of growth preventing the global in time evolution of u. Their characteristic feature is the explosion uniform in x:

$$\lim_{t\to T} u(x, t) = \infty .$$

However, there are some other ways for solutions to (4.1) to blow up.

(4.13) PROPOSITION. *There exist solutions to the one-dimensional equation* (4.1) *with maximal time of existence T such that one of the following holds:*

(i) $u(x, T)$ *is finite for all* x;
(ii) $u(x, T) < \infty$ *for* $x < x_0 \in \mathbf{R}$, $u(y, T) = \infty$ *for* $y \geqslant x_0$;
(iii) $u(x, T) < \infty$ *for* $x \leqslant x_0 \in \mathbf{R}$, $u(y, T) = \infty$ *for* $y > x_0$;
(iv) $u(x, T) = \infty$ *for all* x (*like* (4.9)).

Proof. It suffices to restrict our attention to initial data of type $\sum_{n=0}^{\infty} c_n\delta_n$, with measures of masses c_n located at the points $n \in \mathbf{N}$. From (4.11) it follows that

$$\begin{aligned} u(x, t) &= (4\pi t)^{-1/2} \sum_{n=0}^{\infty} a_n \exp(n^2/4T)\exp(-|x-n|^2/4t) \\ &= (4\pi t)^{-1/2} \sum_{n=0}^{\infty} a_n \exp(n^2(1/4T-1/4t))\exp(-x^2/4t)\exp(nx/2t), \end{aligned}$$

where $c_n = a_n \exp(n^2/4t)$, with the critical time T equal to

$$(4\cdot\inf\{c\colon (4.12) \text{ is finite}\})^{-1}.$$

Clearly, our solution is classical for $t < T$ and it does not exist for $t > T$. Now it is quite easy to construct the power series

$$\sum_{n=0}^{\infty} a_n(e^{x/2T})^n$$

with, say,

$$\exp(-n^{3/2}) \leqslant a_n \leqslant \exp(-n^{1/2})$$

and given behavior at the ends of their intervals of convergence.

Of course, one can produce further examples of solutions with different sets of explosion for $t = T$ in multidimensional case, in particular radial solutions with a prescribed open or closed half-line of infinite values of $u(\cdot, T)$.

(4.14) Remark. Our last observation concerns the blowup of positive solutions to linear uniformly parabolic equations in divergence form. Having an analogue of Widder's representation formula, given by Aronson in [1],

$$u(x, t) = \int_{R^N} G(x-y, t)u_0(y)dy$$

with $G(x, t)$ bounded from below and from above by two Weierstrass kernels for the heat equations with two different coefficients $(u_j)_t = \varkappa_j \Delta u_j$, $j = 1, 2$, one proves that the maxima of blowing up solutions are of order $\exp(c|x|^2)$ with suitable positive constant c depending on the blowup time and the parabolicity constants of the equation. The proof repeats the arguments in (4.8) and the construction (4.13) above for the heat equation.

For the blowup phenomena in boundary value problems it should be noted that the problem is far more delicate as, e.g., the questions of uniqueness occur.

(4.15) EXAMPLE. Consider the heat equation (4.1) in $(0, 1)\times(0, \infty)$ supplemented with the conditions

$$u(0, t) = 0, \quad u(x, 0) = 0,$$
$$u(1, t) = (4\pi)^{-1/2} t^{-3/2} \exp(-1/4t).$$

It is well known (see, e.g., [29]) that there exists a solution of this problem belonging to $C^\infty([0, 1]\times[0, \infty))$. However, if one takes the function

$$v(x, t) = xt^{-1}(4\pi t)^{-1/2} \exp(-x^2/4t),$$

one easily verifies that $v \in C^\infty((0, 1)\times(0, \infty))$ only and $v(x, 0) = 0$ for $x \in (0, 1)$ in the sense of $L^1_{\mathrm{loc}}(0, 1)$ convergence as t tends to zero. Finally, v satisfies the boundary conditions and v is different from the afore-mentioned smooth solution. So we have an example of nonuniqueness of a positive solution to a boundary value problem for (4.1) with a weak but natural definition of assuming the initial data (it is not the distributional convergence because of a measure concentrated in (0, 0)). The situation in the whole space R^N is quite different: positive solutions with the same initial trace (which is a measure) are unique (cf. the Widder formula (4.11)), and $\mathscr{D}'(R^N)$ convergence follows from $L^1_{\mathrm{loc}}(R^N)$ convergence.

An explanation of these effects is given in a recent paper [14] of Dahlberg and Kenig who characterize the positive solutions to initial-Dirichlet problems in cylinders $\Omega\times[0, T]$ for general equations of the type $u_t = \Delta(\varphi(u))$ including the heat and the porous medium equation (such a solution is determined by a measure on Ω and a measure on $\partial\Omega$).

5. Preliminaries on the porous medium equation. We will consider nonnegative solutions to the Cauchy problem for the porous medium equation

$$u_t = \Delta(u^m), \quad m > 1, \tag{5.1}$$

defined in a strip

$$\{(x, t)\colon x \in R^N,\ T_0 < t < T\}, \quad T_0 \in [-\infty, T).$$

For a review of physical motivation and (early) theory of the porous medium equation we refer to [7] and [33]. We will use extensively the results on the solvability and regularity of continuous weak solutions to (5.1) in a suitably chosen class of functions ($P(T)$ in [10]) constructed in [6] and subsequently studied in [10].

A continuous function $u \geqslant 0$ defined on $\{(x, t): x \in \mathbf{R}^N, 0 < t \leqslant T\}$ is said to be in $P(T)$ if u satisfies the integral identity

$$\int_{\mathbf{R}^n \times (t_1, t_2)} (u^m \Delta\varphi + u\varphi_t)dxdt = \int u(x, t_2)\varphi(x, t_2)dx - \int u(x, t_1)\varphi(x, t_1)dx$$

for all $0 < t_1 < t_2 \leqslant T$ and all $\varphi \in C^{2,1}$ with compact support in x.

First we recall that Aronson and Caffarelli proved in [2] the existence of the initial trace (cf. (5.3) below) for any nonnegative solution to (5.1), which is a nonnegative Borel measure of moderate growth (see (5.2) below). In fact, a Fatou theorem holds for the solution to (5.1) in $P(T)$: $u(x, t)$ a.e. tends to the density of the absolutely continuous part of u_0 as t decreases to zero (see [9]). Conversely, for any nonnegative measure μ on $\mathbf{R}^N$ satisfying

$$\sup_{R \geqslant r} R^{-(N+2/(m-1))}\mu(B_R) =: \|\mu\|_r < \infty, \qquad r \geqslant 1, \tag{5.2}$$

there exists a solution to (5.1) (unique in $P(T)$) with the initial trace

$$\lim_{t \to 0} \int_{\mathbf{R}^N} u(x, t)\varphi(x)dx = \int_{\mathbf{R}^N} \varphi(x)d\mu(x) \tag{5.3}$$

for all $\varphi \in C_0(\mathbf{R}^N)$; see Proposition 1.6 in [6] and Theorem 4 in [10].

Note that this result is a complete analogue of Widder's one for the linear heat equation: the nonnegative solutions are characterized by their initial values, modulo a restriction on the function space – here Dahlberg and Kenig's $P(T)$ class – guaranteeing the uniqueness of the solution issued from its trace.

Let us mention an interesting question concerning the a priori regularity assumption in the definition of the class $P(T)$. It was not known for a long time whether the hypothesis $u \in L^m_{\text{loc}}$ (the weakest assumption necessary to define the distributional solution to (5.1)) implies the continuity of u. This problem reduces to that of showing the implication

$$u \in L^m_{\text{loc}} \Rightarrow u \in L^\infty_{\text{loc}},$$

and an affirmative answer was given in [12].

The condition (5.2) can be rewritten as ((0.3), (0.3′) in [6])

$$\sup_{R \geqslant 1} R^{-(N+2/(m-1))} \int_{|x| \leqslant R} u_0(x)dx < \infty \tag{5.2′}$$

or

$$\sup_{R \geqslant 1} R^{-N} \int_{|x| \leqslant R} u_0(x)(1+|x|^2)^{-1/(m-1)}dx < \infty$$

for the initial value u_0 being an L^1_{loc}-function. The length of the maximal interval of existence of the solution with $u(x, 0) = u_0(x)$ in the sense of (5.3) is estimated from below by

$$T = T(u_0) \geqslant C(N, m)/l(u_0)^{m-1},$$

where $l(u_0)$ is the limit of $\|u_0\|_r$ defined in (5.2) when r tends to infinity. This estimate is sharp (up to its order) as the explicit solution

$$u(x, t) = \left(AT^k/(T-t)^k + c|x|^2/(T-t)\right)^{1/(m-1)} \tag{5.4}$$

starting from the initial data

$$u_0(x) = (A + B|x|^2)^{1/(m-1)}$$

(of critical admissible growth, see (5.2′)) shows. Here the notation is consistent with that in [6]:

$$k = N(m-1)/(N(m-1)+2), \qquad c = k/(2mN), \qquad T = c/B, \qquad A \geqslant 0.$$

We would like to show that the special solution (5.4) displays the "typical" way of blowup of positive solutions to (5.1). The significance of this vague statement will be revealed in the sequel.

First we observe that (5.4) written in the form

$$u(x, t) = (T-t)^{-k/(m-1)}\left(AT^k + BT|x|^2/(T-t)^{1-k}\right)^{1/(m-1)} \tag{5.5}$$

is evidently self-similar. We would consider self-similar solutions to (5.1) defined for $t < T$, which blow up at time T, of the scale-invariant form

$$u(x, t) = \lambda^\beta u\left(\lambda x, T - \lambda^\alpha (T-t)\right),$$

so simultaneously

$$u(x, t) = (T-t)^{-\gamma} w(y) \quad \text{with } x = (T-t)^{1/\alpha} y,\ y \in \boldsymbol{R}^N.$$

Clearly, $\beta = (\alpha-2)/(m-1)$ and $\gamma = \beta/\alpha$, which follows from (5.1) and the invariance condition. With this notation the function w defined on $\boldsymbol{R}^N$ is a global solution to the nonlinear elliptic equation

$$\gamma w + \frac{1}{\alpha} y \cdot \nabla w = \Delta(w^m). \tag{5.6}$$

Introducing a new dependent variable $z = w^{m-1}$ we obtain the equation

$$(m-1)\gamma z + \frac{1}{\alpha} y \cdot \nabla z = \frac{m}{m-1}|\nabla z|^2 + mz\Delta z. \tag{5.7}$$

Restricting our attention to the case

$$\alpha = 2/(1-k) = N(m-1)+2, \qquad \beta = N \qquad \text{and} \qquad \gamma = k/(m-1)$$

(the same parameters as for (5.5) with $w(y) = (AT^k + BT|y|^2)^{1/(m-1)}$) we get from (5.7) the equation in divergence form

$$\nabla \cdot \left(z^{1/(m-1)}(\nabla z - 2cy)\right) = 0. \tag{5.8}$$

Observe that $z(y)=c|y|^2+C$, $C\geqslant 0$, satisfies (5.7) if and only if $C=0$ or $\alpha\gamma=N$; the latter condition corresponds exactly to (5.8).

Parallel to a study of self-similar solutions to (5.1) satisfying the nonlinear elliptic equation (5.8) we would consider, using an idea borrowed from [19], an equation satisfied by any solution to (5.1) of the form

$$u(x,t)=(T-t)^{-\gamma}w(y,s),$$

where $x=(T-t)^{1/\alpha}y$, $y\in \mathbf{R}^N$, $(T-t)=e^{-s}$, $t<T$, $s\in\mathbf{R}$. Substituting this expression into (5.1) we obtain the nonlinear parabolic equation

$$w_s+\gamma w+\frac{1}{\alpha}y\cdot\nabla w=\Delta(w^m) \tag{5.9}$$

or, taking $z=w^{m-1}$,

$$z_s+(m-1)\gamma z+\frac{1}{\alpha}y\cdot\nabla z=\frac{m}{m-1}|\nabla z|^2+mz\Delta z. \tag{5.10}$$

For $\gamma=k/(m-1)$ the equation above acquires a simple divergence form

$$\frac{m-1}{m}\frac{d}{ds}(z^{1/(m-1)})=\nabla\cdot\big(z^{1/(m-1)}(\nabla z-2cy)\big). \tag{5.11}$$

Observe an interesting (and crucial in [19]) property of w, the same as remarked after (4.6): rescaling u to u_λ corresponds to shifting $w(y,s)$ in s to $w(y,s-\alpha\log\lambda)$.

For simplicity of the notation we restrict our attention to the solutions with the similarity variables centered at $x_0=0$; compare however (7.2). The center of mass of any solution can be easily defined (and uniquely determined) shifting the balls in the condition (5.2′).

Notice that the factor $(T-t)^{-1/(m-1)}$ determines the maximal growth in time of blowing up solutions:

$$1/(m-1)=\gamma+2/(\alpha(m-1)),$$

where γ is the exponent of our prescribed time asymptotics and $(1/\alpha)(2/(m-1))$ corresponds to the time dependent factor in the similarity variable y and the maximal possible growth of u (cf. Theorem 1 in [10]). So equations (5.8) and (5.11) describe the solutions with the strongest time singularity $(T-t)^{-1/(m-1)}$ characteristic of (5.1) (compare this with the linear heat equation (4.1) where the rate of explosion $\gamma>0$ was arbitrary).

Note that in view of the results in [40] – specific for the one-dimensional case and somewhat similar to Proposition (4.13) (ii)–(iv) – it seems impossible to characterize all blowing up solutions to (5.1) without any supplementary structure assumptions.

6. Some uniqueness results for self-similar blowing up solutions to the porous medium equation. In this section we establish conditional uniqueness results for nonnegative solutions to equation (5.6) or (5.8) defined in $\boldsymbol{R}^N$. The need of introducing some supplementary hypotheses in order to get a local uniqueness or stability result is evident since one can produce many solutions to (5.7) other than $z(y) = c|y|^2$, namely

$$z(y) = c_J|\bar{y}_J|^2, \tag{6.1}$$

where $J = 1,\dots,N-1$, $c_J = c(2+N(m-1))/(2+J(m-1))$, and $\bar{y}_J \in \boldsymbol{R}^N$ has J coordinates of y and $N-J$ zeros. These trivial (lower dimensional) solutions do not have uniform growth in y and they present a serious obstacle in obtaining any general uniqueness result.

We begin with a simple fact concerning radial solutions to (5.8).

(6.2) PROPOSITION. *If $z(y) = g(r)$, $r = |y|$, is a radial solution to* (5.8) *in $\boldsymbol{R}^N$ such that*

$$\lim_{r\to 0} r^{N-1} g'(r) = 0,$$

then $g(r) = cr^2+C$ with an arbitrary (positive) constant C, i.e. any radial self-similar solution to (5.1) *satisfying* (5.8) *is of the special form* (5.5).

Proof. A simple formal argument and the regularity assumption on the behavior of g' at 0 show that g satisfies the ordinary differential equation

$$\left(g^{1/(m-1)} r^{N-1} (g'-2cr)\right)' = 0.$$

Integrating this equation we get $g'(r) = 2cr$ and $g(r) = cr^2+C$ with $C \geqslant 0$ if we look for nonnegative g. As concerns our assumption, it is intuitively clear that g' must even vanish at 0 in order that z be regular at 0.

The second case where it is straightforward to establish the uniqueness of solutions to (5.8) issued from (5.5) is described below.

(6.3) PROPOSITION. *Let $Z = z-c|y|^2$ be a solution to the equation*

$$z\Delta Z + \frac{1}{m-1}|\nabla Z|^2 + \frac{2c}{m-1} y\cdot\nabla Z = 0 \tag{6.4}$$

such that $\Delta Z \geqslant 0$ and

$$\lim_{R\to\infty} \int_{|y|=R} \exp(-\delta|y|^2)\nabla Z(y)\cdot\bar{n}(y)\,dS(y) = 0$$

for some $\delta > 0$. Then Z is a constant.

Proof. Observe that (6.4) in the form

$$z\Delta Z + \frac{1}{m-1}\nabla z\cdot\nabla Z = 0$$

is a direct consequence of (5.8). Using the idea borrowed from [19], we multiply (5.8) by

$$\varrho(y) = \exp\left(-\frac{\delta}{2}|y|^2\right)$$

with a sufficiently large $\delta > 0$ and integrate over the ball of radius R centered at the origin. After some integration by parts, letting R tend to infinity we arrive at

$$\int \varrho\left(z\Delta Z + \frac{2c}{m-1}\frac{1}{\delta}\Delta Z\right) + \frac{1}{m-1}\int \varrho|\nabla Z|^2 = 0$$

as $\nabla\varrho = -\delta\varrho y$ and the boundary integrals vanish (when $R \to \infty$) due to the growth condition imposed on ∇Z. Recall that z, $\Delta Z \geqslant 0$, so $\nabla Z = 0$ almost everywhere and Z is a constant.

The assumption $\Delta Z \geqslant 0$ (in fact, it suffices to have $\Delta Z \geqslant 0$ in the sense of distributions) in (6.3) is restrictive but not unexpected. Namely, from the Aronson–Bénilan inequality for solutions to the Cauchy problem for (5.1) in the class $P(T)$ with initial data imposed at $T_0 = 0$, i.e.,

$$\Delta(u^{m-1})(\cdot, t) \geqslant -\frac{k}{mt} \quad \text{in } \mathscr{D}'(\mathbf{R}^N), \tag{6.5}$$

(cf. (1.19) in [6], (1.4) and Theorem 3 in [10]) it follows that $\Delta z \geqslant 0$ distributionally. In fact, this is a consequence of the stronger inequality $\Delta(u^{m-1}) \geqslant 0$ which holds for self-similar solutions to (5.1) defined for all $t < T$, obtained by shifting T_0 to $-\infty$ in (6.5) and from the identity

$$\Delta_y z(y) = (T-t)\Delta_x(u^{m-1})(x, t).$$

The condition $\Delta Z \geqslant 0$ equivalent to $\Delta z \geqslant 2cN = k/m$ is of course much stronger.

Anyway, (6.5) implies an algebraic rate of growth of u in the spatial variable

$$u(x, t) \leqslant C_t(u)(1+|x|^2)^{1/(m-1)}$$

as proved in [10] (Theorem 1) or in [6] (a corollary to Proposition 1.3). Thus

$$z(y) \leqslant C(z)(1+|y|^2),$$

but we may prove the last estimate completely elementarily – without any recourse to Moser's iteration procedure used for u in [6] or [10].

Formally, using (5.7) and (6.5) we have

$$|\nabla z|^2 \leqslant Cz + \frac{m-1}{m\alpha} y \cdot \nabla z$$

for some positive constant C, hence

$$|\nabla z|^2 \leqslant C(z+|y|^2) + \tfrac{1}{2}|\nabla z|^2$$

for another positive constant still denoted by C. Observe that

$$z(y) = z(0) + \int_0^{|y|} \nabla z(ry/|y|)(y/|y|)dr,$$

so

$$|\nabla z(y)|^2 \leqslant C\Big(1 + |y|^2 + \int_0^{|y|} |\nabla z(ry/|y|)|dr\Big).$$

This integral inequality for $|\nabla z(y)|$ gives finally

(6.6) $$|\nabla z(y)|^2 \leqslant C(1+|y|)^2 \quad \text{and} \quad z(y) \leqslant C(1+|y|^2).$$

More precisely, if

$$R = |y| \quad \text{and} \quad \Psi(r) = |\nabla z(ry/|y|)|,\ 0 \leqslant r \leqslant R,$$

then

$$\Psi^2(R) \leqslant C(1+R^2) + \int_0^R \Psi(r)dr.$$

Denoting the right-hand side of this inequality by Φ and differentiating we get

$$\begin{aligned}\Phi'(R) = 2CR + \Psi(R) &\leqslant (8C^2R^2 + 2\Psi^2(R))^{1/2} \\ &\leqslant \Big(8C^2R^2 + 2C(1+R^2) + 2\int_0^R \Psi\Big)^{1/2} \\ &\leqslant 4(2C+1)\Phi^{1/2}(R)\end{aligned}$$

and after integration we obtain

$$\Psi^2(R) \leqslant \Phi(R) \leqslant C(1+R^2).$$

These formal calculations can be justified for less regular solutions to (5.7) by approximating them in a routine way.

(6.7) Remark. If we consider the most general solutions to (5.6), i.e., the distributional ones with $w \in L^m_{\text{loc}}$, we still can give an estimate of the average growth of w. First observe that

(6.8) $$\frac{d}{dR}\Big(\fint_{B_R} w^m\Big) = \frac{R}{2}\fint_{B_R} (1 - |y|^2/R^2)\Delta(w^m)dy,$$

where

$$\fint_{B_R} = (\omega_N R^N)^{-1} \int_{B_R}$$

denotes the average over the ball B_R. Then recalling (5.6) we have

$$\frac{d}{dR}\Big(\fint_{B_R} w^m\Big) = \frac{R}{2}\fint_{B_R} (1 - |y|^2/R^2)\Big(\gamma w + \frac{1}{\alpha} y\cdot \nabla w\Big)dy,$$

and the right-hand side is equal to

$$\frac{\gamma R}{2} \fint_{B_R} (1-|y|^2/R^2)w - \frac{R^3}{8\alpha} \fint_{B_R} \nabla w \cdot \nabla((1-|y|^2/R^2)^2)$$

$$= \frac{\gamma R}{2} \fint_{B_R} w(1-|y|^2/R^2) + \frac{R^3}{8\alpha} \fint_{B_R} w \Delta((1-|y|^2/R^2)^2).$$

Finally, we obtain

$$\frac{d}{dR}\left(\fint_{B_R} w^m\right) = \frac{\gamma R}{2} \fint_{B_R} w(1-|y|^2/R^2) + \frac{R}{2\alpha} \fint_{B_R} w((N+2)|y|^2/R^2 - N)$$

$$\leqslant CR \fint_{B_R} w$$

with a constant $C = C(\alpha, \gamma, N)$. Now, the Hölder inequality gives

$$\frac{d}{dR}\left(\fint_{B_R} w^m\right) \leqslant CR\left(\fint_{B_R} w^m\right)^{1/m}$$

and after integration we obtain

$$\fint_{B_R} w^m \leqslant CR^{2m/(m-1)}. \tag{6.9}$$

This result is not so precise as (6.6) but consistent with the blowup estimate from [6] (Theorem EU (b), (c))

$$\lim_{t \to T} \|u^{m-1}(t)/(1+|x|^2)\|_\infty = \infty$$

obtained only for solutions from $P(T)$ ($\max w(y) \sim C|y|^{2/(m-1)}$). Moreover, it is easily seen from (6.8) that for solutions satisfying the Aronson–Bénilan inequality (6.5) the average $\fint_{B_R} w^m$ increases with R since $\Delta(w^m) \geqslant 0$, $w^m = z^{m/(m-1)}$, $\Delta z \geqslant 0$.

Finally, observe that subharmonicity of the function $z(y, s)$, which follows also from (6.5), excludes the possibility of strictly localized (confined to a compact set) blowup of solutions to (5.1).

Note that a similar estimate of the averages of u^m over balls shows that

$$\int_{t_0}^{t} u^m(\cdot, \tau)d\tau \in L^\infty_{\mathrm{loc}} \quad \text{for all } 0 < t_0 < t < T$$

(M. Pierre, personal communication). For other L^p regularity results for solutions to the porous medium equation see [5] and [12].

The main result in this section is the uniqueness of sufficiently regular solutions to (5.8) satisfying some uniform estimates on minimal growth in order to separate the special solutions (6.1).

Remark that looking at (5.8) or at the equivalent equation (6.4) as a linear equation with respect to Z (with the coefficients depending however on

$z = Z + c|y|^2$), one might think of a Liouville-type theorem: if Z is of reasonable growth, then Z is a constant. Unfortunately, this approach fails, since the coefficients of (5.8) may grow like a power of $|y|$, and the Liouville theorems for linear elliptic equations in divergence form are generally valid only for the coefficients growing not faster than

$$O((\log\log|y|)^{\varepsilon}), \quad 0 < \varepsilon < 1$$

(cf. [27] for precise statements and examples).

Finally, lack of variational methods for studying the nonlinear elliptic equation (5.6) (cf. [18] and [17], p. 145), which appeared to be very useful in [19], renders our task somewhat difficult.

(6.10) THEOREM. *If $Z \in C^3$ satisfies equation* (6.4) *and is bounded from below, then Z is a constant.*

Remarks. The Bernstein method (and its various modifications) of proving some estimates of ∇Z used here (cf. [29], IV. §17, [34] or [38]) requires our regularity assumption $Z \in C^3$, which is, however, not very restrictive for nondegenerate solutions: $z(y) > 0$. This hypothesis can be weakened to $Z \in C^2$ by considering (6.4) in its weak formulation.

The idea of the proof is based on the classical arguments in [34], Theorem 4.7, remarks on pp. 93, 98, and [38]. See also [24] for a slightly more general situation. Nevertheless, our proof is somewhat different because equations (5.8) and (6.4) do not fit into the schemes in [34] and [38].

A different approach to Liouville-type theorems is given in [26] by Karp. His method of differential inequalities for certain functionals depending on the solution does not require even C^2 regularity assumption (cf. remark in [26], p. 87). It is more flexible than the Bernstein or the Harnack inequality methods (the latter seems to be useless in the situation considered here) and allows us to rederive our result (6.10).

We expect that the one-sided boundedness hypothesis on Z in Theorem (6.10) may be replaced by the condition

$$\min_{|y| \leq R} Z(y) = o(R) \quad \text{for } R \to \infty \tag{6.11}$$

without modifying the conclusion of (6.10). For that reason we present here a proof of a slightly different result.

(6.12) PROPOSITION. *If $Z \in C^3$ satisfies* (6.4) *and $Z(y) = o(|y|)$, then $\nabla Z(y) = o(1)$ for $|y| \to \infty$.*

The proof of (6.12) shows some necessary modifications to be made in Peletier and Serrin's reasoning, it constitutes an ingredient of their proof and it would be useful in the proof of the more general conjecture with the hypothesis (6.11) above.

Proof. Consider the linear elliptic operator

$$Lv = z\Delta v + \frac{1}{m-1}\nabla z \cdot \nabla v$$

and the auxiliary function

$$v = \zeta|\nabla Z|^2 + MZ^2,$$

where Z satisfies (6.4), i.e., $LZ = 0$, $M > 0$, will be determined later. The cutoff function

$$\zeta(y) = (1 - |y - y_0|^2/R^2)_+^2, \quad R > 0, \ |y_0| = 2R, \tag{6.13}$$

is supported on the ball $\{y : |y - y_0| \leqslant R\}$ contained in $\{y : R \leqslant |y| \leqslant 3R\}$.

We calculate

$$Lv = \zeta\Delta(|\nabla Z|^2) + \Delta\zeta|\nabla Z|^2 + 2\nabla\zeta\cdot\nabla(|\nabla Z|^2)$$
$$+ \frac{1}{m-1}\nabla z \cdot (\nabla\zeta|\nabla Z|^2 + \zeta\nabla(|\nabla Z|^2)) + 2M|\nabla Z|^2.$$

Then, taking into account the equalities

$$\nabla(|\nabla Z|^2) = 2(\nabla^2 Z)(\nabla Z),$$
$$\Delta(|\nabla Z|^2) = 2|\nabla^2 Z|^2 + 2\nabla Z\cdot\nabla(\Delta Z)$$

and

$$z\nabla Z\cdot\nabla(\Delta Z) + (\nabla Z\cdot\nabla z)\Delta Z + \frac{1}{m-1}(\nabla^2 z)(\nabla Z)(\nabla Z) + \frac{1}{m-1}(\nabla^2 Z)(\nabla z)(\nabla Z) = 0$$

(a consequence of (6.4) differentiated and then multiplied by ∇Z) we get

$$zLv = 4z(\nabla^2 Z)(\nabla Z)(\nabla\zeta) + 2\zeta z|\nabla^2 Z|^2 - \frac{2}{m-1}\zeta(\nabla^2 z)(\nabla Z)(\nabla Z) \tag{6.14}$$
$$+ 2\zeta z(m-1)|\Delta Z|^2 + \left(\frac{1}{m-1}\nabla\zeta\cdot\nabla z + z\Delta\zeta + 2Mz\right)|\nabla Z|^2.$$

We wish to estimate zLv from below using positivity of the second, fourth and the seventh terms.

The absolute value of the first term can be estimated using (6.13) by

$$\zeta z|\nabla^2 Z|^2 + CR^{-2}z|\nabla Z|^2$$

(note that $|\nabla\zeta|^2 \leqslant CR^{-2}\zeta$).

The fifth term is not less than

$$-CR^{-1}|\nabla z|\,|\nabla Z|^2.$$

The sixth term is simply greater than

$$-CR^{-2}z|\nabla Z|^2$$

(as $|\Delta\zeta| \leqslant CR^{-2}$).

Finally, observing that $\nabla^2 z = \nabla^2 Z + 2cI$, the absolute value of the third term can be estimated by

$$\zeta z|\nabla^2 Z|^2 + \zeta z^{-1}|\nabla Z|^4 + \frac{4c}{m-1}\zeta|\nabla Z|^2.$$

Now, choosing $M = C_1 R^{-2}$ with a sufficiently large constant C_1 independent of R, recalling that (6.6) implies $\nabla z(y) = O(|y|)$, $\nabla Z(y) = \nabla z(y) - 2cy = O(|y|)$, $z(y) \sim c|y|^2$ as $Z(y) = o(|y|)$, we get $Lv \geqslant 0$. From the maximum principle applied to L and v we have

$$|\nabla Z(y_0)|^2 \leqslant \sup_{|y-y_0|=R} v(y) = M \sup_{|y-y_0|=R} |Z(y)|^2$$
$$= o(R^{-2}R^2) = o(1) \quad \text{for } R \to \infty,$$

as asserted. In particular, $|\nabla Z|^2$ is globally bounded.

In fact, the more precise estimate

$$|\nabla Z(y_0)| \leqslant CR^{-1} \sup_{R \leqslant |y| \leqslant 3R} (-Z(y) + Z(y_0))$$

from Theorem 4.6 in [34] is obtained using a more complicated auxiliary function

$$v = \zeta|\nabla(\varphi^{-1}(Z))|^2 \quad \text{with } \varphi(r) = \text{const} - e^{-r}.$$

Our modifications consist mainly in the a priori estimate of the term similar to the third term in (6.14) using the hypothesis $Z(y) = o(|y|)$ or Z one-sided bounded. This permits us to reduce rather lengthy computations concerning $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ in Theorem 4.6. Then the conclusion of Theorem (6.10) follows from the proof of Theorem 4.7 in [34].

(6.15) Remark. The assumption $Z(y) = o(|y|)$ can be easily justified in the case of the one-dimensional equation (5.8), i.e., $(wZ')' = 0$ with a strongly nondegenerate solution:

$$\inf_{y \in \mathbf{R}} z(y) > 0.$$

In fact, we have immediately $wZ' = C$ with a constant C. If $C = 0$, then obviously $Z = \text{const}$. If $C \neq 0$, say $C > 0$, then for $y > 0$ we get

$$0 \leqslant Z(y) - Z(0) = \int_0^y C/w \leqslant C\int_0^y (z(0) + cy^2)^{-1/(m-1)} dy$$
$$\leqslant C\int_0^y (1 + y^{2/(m-1)})^{-1} dy = o(y) \quad \text{for } y \to \infty$$

in all the cases:

$$2/(m-1) > 1, \quad Z(y) = O(1);$$
$$2/(m-1) = 1, \quad Z(y) = O(\log y) = o(y);$$
$$2/(m-1) < 1, \quad Z(y) = O(y^{1-2/(m-1)}) = o(y).$$

For $y < 0$ we have $z(y) \geqslant A > 0$, so $Z'(y) \leqslant C$ for a constant C and after integration we obtain

$$Z(y) \geqslant Cy, \quad z(y) = Z(y) + cy^2 \geqslant \varepsilon y^2$$

for some positive ε. Then $Z'(y) \leqslant C|y|^{-2/(m-1)}$ and similarly as before $Z(y) = o(|y|)$. The case $wZ' = C < 0$ is analogous.

Let us recall Example 4 from [26] due to Gilbarg and Serrin, which shows that our supplementary hypotheses on Z are reasonable.

(6.16) EXAMPLE. The equation $\Delta Z + b \cdot \nabla Z = 0$ in $\boldsymbol{R}^N$ has no nonconstant bounded solution if $b(y) = O(|y|^{-1})$. This condition is satisfied for any nondegenerate solution to our equation (6.4) written as

$$\Delta Z + \frac{2c}{m-1} \frac{\nabla z}{z} \cdot \nabla Z = 0$$

since (6.6) and assumption (6.10) or (6.11) give $\nabla z/z = O(|y|^{-1})$ for $|y| \to \infty$. Moreover, in general the condition imposed on the vector field $b(y)$ cannot be significantly relaxed.

7. Asymptotic uniqueness for the nonlinear parabolic equation for the blowing up solutions. Removing the hypothesis on self-similarity of the blowing up solutions to the porous medium equation with the maximal existence time T, we saw that they satisfy the nonlinear parabolic equation (5.9) or (5.10) on $\boldsymbol{R}^N \times \boldsymbol{R}$, rewritten as

(7.1) $$\frac{m-1}{m} \frac{d}{ds}(z^{1/(m-1)}) = \nabla \cdot (z^{1/(m-1)} \nabla Z),$$

where $Z(y, s) = z(y, s) - c|y|^2$. Here the similarity variables y, s are used to shift the singularity to $s = \infty$, and the study of asymptotics of blowing up solutions is reduced to the study of the behavior of translations of z in s, $s \to \infty$; cf. the remark after (5.11).

Equation (5.1) does not depend on x, and therefore we obtain, translating the special solution (5.5) in x, a lot of new solutions (which are self-similar only after centering in x) of the form

(7.2) $$v(x, t) = (T-t)^{-k/(m-1)} (AT^k + c|x - x_0|^2/(T-t)^{1-k})^{1/(m-1)}$$

with a fixed $x_0 \in \boldsymbol{R}^N$. They lead to

$$z(y, s) = AT^k + c|y - e^{s/\alpha} x_0|^2$$

satisfying (7.1) with

$$Z(y, s) = AT^k + c(e^{2s/\alpha}|x_0|^2 - 2e^{s/\alpha} x_0 \cdot y) = O(|y|)$$

uniformly for all $s \leqslant s_0$, $s_0 \in \mathbf{R}$ fixed. Analogously, (6.1) translated in x also gives us new nonstationary solutions to (7.1).

Having in mind a Liouville-type result for (7.1), first we recall that all remarks preceding the proof of Theorem (6.10), concerning the Liouville theorem for linear equations and lack of variational methods, apply now in the context of the nonlinear parabolic equation (7.1). Moreover, comparing with the statement of Theorem (6.10), it is well known that the one-sided Liouville theorems fail for solutions to parabolic equations. Hence the best uniqueness result we can expect is

(7.3) *If* Z *satisfies* (7.1) *for* $s \leqslant 0$, $y \in \mathbf{R}^N$ *and*

$$Z(y, s) = o(|y|) \quad \textit{for } |y| \to \infty,$$

then $Z = \text{const}$.

Unfortunately, the standard tools like pointwise Bernstein type estimates of ∇Z similar to those in the preceding section (and in [38]) — with a suitable modification of the cutoff function $\xi(s)\zeta(y)$ — even with some extra hypotheses on z ($\nabla z = O(|y|)$), give only weaker results, e.g.,

(7.4) Z *bounded for* $y \in \mathbf{R}^N$, $s \leqslant 0$, *implies* $Z = \text{const}$.

We do not repeat a cumbersome but a rather standard proof of (7.4), since we would like to present a different approach to this asymptotic problem of stability (as it might be interpreted) for (7.1) based on a fairly standard method of proving the uniqueness of weak solutions. The crucial thing in this method is a suitable choice of test functions and good estimates of the (approximate) solutions of the conjugate problem. The idea in the context of the porous medium equation goes back to Kalashnikov [25]. We give the proof based on some calculations presented in [6], Proposition 2.1, where the method of [25] was generalized to a multidimensional situation. We return to the original variables in (5.1) to apply directly some of ideas from [6] and [25], without unnecessary repetitions of the fragments of their proofs.

To formulate a reasonable hypothesis guaranteeing an asymptotic uniqueness result, let us observe first that rewriting (7.3) for v in (7.2) and the corresponding centered self-similar solution u in (5.5) we have

$$u^{m-1}(x, t) - v^{m-1}(x, t) = O\big((T-t)^{-k}\big) + O\big(|x|/(T-t)\big), \tag{7.5}$$

so (a weaker condition)

$$u^{m-1}(x, t) - v^{m-1}(x, t) = O\big((T-t)^{-k}\big) + O\big(|x|/(T-t)^{(1+k)/2}\big)$$

for $x \in \mathbf{R}^N$ and $t \to -\infty$.

Now we can state our

(7.6) THEOREM. *Let $N \geqslant 2$ and u, v be solutions to* (5.1) *on $\boldsymbol{R}^N \times (-\infty, T)$, both blowing up at time T, such that*

$$u^{m-1}(x, t) - v^{m-1}(x, t) = o\big((T-t)^{-k}\big) + o\big(|x|/(T-t)^{(1+k)/2}\big),$$
$$u(x, t) - v(x, t) = o\big(|x|^{2/(m-1)-2}(T-t)^{2/\alpha - 1/(m-1)}\big) + o\big(|x|^{2/(m-1)-1}(T-t)^{1/\alpha - 1/(m-1)}\big)$$

for $x \in \boldsymbol{R}^N$, $t \leqslant T-1$ (we may assume for simplicity that $T = 1$; the Landau symbols o are understood here for $t \to -\infty$ and $x/(T-t)^{1/\alpha} = y$ bounded). Then u is identically equal to v.

(7.7) Remarks. Observe that the hypotheses in (7.6) are reasonable compared with the pair of explicit solutions satisfying (7.5). The first terms in (7.6) correspond to adjusting a constant A in (5.5), the second ones control the behavior for large x. The relation with (7.3) is also obvious: the similarity variable y enters here explicitly.

The latter hypothesis (on $u - v$) is a consequence of the former one for $m \leqslant 2$ or for solutions regularly growing in the spatial variable. Namely, .

$$u - v = \frac{1}{m-1}(\tilde{u}^{m-1})^{1/(m-1)-1}(u^{m-1} - v^{m-1})$$

with $\tilde{u}$ lying between u and v, so for the corresponding

$$\tilde{z}(y, s) = (T-t)^k \tilde{u}(x, t)^{m-1} \geqslant \varepsilon |y|^2$$

for a positive ε. Hence $\tilde{u}^{2-m}$ is bounded from above by

$$C\big((T-t)^{-k}|x|^2/(T-t)^{2/\alpha}\big)^{1/(m-1)-1} = C\big(|x|^2/(T-t)\big)^{1/(m-1)-1}.$$

This gives the second hypothesis of our Theorem (7.6).

The case $N = 1$ requires an argument different from that given below.

Proof. Let us rewrite (5.1) on $\boldsymbol{R}^N \times [t, 0]$ for u and v in the weak form

$$\int_t^0 \int \big((u-v)\zeta_t + (u^m - v^m)\Delta\zeta\big) = \int (u-v)(0)\zeta(0) - \int (u-v)(t)\zeta(t)$$

for all $\zeta \in C_0^\infty(\boldsymbol{R}^N \times [t, 0])$. Defining

$$a(x, t) = \begin{cases} \big(u^m(x, t) - v^m(x, t)\big)/\big(u(x, t) - v(x, t)\big) & \text{if } u(x, t) \neq v(x, t), \\ m u^{m-1}(x, t) & \text{if } u(x, t) = v(x, t), \end{cases}$$

we arrive at

$$\int_t^0 \int (u-v)(\zeta_t + a\Delta\zeta) + \int (u-v)(t)\zeta(t) = \int (u-v)(0)\zeta(0). \tag{7.8}$$

We would like to prove that

$$\int (u-v)(0)\theta = 0 \quad \text{for all } \theta \in C_0^\infty(\boldsymbol{R}^N)$$

or that the left-hand side of (7.8) can be made arbitrarily small for a suitable $t \ll 0$.

Let the support of θ be contained in the ball $\{x\colon |x| \leqslant R_0\}$ and $R \geqslant 2R_0$. Assume that ψ is a solution to the initial-boundary value problem

$$\psi_t + a\Delta\psi = 0 \quad \text{on } \{x\colon |x| < R\} \times (t, 0),$$
$$\psi|_{|x|=R} = 0, \quad \psi(x, 0) = \theta(x). \tag{7.9}$$

The suitable hypothesis for the existence and C^∞ regularity of a solution to this problem is $a \in C^\infty$, $a \geqslant a_0 > 0$. This follows easily from the auxiliary non-degeneracy assumption on u, v: $u(x, t) > 0$, $v(x, t) > 0$ and their continuity. The general case should be treated using the similar smooth and strictly positive approximations of a as in [6]. Their arguments from (2.7), (2.19), (2.20) therein can be adopted without any essential change in our situation, so we omit this fragment of proof.

Now let us take, as a test function in (7.8), $\zeta = \varphi_\varepsilon \psi$, where ψ solves (7.9) and $\varphi_\varepsilon \in C_0^\infty(\mathbf{R}^N)$, $0 \leqslant \varphi_\varepsilon \leqslant 1$, $\varphi_\varepsilon = 1$ on $\{x\colon |x| \leqslant R - 2\varepsilon\}$, $\varphi_\varepsilon = 0$ on $\{x\colon |x| \geqslant R - \varepsilon\}$ and $|\nabla \varphi_\varepsilon| \leqslant C/\varepsilon$, $|\Delta \varphi_\varepsilon| \leqslant C/\varepsilon^2$. The integral identity (7.8) is now rewritten as

$$\int (u-v)(t)\varphi_\varepsilon \psi(t) + \int_t^0 \int (u-v)\varphi_\varepsilon(\psi_t + a\Delta\psi)$$
$$+ \int_t^0 \int (u-v)(2\nabla\varphi_\varepsilon \cdot \nabla\psi + \psi\Delta\varphi_\varepsilon) = \int (u-v)(0)\theta. \tag{7.10}$$

Denote the first integral by I. The second term obviously vanishes due to (7.9). The main difference between our task in proving the asymptotic uniqueness result and Proposition 2.1 in [6] is of course the appearance of I. We would like to show that $|I|$ is small for $t \ll 0$ and suitably chosen $R \gg R_0$. Hence it is necessary to estimate from above the solution to (7.9) for $t \ll 0$ and this estimate should be totally different compared to that in [6]: much better in t, possibly slightly worse in x. To execute this idea we construct a supersolution to problem (7.9) of the form

$$\Psi(x, t) = \lambda\eta(x), \quad \text{where } \lambda > 0,\ -\Delta\eta = c\eta,$$
$$\eta|_{|x|=R} = 0, \quad \eta|_{|x|<R} > 0, \tag{7.11}$$

η is the first (normalized) eigenfunction of the laplacian on the ball of radius R with homogeneous Dirichlet conditions. Taking a suitably large λ which depends only on θ (not on $R \geqslant 2R_0$), we obtain a supersolution to problem (7.9):

$$\Psi_t + a(x)\Delta\Psi \leqslant -ca(x)\Psi \leqslant 0, \quad \Psi|_{|x|=R} = 0,$$

$\Psi(x, 0) \geqslant \theta(x)$ and the maximum principle applies. Note that if our problem is nondegenerate: $a(x) \geqslant a_0 > 0$, then one can take Ψ even with time decay:

$$\Psi(x, t) = \lambda \exp(ca_0 t)\eta(x),$$

with c of order R^{-2N}.

Thus, using the hypotheses in (7.5), we get

$$(7.12)\qquad |I| \leqslant \int_{B_R} |(u-v)(t)|\Psi = o\big(R^{N+2/(m-1)-2}(T-t)^{2/\alpha-1/(m-1)}\big)$$
$$+o\big(R^{N+2/(m-1)-1}(T-t)^{1/\alpha-1/(m-1)}\big)$$
$$= o\big((R/(T-t)^{1/\alpha})^{N-2+2/(m-1)}\big)+o\big((R/(T-t)^{1/\alpha})^{N-1+2/(m-1)}\big)$$

for bounded $R/(T-t)^{1/\alpha}$ and $t \to -\infty$.

Return to the third integral in (7.10) and denote it by J. Proceeding similarly as in the proof of Proposition 2.1 in [6] we arrive at

$$|J| \leqslant CR^{N-1}\Big\{ \sup_{\substack{|x|=R \\ t_1 \leqslant \tau \leqslant 0}} \left|\frac{\partial \psi}{\partial \nu}(x,t)\right| \int_{t_1}^{0} |(u^m-v^m)(x,\tau)|\,d\tau$$
$$+ \sup_{\substack{|x|=R \\ t \leqslant \tau \leqslant t_1}} \left|\frac{\partial \psi}{\partial \nu}(x,\tau)\right| \int_{t}^{t_1} |(u^m-v^m)(x,\tau)|\,d\tau\Big\} =: J_1+J_2.$$

The integral J_1 is estimated exactly as in [6] by

$$(7.13)\qquad C(t_1,\beta)R^{N-1+2/(m-1)-2\beta},$$

where β is an arbitrary positive number (the suitable supersolution used for comparison is $Ce^{-C(\beta)t}(1+|x|^2)^{-\beta}$). The term J_2 can be estimated with the aid of our supersolution (7.11) using the similar procedure as in (2.11)–(2.14) in [6] by

$$CR^{N-1}R^{1-N}\int_{t}^{t_1} |(u^m-v^m)(x,\tau)|\,d\tau.$$

More precisely: $\psi \leqslant \Psi$ is dominated by a function g on $\{x\colon R_0 \leqslant |x| \leqslant R\}$ of the form $d|x|^{2-N}+\tilde{d}$ (or $d\log|x|+\tilde{d}$ if $N=2$), where d depends only on θ, λ, η (cf. (7.10), remember that $R \geqslant 2R_0$) and $\tilde{d} \leqslant 0$. Hence, by the normal derivative lemma,

$$\left|\frac{\partial \psi}{\partial \nu}\right| \leqslant \left|\frac{\partial g}{\partial \nu}\right| \leqslant CR^{1-N}.$$

From our assumption in (7.5) we have, analogously as in Remark (7.7), the asymptotic estimate

$$(u^m-v^m)(x,t) = m\tilde{u}^{m-1}(u-v)$$
$$= o\big(|x|^{2/(m-1)}(T-t)^{-k-1/(m-1)}\big)+o\big(|x|^{2/(m-1)+1}(T-t)^{-1-1/(m-1)}\big)$$

with the same meaning of o as in (7.6). Integrating this from t to t_1 gives finally

$$J_2 = o\big((R/(T-t_1)^{1/\alpha})^{2/(m-1)}\big)(T-t_1)^{2/(\alpha(m-1))+1-k-1/(m-1)}$$
$$+o\big((R/(T-t_1)^{1/\alpha})^{2/(m-1)+1}\big)(T-t_1)^{2/(\alpha(m-1))+1/\alpha-1/(m-1)}$$

valid for $t_1 \ll 0$ (the exponents in $(T-t_1)$ are both nonpositive).

Given arbitrary $\varepsilon > 0$ and $C > 0$, there exists t_1 satisfying $J_2 < \varepsilon/4$ for all $t \leqslant t_1$ and all R such that

$$|y| \leqslant R/(T-t)^{1/\alpha} \leqslant C.$$

Then, taking sufficiently large $R \geqslant 2R_0$, J_1 can be made less than $\varepsilon/4$ (see (7.13)). Recalling (7.12), the integral $|I|$ would be less than $\varepsilon/2$ if we take $t \ll 0$ of a sufficiently large absolute value. This completes the proof as $\int (u-v)(0)\theta = 0$ in (7.8) for all $\theta \in C_0^\infty(\boldsymbol{R}^N)$.

Recall that our proof applies to the rescaled problem (7.1) considered for $|y| \leqslant C$, C – arbitrary, and $s \leqslant 0$. We used the notation from [6] and [25] for the original problem (5.1) to avoid inessential changes of variables and to use as much as possible the calculations from these papers.

(7.14) Remarks. The difficulties in establishing the stability properties of the special solutions (5.5) are connected with the spatial behavior (different from that in [19] and [20]) of the solutions in similarity variables. Here the interesting solutions must be unbounded in x, while in [19] and [20] they are bounded and stabilize to constants.

A great variety of solutions of type (7.2) (and (6.1) translated in x) makes the dynamical systems approach in [19] hopeless (Giga and Kohn considered the analogue of (7.1) constructed for (2.1) in a weighted Sobolev space, with a special structure of linear higher order terms which played a crucial role and which is lacking in (7.2)). The study of attractors corresponding to these special solutions is essential in establishing the fine description of blowing up solutions. These attractors seem to have a fairly complicated structure.

REFERENCES

[1] D. G. Aronson, *Widder's inversion theorem and the initial distribution problem*, SIAM J. Math. Anal. 12 (1981), pp. 639–651.

[2] — and L. A. Caffarelli, *The initial trace of a solution of the porous medium equation*, Trans. Amer. Math. Soc. 280 (1983), pp. 351–366.

[3] P. Baras, *Non-unicité des solutions d'une équation d'évolution non-linéaire*, Ann. Fac. Sci. Toulouse Math. (5) 5 (1983), pp. 287–302.

[4] — and L. Cohen, *Complete blow-up after $T_{\max}$ for the solution of a semilinear heat equation*, J. Funct. Anal. 71 (1987), pp. 142–174.

[5] P. Bénilan, *A strong regularity L^p for solution of the porous media equation*, pp. 39–58 in: C. Bardos, A. Damlamian, J. I. Diaz and J. Hernandez (eds.), *Contributions to Nonlinear Partial Differential Equations*; *Proceedings of an International Meeting Held in Madrid on December 14–17, 1981*; Research Notes in Math. 89, Pitman, Boston 1983.

[6] — M. G. Crandall and M. Pierre, *Solutions of the porous medium equation in $\boldsymbol{R}^N$ under optimal conditions on initial values*, Indiana Univ. Math. J. 33 (1984), pp. 51–87.

[7] M. Bertsch and L. A. Peletier, *Porous media type equations — an overview*, preprint, Leiden 1983/7.

[8] Y. G. Chen, *Asymptotic behaviours of blowing-up solutions for finite difference analogue of* $u_t = u_{xx} + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 33 (1986), pp. 541–574.

[9] B. E. J. Dahlberg, E. B. Fabes and C. E. Kenig, *A Fatou theorem for solutions of the porous medium equation*, Proc. Amer. Math. Soc. 91 (1984), pp. 205–212.

[10] B. E. J. Dahlberg and C. E. Kenig, *Non-negative solutions of the porous medium equation*, Comm. Partial Differential Equations 9 (1984), pp. 409–437.

[11] — *Nonnegative solutions of the generalized porous medium equations*, Rev. Mat. Iberoamericana 2 (1986), pp. 267–305.

[12] — *Weak solutions of the porous medium equation*, preprint, Göteborg 1987/34.

[13] — *Weak solutions of the porous medium equation in a cylinder*, preprint, Göteborg 1987/33.

[14] — *Nonnegative solutions of the initial-Dirichlet problem for the generalized porous medium equations in cylinders*, J. Amer. Math. Soc. 1 (1988), pp. 401–412.

[15] A. Friedman and B. McLeod, *Blow-up of positive solutions of semilinear heat equation*, Indiana Univ. Math. J. 34 (1985), pp. 425–447.

[16] V. A. Galaktionov, V. A. Dorodnitsyn, G. G. Elenin, S. P. Kurdyumov and A. A. Samarskiĭ, *Quasilinear heat equation with source: strained regime, localization, symmetry, exact solutions, asymptotics, structures* (in Russian), pp. 95–205 in: R. V. Gamkrelidze (ed.), *Itogi nauki i tekhniki, Sovremennye problemy matematiki — noveĭshie dostizheniya* 28, VINITI, Moscow 1986.

[17] V. A. Galaktionov, S. P. Kurdyumov, S. A. Posashkov and A. A. Samarskiĭ, *Quasilinear parabolic equation with a complicated spectrum of unbounded self-similar solutions* (in Russian), pp. 142–182 in: A. A. Samarskiĭ, S. P. Kurdyumov and V. A. Galaktionov (eds.), *Matematicheskoe modelirovanie, Protsessy v nelineĭnykh sredakh*, Nauka, Moscow 1986.

[18] V. A. Galaktionov, S. P. Kurdyumov and A. A. Samarskiĭ, *Asymptotic "eigenfunctions" of the Cauchy problem for a nonlinear parabolic equation* (in Russian), Mat. Sb. (N. S.) 126 (1985), pp. 435–472. See also letter to the editors: *Asymptotic "eigenfunctions"* ... (in Russian), ibidem 131 (1986), p. 413.

[19] Y. Giga and R. V. Kohn, *Asymptotically self-similar blow-up of semilinear heat equations*, Comm. Pure Appl. Math. 38 (1985), pp. 297–319.

[20] — *Characterizing blowup using similarity variables*, Indiana Univ. Math. J. 36 (1987), pp. 1–40.

[21] — *Removability of blowup points for semilinear heat equation*, Proceedings Equadiff 1987.

[22] A. Haraux and F. B. Weissler, *Non-uniqueness for a semilinear initial value problem*, Indiana Univ. Math. J. 31 (1982), pp. 167–189.

[23] M. A. Herrero and M. Pierre, *The Cauchy problem for $u_t = \Delta u^m$ when $0 < m < 1$*, Trans. Amer. Math. Soc. 291 (1985), pp. 145–158.

[24] A. V. Ivanov, *Second order quasilinear degenerate and nonuniformly elliptic and parabolic equations* (in Russian), Trudy Mat. Inst. Steklov. 160 (1982), pp. 1–285.

[25] A. S. Kalashnikov, *The Cauchy problem in the class of increasing functions for equations of non-stationary filtration type* (in Russian), Vestnik Moskov. Univ. Ser. I Mat. Mekh. 18, No. 6 (1963), pp. 17–27.

[26] L. Karp, *Asymptotic behavior of solutions of elliptic equations, I: Liouville type theorems for linear and nonlinear equations on $\mathbf{R}^n$, II: Analogues of Liouville's theorem for solutions of inequalities on $\mathbf{R}^n$, $n \geqslant 3$*, J. Analyse Math. 39 (1981), pp. 75–102 and 103–115.

[27] Yu. T. Kuz'menko, *On the Liouville theorem for degenerate elliptic and parabolic equations* (in Russian), Mat. Zametki 29 (1981), pp. 397–408. English translation: Math. Notes 29 (1981), pp. 203–209.

[28] A. A. Lacey, *The form of blow-up for nonlinear parabolic equations*, Proc. Roy. Soc. Edinburgh Sect. A 98 (1984), pp. 183–202.

[29] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, Trans. Math. Monographs 23 (1968).

[30] H. A. Levine and P. E. Sacks, *Some existence and nonexistence theorems for solutions of degenerate parabolic equations*, J. Differential Equations 52 (1984), pp. 135–161.

[31] C. E. Mueller and F. B. Weissler, *Single point blow-up for a general semilinear heat equation*, Indiana Univ. Math. J. 34 (1985), pp. 881–913.

[32] W.-M. Ni, P. E. Sacks and J. Tavantzis, *On the asymptotic behavior of solutions of certain quasilinear parabolic equations*, J. Differential Equations 54 (1984), pp. 97–120.
[33] L. A. Peletier, *The porous media equation*, pp. 229–241 in: H. Amann, N. Bazley and K. Kirchgässner (eds.), *Applications of Nonlinear Analysis in the Physical Sciences*, Pitman, Boston 1981.
[34] — and J. Serrin, *Gradient bounds and Liouville theorems for quasilinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 5 (1978), pp. 65–104.
[35] M. Pierre, *Nonlinear fast diffusion with measures as data*, pp. 179–188 in: L. Boccardo and A. Tesei (eds.), *Nonlinear Parabolic Equations — Qualitative Properties of Solutions*, Rome 1985, Pitman Research Notes in Math. 149, Longman, Harlow 1987.
[36] P. E. Sacks, *Global behavior for a class of nonlinear evolution equations*, SIAM J. Math. Anal. 16 (1985), pp. 233–250.
[37] A. A. Samarskiĭ, V. A. Galaktionov, S. P. Kurdyumov and A. P. Mikhaĭlov, *Strained Regimes in Problems for Quasilinear Parabolic Equations* (in Russian), Nauka, Moscow 1987.
[38] I. N. Tavkhelidze, *Liouville theorems for elliptic and parabolic equations of second order* (in Russian), Vestnik Moskov. Univ. Ser. I Mat. Mekh. 31, No. 4 (1976), pp. 28–35. English translation: Moscow Univ. Math. Bull. 31, No. 3–4 (1976), pp. 70–76.
[39] W. Troy, *Bounded solutions of* $\Delta w - \frac{y}{2}\cdot\nabla w + |w|^{p-1}w - \frac{w}{p-1} = 0$, SIAM J. Math. Anal. 18 (1987), pp. 332–336.
[40] J. L. Vázquez, *The interfaces of one-dimensional flows in porous media*, Trans. Amer. Math. Soc. 285 (1984), pp. 717–737.
[41] F. B. Weissler, *An L^∞ blow-up estimate for a nonlinear heat equation*, Comm. Pure Appl. Math. 38 (1985), pp. 291–295.

INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW, POLAND

Reçu par la Rédaction le 15.9.1988