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SOME APPLICATIONS OF GEOMETRY OF BANACH SPACES TO HARMONIC ANALYSIS

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1. Introduction. In this paper we investigate some applications of geometry of Banach spaces to harmonic analysis. We will focus in particular on the stability of absolutely continuous linear forms on subspaces of $L^\infty(G)$ with respect to w^*-limits of weak Cauchy sequences.

Our work is a natural continuation of the three consecutive papers ([PSW], [K], [He]), of the Mooney–Havin theorem on L^1/H^1 ([M], [Hav]) and its extensions ([G1], [G2]). The techniques from harmonic analysis we use are classical; the use of Baire category methods in this context might be less classical. A precursor for our approach is Bishop's generalized Rudin–Carleson theorem ([B]; see [P], p. 19), which is dual to the result of Heard [He] which we extend.

Let us now describe the content of this paper. In Section II we recall some previous abstract results of the first author [G3], namely Theorems II.1 and Corollary II.2 for which we give easy and self-contained proofs. Theorem II.1 is a result on sequential "very weak" completeness for subspaces L of dual spaces X^* which are fixed spaces of isometric bijections. From Theorem II.1 we derive Theorem II.3, which extends a result of [He]. Let us mention that the isometries we use in the applications are ℓ^1-symmetries, for which some proofs can be simplified. However, the more general statements of Section II still have simple proofs and could be used in more general — e.g. noncommutative — situations.

In Section III we apply Theorem II.3 to multipliers; our main result is Theorem III.5: if G is a compact abelian group with $\hat{G} = \Gamma$ and if $\Lambda \subseteq \Gamma$ is such that

$$(*) \qquad M_{\Lambda^c}(G) = L^1_{\Lambda^c}(G) \oplus (M_s)_{\Lambda^c}(G)$$

then any multiplier $L^\infty_\Lambda(G) \to C_\Lambda(G)$ is the restriction to L^∞_Λ of the con-

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volution with an L^1 function. The condition $(*)$ is shown to hold in many situations. Corollary III.7 extends a result of Dressler and Pigno [DP].

In Section IV we compare different approaches to proving the Mooney–Havin theorem, stating that L^1/H^1 is weakly sequentially complete (and the abstract extensions of this result). One approach uses Corollary II.2.

In Section V we gather some remarks and problems.

Notation. Throughout this paper G denotes an abelian metrizable compact group and $\Gamma = \hat{G}$ its discrete dual group. A ***trigonometric polynomial*** is a finite linear combination of characters $\gamma \in \Gamma$. $L^1(G)$ is the space of integrable functions with respect to the Haar measure of G. An ***approximate identity*** $\{\varphi_n\}$ in $L^1(G)$ is a sequence of positive functions of norm 1 in $L^1(G)$ such that for every $g \in L^1(G)$

$$\lim_n \|\varphi_n * g - g\|_1 = 0.$$

An example is given by $\varphi_n = m(V_n)^{-1} 1_{V_n}$ where $\{V_n\}$ is a basis of neighborhoods of 0 in G.

If Λ is a subset of Γ, $L^1_\Lambda(G)$ denotes the space

$$L^1_\Lambda(G) = \{f \in L^1(G) \mid \hat{f}(n) = 0 \quad \forall n \in \Gamma \backslash \Lambda\}.$$

We define similarly $L^\infty_\Lambda(G)$, $C_\Lambda(G)$ and $(M_s)_\Lambda(G)$, the space of measures singular with respect to Haar measure whose Fourier transform is supported by Λ; the letter G will often be omitted. We write $f_x = f(x + \cdot)$. The duality between $L^1(G)$ and $L^\infty(G)$ is given by

$$\langle f, g \rangle = \int f(t) g(-t)\, dm(t).$$

We set $\Gamma \backslash \Lambda = \Lambda^c$. We recall that Λ is a *Riesz set* if $M_\Lambda(G) = L^1_\Lambda(G)$, and a *Rosenthal set* if $L^\infty_\Lambda(G) = C_\Lambda(G)$. The weak-star topology on a dual space X^* is denoted by w^*. If $A \subseteq X$, the orthogonal of A in X^* is $A^\perp$; if $H \subseteq X^*$, the (pre-)orthogonal of H in X is $H_\perp$. The closed unit ball of a Banach space Y is denoted by Y_1.

The other notations are classical or will be defined before use.

II. Two basic results.

Our first statement follows from ([G3], Prop. 8). For the sake of completeness we will give an easy and self-contained proof.

THEOREM II.1. *Let X be a Banach space, and let $S : X^* \to X^*$ be an isometric bijection. Let $\{y_n\}$ be a weak Cauchy sequence of fixed points under S. Then $y = w^*\text{-}\lim y_n$ is also a fixed point.*

Proof. By assumption, we have $S(y_n) = y_n$ for every n and we have to show that $S(y) = y$. We let t denote the limit of $\{y_n\}$ in (X^{***}, w^*); this limit exists since $\{y_n\}$ is weak Cauchy; clearly, the restriction of t to X is y;

hence $\mathrm{Ker}\,(y-t)\cap X_1^{**}$ is w^*-dense in X_1^{**} since it contains X_1. It is even a w^*-dense G_δ of X_1^{**} since we have

$$\mathrm{Ker}\,(y-t) = \bigcap_{n,k\geq 1} \{z \in X^{**} \mid \exists p \geq n : |z(y-y_p)| < k^{-1}\}.$$

Since $S(y_n) = y_n$ for every n, $S^{**}(t) = t$; therefore

$$S(y) - t = S(y) - S^{**}(t) = S^{**}(y-t),$$

hence

$$\mathrm{Ker}(S(y)-t) = (S^*)^{-1}(\mathrm{Ker}\,(y-t))$$

and since $(S^*)^{-1} = (S^{-1})^*$ is w^*-continuous, it follows that $\mathrm{Ker}\,(S(y)-t)\cap X_1^{**}$ is also a w^*-dense G_δ of X_1^{**}. Now by Baire's theorem,

$$\Omega = \mathrm{Ker}\,(S(y)-t)\cap \mathrm{Ker}\,(y-t)$$

is also w^*-dense in X_1^{**}; as Ω is contained in $\mathrm{Ker}\,(S(y)-y)$, $\mathrm{Ker}\,(S(y)-y)$ is w^*-dense in X_1^{**}, hence $S(y)-y=0$ since $S(y)-y$ is continuous on (X_1^{**}, w^*). This concludes the proof. ∎

Theorem II.1 implies the following corollary ([G3], Prop. 10):

COROLLARY II.2. *Let Y be a Banach space. If there exists an isometric bijection $S : Y^{**} \to Y^{**}$ such that* $\mathrm{Ker}\,(S-I) = Y$ *then Y is weakly sequentially complete.*

Proof. We apply Theorem II.1 to $X = Y^*$: let $\{x_n\}$ be a weak Cauchy sequence in Y and $z = \lim x_n$ in (Y^{**}, w^*). $\{x_n\}$ is also a weak Cauchy sequence in Y^{**}, hence, by II.1, z is a fixed point for S, i.e. $z \in Y$. ∎

In the applications S is a symmetry whose set of fixed points is Y (see [G1]).

Our next statement is an extension of a result of [He]. An example where this result applies is $X = C(\mathbf{T})$, $A = A(D)$ the disc algebra and $\pi: M(\mathbf{T}) \to L^1(\mathbf{T})$ the Radon-Nikodym projection. Note that $\pi^*(A^{\perp\perp})$ is isometric to $H^\infty(D)$ and a stronger result holds since $L^1(\mathbf{T})/H_0^1(D)$ is weakly sequentially complete (see Section IV below). However, Theorem II.3 is optimal—take for instance $\pi = I$.

Note that $S = 2\pi - I$ is a symmetry (in particular an isometric bijection) which is naturally associated to π.

THEOREM II.3. *Let X be a Banach space, and let A be a closed subspace of X. Let $\pi: X^* \to X^*$ be a linear projection such that $\|2\pi - I\| = 1$, and such that $A^\perp = \pi(A^\perp)\oplus(I-\pi)(A^\perp)$. Let $H = \pi^*(A^{\perp\perp})$, and let $y_n \in \pi(X^*)$ be such that*

$$\lim_{n\to\infty} z(y_n)$$

exists for every $z \in H$. Then $(I - \pi)(y) \in A^{\perp}$ for every w^*-cluster point y of $\{y_n\}$. Moreover, there exists $y \in \pi(X^*)$ such that

$$\lim_{n\to\infty} y_n(x) = y(x)$$

for every $x \in A$.

Note that for $A = X$, Theorem II.3 is a particular case of Theorem II.1. Theorem II.3 will be a consequence of Theorem II.1 applied to $A^* = X^*/A^{\perp}$.

Proof. First we notice that

$$(1) \qquad A^* = X^*/A^{\perp} = \pi(X^*)/\pi(A^{\perp}) \oplus (I-\pi)(X^*)/(I-\pi)(A^{\perp}).$$

Hence π induces a projection $\dot{\pi}$ from A^* onto $\pi(X^*)/\pi(A^{\perp})$; we let $\dot{S} = 2\dot{\pi} - I$.

We check that $\|\dot{S}\| = 1$; indeed, if $\dot{y} = \dot{y}_1 + \dot{y}_2$ is the decomposition of $\dot{y} \in A^*$ given by (1), we have $\dot{S}(\dot{y}) = \dot{y}_1 - \dot{y}_2$, and

$$\begin{aligned}\|\dot{y}_1 - \dot{y}_2\| &= \inf\{\|y_1 - y_2 + y'\| \mid y' \in A^{\perp}\}\\ &= \inf\{\|y_1 - y_2 + y_1' + y_2'\| \mid y_1' \in \pi(A^{\perp}),\ y_2' \in (I-\pi)(A^{\perp})\}.\end{aligned}$$

Since $\|2\pi - I\| = 1$, we have

$$\|y_1 - y_2 + y_1' + y_2'\| = \|y_1 + y_2 + y_1' - y_2'\|,$$

and the same computation shows

$$\|\dot{y}_1 - \dot{y}_2\| = \|\dot{y}_1 + \dot{y}_2\|.$$

We now describe the predual of H; since

$$H = \pi^*(A^{\perp\perp}) = \overline{\pi^*(A)}^{\,w^*},$$

we have for $y \in X^*$,

$$\begin{aligned} y \in H_{\perp} &\Leftrightarrow z(y) = 0 \quad \forall z \in \pi^*(A)\\ &\Leftrightarrow \langle \pi(y), x\rangle = 0 \quad \forall x \in A\\ &\Leftrightarrow \pi(y) \in A^{\perp}\end{aligned}$$

and thus

$$\begin{aligned} H_{\perp} &= \pi^{-1}(A^{\perp}) = \pi(A^{\perp}) \oplus (I-\pi)(X^*),\\ H &= (X^*/H_{\perp})^* \simeq (\pi(X^*)/\pi(A^{\perp}))^*.\end{aligned}$$

Therefore the assumption we made on $\{y_n\}$ means that the corresponding sequence $\{\dot{y}_n\}$ in A^* is a weak Cauchy sequence.

We may now apply Theorem II.1 to the isometry $\dot{S}$ to get $\dot{S}(\dot{y}_0) = \dot{y}_0$, where $\dot{y}_0 = w^*\text{-}\lim \dot{y}_n$ in (A^*, w^*). If y is any w^*-cluster point of $\{y_n\}$ in X^*, we have $\dot{y} = \dot{y}_0$, and thus $S(\dot{y}) = \dot{y} = (2\dot{\pi} - I)(\dot{y})$, hence $(\dot{\pi} - I)(\dot{y}) = 0$, and this means exactly that $(I - \pi)(y) \in A^{\perp}$.

To conclude the proof, we note that the sequence $\{\dot{y}_n\}$ is weak Cauchy in A^* and therefore bounded in A^*; thus we may find a bounded sequence $\{y'_n\}$ in X^* such that $\dot{y}_n = \dot{y}'_n$. Now we may pick a cluster point y' of $\{y'_n\}$. What precedes applies to $\{y'_n\}$, and thus we have $\dot{\pi}(\dot{y}') = \dot{y}'$; if we let $y = \pi(y')$, we have

$$y(x) = \lim_{n\to\infty} y_n(x)$$

for every $x \in A$, and of course $y \in \pi(X^*)$. ∎

Theorem II.3 applies as soon as we have nontrivial isometric symmetries in dual spaces, and this happens in a variety of situations (C^*-algebras, Banach lattices, M-ideals, ...). In the next section it will be applied to harmonic analysis, and the projections π will stem from the Radon–Nikodym decompositions of measures.

III. Applications to multipliers. Throughout this section, G will denote a metrizable compact abelian group, and Λ a subset of the dual group Γ. We recall

DEFINITION III.1. A bounded operator $T : L^\infty_\Lambda(G) \to L^\infty_\Lambda(G)$ is called a *multiplier* if there exists a sequence $\{a_\gamma \mid \gamma \in \Lambda\}$ of complex numbers such that

$$\widehat{T(h)}(\gamma) = a_\gamma \hat{h}(\gamma)$$

for every $h \in L^\infty_\Lambda$ and every $\gamma \in \Lambda$.

In particular, $T(h_x) = (T(h))_x$ for every $x \in G$; i.e. T commutes with translations.

The following result is well known (see for instance [Har]). We outline a proof for completeness.

LEMMA III.2. *Let $T : L^\infty_\Lambda(G) \to L^\infty_\Lambda(G)$ be a bounded linear operator. The following assertions are equivalent:*

(i) *T is a multiplier;*

(ii) *there exists a measure $\mu \in M(G)$ such that $T(h) = \mu * h$ for every $h \in L^\infty_\Lambda(G)$.*

If these conditions are satisfied then

$$\|T : L^\infty_\Lambda \to L^\infty_\Lambda\| = \|\dot{\mu}\|_{M(G)/M_{\Lambda^c}}.$$

Moreover, T is compact iff there exists $g \in L^1(G)$ such that $T(h) = g * h$ for every $h \in L^\infty_\Lambda(G)$.

Proof. (ii) ⇒ (i) follows immediately from the formula

$$\widehat{\mu * h}(\gamma) = \hat{\mu}(\gamma)\hat{h}(\gamma).$$

(i) $\Rightarrow$ (ii). We note first that every $\gamma \in \Lambda$ is an eigenvector of T and thus the space of trigonometric polynomials supported by Λ is stable under T; it follows that T is a bounded operator $C_\Lambda \to C_\Lambda$.

We observe now that if $\gamma \in \Lambda$, $\hat{\mu}(\gamma)$ depends only on the coset $\dot{\mu}$ of μ in $M(G)/M_{\Lambda^c}$; therefore the notation $\hat{\dot{\mu}}(\gamma)$, or $\dot{\mu} * f$ for $f \in L^\infty_\Lambda$, makes sense. The operator

$$T^* : M(G)/M_{\Lambda^c} \to M(G)/M_{\Lambda^c}$$

satisfies

$$\widehat{T^*(\dot{\mu})}(\gamma) = a_\gamma \hat{\dot{\mu}}(\gamma)$$

for every $\gamma \in \Lambda$ and every $\dot{\mu}$; in particular, by taking $\dot{\mu} = \dot{\delta}_0$ we find $\dot{\nu} = T^*(\dot{\delta}_0)$ such that for every $\gamma \in \Lambda$, $\hat{\dot{\nu}}(\gamma) = a_\gamma$. It follows that for every $\gamma \in \Lambda$ and every $h \in L^\infty_\Lambda$,

$$\widehat{T(h)}(\gamma) = a_\gamma \hat{h}(\gamma) = \widehat{\dot{\nu} * h}(\gamma),$$

which proves that $T(h) = \dot{\nu} * h$; that is, every measure $\mu \in \dot{\nu}$ represents T in the sense of (ii).

Finally, we note that

$$\|\dot{\nu}\|_{M(G)/M_{\Lambda^c}} = \|T : C_\Lambda \to C_\Lambda\| \leq \|T : L^\infty_\Lambda \to L^\infty_\Lambda\|,$$

and on the other hand

$$\|\dot{\nu} * h\|_{L^\infty_\Lambda} \leq \|\dot{\nu}\|_{M(G)/M_{\Lambda^c}} \|h\|_{L^\infty_\Lambda}$$

for every $h \in L^\infty_\Lambda$; this completes the proof of (i) $\Rightarrow$ (ii).

Every trigonometric polynomial defines a finite-dimensional ranged multiplier on L^∞_Λ, hence every $\varphi \in L^1(G)$ defines a compact multiplier on L^∞_Λ. Conversely, let $\dot{\mu} \in M(G)/M_{\Lambda^c}$ define a compact multiplier on L^∞_Λ. It also defines a compact multiplier on $L^1(G)/L^1_{\Lambda^c}$; let $\{\varphi_n\}$ be an approximate identity in $L^1(G)$; then $\widehat{\mu * \varphi_n}(\gamma) \to \hat{\mu}(\gamma)$ for every $\gamma \in \Lambda$ and $\{(\mu \mathbin{\dot{*}} \varphi_n)\}$ is norm-converging in $L^1(G)/L^1_{\Lambda^c}$, hence $\dot{\mu} \in L^1(G)/L^1_{\Lambda^c}$. ∎

We now consider multipliers $L^\infty_\Lambda(G) \to C_\Lambda(G)$.

LEMMA III.3. *Let $T(h) = h * \mu$ be a multiplier on $L^\infty_\Lambda(G)$ and $\dot{\mu} \in M(G)/M_{\Lambda^c}$ be the coset of μ. The following assertions are equivalent:*

(i) $T : L^\infty_\Lambda \to C_\Lambda$;

(ii) *for every approximate identity $\{\varphi_n\}$ in $L^1(G)$, $\{(\mu \mathbin{\dot{*}} \varphi_n)\}$ is a weak Cauchy sequence in $L^1(G)/L^1_{\Lambda^c}$.*

In what follows we will use only the easier implication (i) $\Rightarrow$ (ii).

Proof. (i) $\Rightarrow$ (ii). By assumption $\dot{\mu} * h \in C_\Lambda$ for every $h \in L^\infty_\Lambda$. It follows that for every approximate identity $\{\varphi_n\}$ one has

$$\lim_{n\to\infty} \|\dot{\mu} * h * \dot{\varphi}_n - \dot{\mu} * h\|_\infty = 0.$$

In particular, for every $h \in L^\infty_\Lambda$,

$$\dot\mu * \dot\varphi_n * h(0) = \langle \dot\mu * \dot\varphi_n, h \rangle \underset{n\to\infty}{\longrightarrow} \dot\mu * h(0)$$

and thus $\{\dot\mu * \dot\varphi_n\}$ is a weak Cauchy sequence in the predual $L^1(G)/L^1_{\Lambda^c}$ of $L^\infty_\Lambda(G)$.

(ii) $\Rightarrow$ (i). Let $\dot\mu \in M(G)/M_{\Lambda^c}$ and let $f \in L^\infty_\Lambda(G)$ be such that $\dot\mu * f$ is not continuous; then for any approximate identity $\{\varphi_n\}$ in $L^1(G)$, the sequence $\{\dot\mu * f * \dot\varphi_n\}$ does not converge uniformly. Hence we may find:

a) $\epsilon > 0$;

b) two sequences $\{n_k\}$, $\{n'_k\}$ of integers such that $n_k < n'_k < n_{k+1}$;

c) a sequence $\{t_k\}$ in G such that

$$\|\dot\mu * f * (\dot\varphi_{n_k} - \dot\varphi_{n'_k})\|_\infty = |\langle \dot\mu * (\dot\varphi_{n_k} - \dot\varphi_{n'_k}) * \delta_{t_k}, f\rangle| \geq \epsilon.$$

By taking a subsequence and replacing f by a translate we may and do assume that $\{t_k\}$ converges to 0. If we let now

$$\psi_{2k} = \varphi_{n_k} * \delta_{t_k}, \qquad \psi_{2k+1} = \varphi_{n'_k} * \delta_{t_k},$$

the sequence $\{\psi_k\}$ is an approximate identity in $L^1(G)$ but $\{\dot\mu * \dot\psi_k\}$ is not a weak Cauchy sequence in $L^1(G)/L^1_{\Lambda^c}$. ∎

Though we will not use it we mention the following result:

LEMMA III.4. *Let $\Lambda \subset \Gamma$. Let $T : L^\infty_\Lambda \to L^\infty_\Lambda$ be a bounded linear operator which commutes with translations. The following assertions are equivalent:*

(i) $T : L^\infty_\Lambda \to C_\Lambda$;

(ii) *T has a separable range.*

Proof. (i) $\Rightarrow$ (ii) is obvious. Let $f \in L^\infty_\Lambda$ and $h = T(f)$; by (ii), $\{h_x\}_{x\in G} = \{T(f_x)\}_{x\in G}$ is norm-separable in $L^\infty(G)$. Hence h is continuous by ([E], Corollary 1). ∎

It follows from Lemma III.3 that if $L^1/L^1_{\Lambda^c}$ is weakly sequentially complete then every multiplier $L^\infty_\Lambda \to C_\Lambda$ is a convolution with $g \in L^1(G)$. The finer property of II.3 will be used to show

THEOREM III.5. *Let G be a metrizable compact abelian group, and let Λ be a subset of Γ. If*

$$(*) \qquad M_{\Lambda^c}(G) = L^1_{\Lambda^c}(G) \oplus (M_s)_{\Lambda^c}(G)$$

then every multiplier T from $L^\infty_\Lambda(G)$ to $C_\Lambda(G)$ is the restriction to $L^\infty_\Lambda(G)$ of the convolution with a function in $L^1(G)$.

Proof. By assumption, we have

$$C_\Lambda(G)^\perp = M_{\Lambda^c}(G) = L^1_{\Lambda^c}(G) \oplus (M_s)_{\Lambda^c}(G).$$

If π denotes the Radon–Nikodym projection from $M(G)$ onto $L^1(G)$, then $\|2\pi - I\| = 1$ since the norm is additive on orthogonal measures; moreover $L^\infty_\Lambda(G)$ is isometric to $\pi^*(C_\Lambda(G)^{\perp\perp})$.

Let now $\mu \in M(G)$ be a measure, provided by Lemma III.2, such that $T(h) = h * \mu$. We let now $y_n = \mu * \varphi_n$, where $\{\varphi_n\}$ is an approximate identity in $L^1(G)$. By Lemma III.3, the assumptions of Theorem II.3 are satisfied, with $A = C_\Lambda(G)$ and $X = C(G)$; thus there is $g \in L^1(G)$ such that

$$\lim_{n\to\infty} \langle \mu * \varphi_n, f\rangle = \langle g, f\rangle$$

for every $f \in C_\Lambda$; applying this to $f = \gamma$ ($\gamma \in \Lambda$) leads to

$$\lim_{n\to\infty} \widehat{\mu * \varphi_n}(\gamma) = \hat{\mu}(\gamma) = \hat{g}(\gamma)$$

and this concludes the proof. ∎

Remark III.6. Note that (*) implies

$$M(G)/M_{\Lambda^c} = L^1(G)/L^1_{\Lambda^c} \oplus M_s(G)/(M_s)_{\Lambda^c}$$

and the norm in this decomposition is additive by the proof of Theorem II.3. Note also that for any Λ, $M(G)/M_{\Lambda^c}$ is the space of multipliers of L^∞_Λ and $L^1(G)/L^1_{\Lambda^c}$ is the space of compact multipliers of L^∞_Λ (Lemma III.2).

We now give examples where Theorem III.5 applies.

EXAMPLES III.7.

1. If Λ^c is a Riesz set, that is, if $M_{\Lambda^c} = L^1_{\Lambda^c}$, then (*) trivially holds. In this case, Theorem III.5 follows from the result of Heard ([He]; see [DP]).

2. Recall that the *Bohr topology* is the topology induced on $\Gamma \subseteq C(G)$ by pointwise convergence on G. The property (*) of III.5 is "local" in the following sense: if $\Lambda \subseteq \Gamma$ is such that every $\gamma \in \Lambda$ has a neighborhood V_γ for the Bohr topology such that $V_\gamma \cap \Lambda^c$ satisfies (*), then Λ^c satisfies (*). The proof goes as in [Me]: we have to show that for every $\mu \in M_{\Lambda^c}$, $\mu_a \in M_{\Lambda^c}$. Pick $\gamma \in \Lambda$; since the algebra $(\ell^1(G), *)$ is regular, there is a discrete measure ν on G such that $\hat{\nu}(\gamma) = 1$ and $\hat{\nu}(\gamma') = 0$ for every $\gamma' \notin V_\gamma$. We have

$$\mu * \nu \in M_{V_\gamma \cap \Lambda^c}.$$

By assumption it follows that $(\mu * \nu)_a \in M_{V_\gamma \cap \Lambda^c}$. But clearly $(\mu * \nu)_a = \mu_a * \nu$, and thus $\hat{\mu}_a(\gamma) = \widehat{\mu_a * \nu}(\gamma) = 0$; this shows $\mu_a \in M_{\Lambda^c}$.

The above proof shows in particular that any subset Λ of Γ which is open for the Bohr topology satisfies the conclusion of III.5: take $V_\gamma = \Lambda$ for every $\gamma \in \Lambda$; then $V_\gamma \cap \Lambda^c = \emptyset$ satisfies (*). In this special case, however, more can be said (see 3 below). We refer to ([Me], [LP2], [G2]) for arithmetical examples involving the Bohr topology.

3. Every set Λ which is closed for the Bohr topology is "nicely placed" in the sense of [G1], that is, the unit ball of L^1_Λ is closed for the topology of convergence in measure. If Λ^c is "nicely placed", then $(*)$ holds: indeed, if $\{\varphi_n\}$ is an approximate identity in $L^1(G)$, then $\lim \|\mu * \varphi_n - \mu\|_1 = 0$ if μ is absolutely continuous, and $\lim \|\mu * \varphi_n\|_p = 0$ if $0 < p < 1$ and μ is singular ([BO]); if now $\mu \in M_{\Lambda^c}$, the bounded sequence $\{\mu * \varphi_n\}$ in $L^1_{\Lambda^c}$ converges in measure to μ_a and thus $\mu_a \in L^1_{\Lambda^c}$. However, if Λ^c is nicely placed, then $L^1/L^1_{\Lambda^c}$ is weakly sequentially complete ([G2], Lemma 1.8) and then Lemma III.3 implies immediately the conclusion of Theorem III.5. We refer to [G2] for examples of nicely placed sets; let us mention that there are Riesz sets, and even Rosenthal sets, which are not nicely placed ([G2], Example 3.8).

4. We conclude this list by examples where the conclusion of III.5 does not hold. Recall that Λ is called a Rosenthal set if $L^\infty_\Lambda(G) = C_\Lambda(G)$ (see [Ro], [DP], [LP1], [G4]); if Λ is Rosenthal then clearly any $\mu \in M(G)$ will define by convolution a multiplier from L^∞_Λ to C_Λ; in particular, if Λ is Rosenthal and infinite, the identity operator is a multiplier which does not stem from a function $f \in L^1(G)$ since $\hat{f} \in C_0(G)$ for $f \in L^1$. It is possible to elaborate on this example: for instance, if Λ_0 is a Rosenthal infinite subset of $2\mathbb{Z}$ and $\Lambda = \Lambda_0 \cup \{2\mathbb{Z}+1\}$, then the operator $T(f) = f(x) + f(x+\pi)$ is a multiplier from L^∞_Λ to $L^\infty_{\Lambda_0} = C_{\Lambda_0} \subseteq C_\Lambda$ which is not given by $g \in L^1$.

We now give an application. The following corollary improves a result of Dressler and Pigno ([DP]) and was proved by the authors in ([G4], Appendix).

COROLLARY III.8. *Let Λ_0 be a Rosenthal subset of Γ, and let Λ be such that*

$$M_\Lambda = L^1_\Lambda \oplus (M_s)_\Lambda.$$

Then

$$M_{\Lambda_0 \cup \Lambda} = L^1_{\Lambda_0 \cup \Lambda} \oplus (M_s)_\Lambda.$$

Proof. Since every subset of a Rosenthal set is Rosenthal, we may and do assume that $\Lambda_0 \cap \Lambda = \emptyset$. We pick $\mu \in M_{\Lambda_0 \cup \Lambda}$, and we define

$$C_\mu : L^\infty_{\Lambda^c} \to L^\infty_{\Lambda_0} = C_{\Lambda_0} \subseteq C_{\Lambda^c}$$

by $C_\mu(h) = \mu * h$. By our assumption on Λ Theorem III.5 provides $g \in L^1(G)$ such that $\mu * h = g * h$ for every $h \in L^\infty_{\Lambda^c}$; in particular, $\hat{\mu}(\gamma) = \hat{g}(\gamma)$ for every $\gamma \in \Lambda^c$ and thus $(\mu - g) \in M_\Lambda$. Since $\mu \in M_{\Lambda \cup \Lambda_0}$, it follows that

$$g \in L^1 \cap M_{\Lambda \cup \Lambda_0} = L^1_{\Lambda \cup \Lambda_0};$$

moreover, $\mu - g = g' + \nu$ with $g' \in L^1_\Lambda$ and $\nu \in (M_s)_\Lambda$ and therefore

$$\mu = (g + g') + \nu$$

belongs to $L^1_{\Lambda\cup\Lambda_0} \oplus (M_s)_\Lambda$.

IV. Around the Mooney–Havin theorem. Within the setting of Banach algebras, results of weak sequential completeness are classically shown by using "peak sets" (see [A], [Ch1, 2], [P]). In fact, they can also be obtained through Corollary II.2. Indeed, the following holds:

PROPOSITION IV.1. *Let (S, Σ, μ) be a probability space, and let H be a subalgebra of $L^\infty(\mu)$ containing the constants such that for every closed subset F of the spectrum L of $L^\infty(\mu)$ satisfying $\tilde{\mu}(F) = 0$, there is $f \in H$ with $\|f\| = 1$, $f_{|F} = 1$ and $\tilde{\mu}\{|f| = 1\} = 0$. Then for every H-submodule M of $L^\infty(\mu)$, one has*

$$M^\perp = (M^\perp \cap L^1(\mu)) \oplus (M^\perp \cap M_s(\tilde{\mu})).$$

Here we denote by $\tilde{\mu}$ the probability measure induced by μ on L and $M_s(\tilde{\mu})$ the space of measures on L which are singular with respect to $\tilde{\mu}$.

Proof. Pick $\nu \in M^\perp$, and write $\nu = \nu_a + \nu_s$, with $\nu_a \in L^1(\tilde{\mu})$ and $\nu_s \in M(L)$ orthogonal with respect to $\tilde{\mu}$. Since $\tilde{\mu}$ is a normal measure, there is a closed subset F of L with $\tilde{\mu}(F) = 0$ and $|\nu_s|(L\backslash F) = 0$. Let $f \in H$ be such that $\|f\| = 1$, $f_{|F} = 1$ and $\tilde{\mu}(\{|f| = 1\}) = 0$. Replacing f by $(1 + f)/2$ if necessary, we may assume that $\{|f| = 1\} = \{f = 1\}$ and we denote this set by A. Then clearly

$$1_A = \lim_{n\to\infty} f^n$$

pointwise on L. Now for every $x \in M$, we may write

$$\nu_s(x) = \nu(1_A x) = \nu\left(\lim_{n\to\infty} f_n x\right)$$

and by Lebesgue's theorem,

$$\nu(\lim_{n\to\infty} f_n x) = \lim_{n\to\infty} \nu(f_n x) = 0$$

and thus $\nu_s \in M^\perp$.

COROLLARY IV.2. *In the notation of Proposition* IV.1, *if M is w^*-closed, then*

(i) *the unit ball of the space $M_\perp = L^1 \cap M^\perp$ is closed for the topology of convergence in measure;*

(ii) *the predual $M_* = L^1/M_\perp$ is weakly sequentially complete.*

Proof. We set $X = L^1 \cap M^\perp = M_\perp$; then by the bipolar theorem $X^{\perp\perp} = M^\perp$, and thus by IV.1

$$X^{\perp\perp} = X \oplus (M_s(\tilde{\mu}) \cap X^{\perp\perp}). \tag{1}$$

It follows from (1) and a theorem of Bukhvalov–Lozanovskiĭ [BL] that the unit ball of X is closed in measure, which is (i). For (ii), we observe that (1) implies that

$$M^* = L^{\infty *}/M^\perp = L^1/M_\perp \oplus_1 M_s(\tilde\mu)/M_s(\tilde\mu) \cap M^\perp$$

and thus the predual $M_* = L^1/M_\perp$ is ℓ^1-complemented in M^*; therefore there is an isometric bijection S—in fact, a symmetry—of M^* with $\mathrm{Ker}(S - I) = M_*$; and then Corollary II.2 shows that M_* is weakly sequentially complete. ■

Let us mention that it is proved in [G1] by using the converse implication in [BL] that (i) implies (ii); this provides a direct approach to the theorem of Mooney–Havin and its extensions ([G1], [G2]) which does not use peak sets, when it is possible to prove (i) without using them; it is so for instance when $H = H^\infty(D)$ [G1]. Note also that Proposition IV.1 applies to $H = H^\infty(\mu)$ when μ is a unique representing measure for a uniform algebra.

We described here a short way to get weak sequential completeness; more information on $L^1/M_\perp$ can be obtained if peak sets are studied more carefully (see e.g. [Ch2], [P], [G2]).

V. Miscellaneous remarks and questions

V.1. Let us assume that $L^1/L^1_{\Lambda^c}$ does not contain a subspace isomorphic to $\ell^1(\mathbb{N})$. Then by Odell–Rosenthal's theorem [OR] for every approximate identity $\{\varphi_n\}$ in $L^1(G)$, $\{\dot\varphi_n\}$ has a weak Cauchy subsequence in $L^1/L^1_{\Lambda^c}$; moreover, every element in $(L^\infty_\Lambda)^*$ is w^* of the first Baire class. In particular, for every $\dot\mu \in M(G)/M_{\Lambda^c}$, $\{\dot\mu * \varphi_n\}$ has a weak Cauchy subsequence in $L^1(G)/L^1_{\Lambda^c}$. This does not imply a priori that $\{\dot\mu * \varphi_n\}$ is a weak Cauchy sequence (which, in view of Lemma III.3, would imply that every multiplier $L^\infty_\Lambda \to L^\infty_\Lambda$ is a multiplier $L^\infty_\Lambda \to C_\Lambda$, hence by considering $\mu = \delta_0$ that $L^\infty_\Lambda = C_\Lambda$, i.e. Λ is a Rosenthal set). We ask again

QUESTION 1 [LP1]. If $L^1/L^1_{\Lambda^c}$ does not contain $\ell^1(\mathbb{N})$ as a closed subspace, is Λ a Rosenthal set?

Under this assumption, Λ is a Riesz set [LP1] and every $f \in L^\infty_\Lambda$ is Riemann integrable ([LPS], Corollary IV.4).

Question 1 can be strengthened as follows:

DEFINITION V.1. $\Lambda \subset \Gamma$ has *property* (ρ) if there exists $\rho \in L^{\infty *}_\Lambda$ such that

(i) ρ is Borel on $(L^{\infty *}_\Lambda, w^*)$;
(ii) $\rho(f) = f(0)$ for every $f \in C_\Lambda$.

Note that if $L^1/L^1_{\Lambda^c}$ does not contain ℓ^1 as a closed subspace Λ has property (ρ). (Take for ρ a cluster point of $\{\varphi_n\}$ where $\{\varphi_n\}$ is an approximate identity in $L^1(G)$.)

QUESTION 2. If Λ has property (ρ), is Λ a Rosenthal set?

PROPOSITION V.2. *Let $\Lambda \subset \Gamma$ and assume that Λ has property (ρ). Then Λ is a Riesz set.*

Proof. Pick $\mu \in M_\Lambda$, and consider

$$C_\mu : L^\infty \to L^\infty_\Lambda, \qquad f \to f * \mu.$$

Clearly, the multiplier C_μ is (w^*-w^*)-continuous, and thus $\rho \circ C_\mu$ is a w^*-Borel linear form in $L^{\infty *}$. Now by a result of Christensen ([C], Th. 5.8), $\rho \circ C_\mu$ is w^*-continuous; that is, there is $g \in L^1$ such that $\rho(\mu * f) = \langle g, f\rangle$ for every $f \in L^\infty$. It follows that $\hat{\mu} = \hat{g}$ and thus $\mu = g$ is absolutely continuous. ∎

There are Riesz sets for which such a lifting ρ does not exist. For instance, if $\Lambda = \mathbb{N}$ then $L^1/L^1_{\Lambda^c}$ $(= L^1/\overline{H^1_0})$ has the property (X) [GT] and thus any $\rho \in H^{\infty *}$ which is w^*-Borel is given by $g \in L^1$ (see IV.2); and since $\hat{g} \in c_0$, g cannot coincide on $A(D)$ with a Dirac measure.

Let us consider the families $\mathcal{F}_1$ (respectively $\mathcal{F}_2$) of subsets Λ of Γ such that $L^1/L^1_{\Lambda^c}$ does not contain ℓ^1 (resp. Λ has property (ρ)). Then $\mathcal{F}_1 \subset \mathcal{F}_2$ and $\mathcal{F}_1$ contains the family of Rosenthal sets. As $\mathcal{F}_2$ is contained in the family of Riesz sets the assumptions of ([G4], Theorem 2) are satisfied, hence neither $\mathcal{F}_1$ nor $\mathcal{F}_2$ is an analytic subset of 2^Γ.

V.2. There exist means on $L^\infty(G)$ (G is a compact infinite abelian group), i.e. translation invariant linear forms φ with $\|\varphi\| = \varphi(1) = 1$, which are distinct from the Haar measure [Ru]. It follows that there exist operators from $L^\infty(G)$ to $C(G)$, with rank one, which commute with translations and which are not multipliers. In fact, it is well known and easy to show that multipliers are exactly those operators which commute with convolutions with continuous functions. In some cases, however, any "reasonable" operator from L^∞_Λ to C_Λ which commutes with translations is a multiplier. Here, "reasonable" means that $f \to T(f)(0) \in L^{\infty *}_\Lambda$ is w^*-Borel. In fact, if $L^1/L^1_{\Lambda^c}$ has property (X) (see [GT]) then there is a function $g \in L^1$ such that $T(f)(0) = \langle f, g\rangle$, hence $T(f) = f * g$. It will be so e.g. if $\Gamma \backslash \Lambda$ is a $\Lambda(1)$ set (by [G1], Th. 30), and if $\Lambda = \mathbb{N}$ since L^1/H^1 has (X) [GT]. Such sets Λ are quite "big"; on the other hand, if Λ is Sidon, then $L^\infty_\Lambda = C_\Lambda \simeq \ell^1(\Lambda)$ and it is easily seen that every operator on L^∞_Λ which commutes with translations is a multiplier. So it is not clear whether "reasonable" translation invariant operators from L^∞_Λ to C_Λ are always multipliers.

V.3. Theorem II.1 and Corollary II.2 have been sufficient so far for applications but they are not optimal. For instance, if $S : X^* \to X^*$ is such that $\operatorname{Ker}(S)$ and $S(X^*_1)$ are w^*-closed, then for any weak Cauchy sequence $\{y_n\}$, $S(w^*\text{-}\lim y_n) = w^*\text{-}\lim S(y_n)$ ([G5], Th. VII.4). Also, by using the ball topology ([GK], Th. 3.3 and the proof of Th. 9.3), it can be shown that if $X \subseteq Y \subseteq X^{**}$ and if S is an isometric bijection of Y whose restriction to X is the identity, then $S(z) = z$ for every $z \in Y$ which is the limit in (X^{**}, w^*) of a weak Cauchy sequence in X.

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