COLLOQUIUM MATHEMATICUM

VOL. XLIV 1981 FASC. 2

A COMBINATORIAL CHARACTERIZATION OF PLANAR 2-COMPLEXES

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1. Introduction. Harary and Rosen [3] consider the old result that a connected simplicial 2-complex has an imbedding in the 2-sphere if and only if it contains no subset homeomorphic to K_5, to $K_{3,3}$, or to F^2, where K_5 and $K_{3,3}$ are the Kuratowski non-planar graphs [5] and F^2 is the thumbtack-shaped subset of R^3, called a *disk with feeler*, defined as follows:

$$F^2 = \{(x, y, 0) : x^2+y^2 \leqslant 1\} \cup \{(0, 0, z) : 0 \leqslant z \leqslant 1\}.$$

Such a 2-complex is planar, moreover, if and only if it contains no 2-sphere.

Observing that this characterization of planar 2-complexes is topological, Harary and Rosen ask at the end of their paper for a completely combinatorial characterization of planar 2-complexes (P 988). In two counterexamples containing topological copies of the Kuratowski graphs, they demonstrate that it is not sufficient to require that the 2-complex be locally planar and its 1-skeleton planar. Harary and Rosen also prove that a 2-complex is locally planar if and only if it contains no disk with feeler.

It is the purpose of this note* to provide a purely combinatorial characterization of spherical and planar 2-complexes. This combinatorial characterization also yields a new proof of the topological characterization of spherical 2-complexes mentioned above. In a sequel, Gross and Rosen [2] derive an essentially optimal algorithm to decide whether a 2-complex is planar.

* The research of the first-named author was partially supported by NSF Contract MCS 76 05850 at the Columbia University.

2. Augmenting the 1-skeleton. In this section it is proved that it can be decided whether a locally planar 2-complex C has an imbedding in the 2-sphere by constructing a graph from C and then deciding whether that graph is planar. It was noted in the previous section that the 1-skeleton of C would not be a satisfactory graph for this purpose. However, it will be shown that by retaining from C not only the 1-skeleton, but also an appropriate 1-dimensional subset of the interior of each 2-cell, an appropriate graph is obtained. In what follows, the first barycentric subdivision of a complex K is denoted by K', and its 1-skeleton by $K^{(1)}$.

THEOREM 1. *Let C be a compact connected simplicial 2-complex. Then C may be imbedded in the 2-sphere if and only if C is locally planar and the 1-skeleton $C'^{(1)}$ of its first barycentric subdivision is planar.*

Proof. Any imbedding of the 2-complex C in the 2-sphere S^2 induces an imbedding of the 1-complex $C'^{(1)}$ into S^2, so the "only if" part of the theorem is obvious.

Conversely, suppose that $f\colon C'^{(1)} \to S^2$ is an imbedding. Let R be any 2-simplex of C, and let x be the vertex of C' at the barycenter of R. The image $f(\mathrm{bd}(R'))$ of the boundary of the subdivision of R is a simple circuit separating S^2 into two disks D_R and E_R. Let D_R be the disk containing the vertex image $f(x)$.

If the imbedding $f: C'^{(1)} \to S^2$ places no edges of $C'^{(1)}$ into the interior of the disk D_R except for the 6 edges incident on the barycenter x of the 2-simplex R, then the imbedding f may be extended so that it maps the subdivided 2-simplex R' onto the disk D_R. In this case, the disk D_R is called *uncluttered.* Otherwise, it is called *cluttered.*

If for every 2-simplex of the 2-complex C the associated disk in S^2 containing the image of its barycenter is uncluttered, then there is no problem in extending the imbedding $f: C'^{(1)} \to S^2$ to an imbedding of C', thereby inducing an imbedding of C. On the other hand, if some disks are cluttered, then one must first remove their clutter by modifying the imbedding f before extending the imbedding to all of C'. The rest of the proof is concerned with establishing the validity of the modification procedure illustrated in Fig. 1.

The clutter removal begins with the selection of a 2-simplex R of the 2-complex C such that the disk D_R in S^2 is maximal, that is, not contained in the disk D_Q for any other 2-simplex Q of the 2-complex C. Let u, v, and w be the vertices of the 2-simplex R, let a, b, and c be the midpoints of the edges uv, vw, and wu, respectively, and let x be the barycenter of R. This is illustrated in Fig. 1, with $\bar{u}$ denoting $f(u)$, $\bar{v}$ denoting $f(v)$, etc.

Let uax denote the 2-simplex of C' with vertices u, a, and x, and let $|\bar{u}\bar{a}\bar{x}|$ denote the subdisk of D_R bounded by $f(\mathrm{bd}(uax))$. Now suppose that X is the clutter in the interior of $|\bar{u}\bar{a}\bar{x}|$. If $\bar{x} \in \mathrm{cl}(X)$, where $\mathrm{cl}(X)$ denotes the closure of X, it would be impossible to remove the clutter from D_R, since some clutter would necessarily be attached to the interior point $\bar{x}$. If $\bar{a} \in \mathrm{cl}(X)$, it might be possible to flip the clutter over the edge image $|\bar{u}\bar{a}|$, provided $\bar{x} \notin \mathrm{cl}(X)$. However, if some other 2-simplex Q of the 2-complex C meets the 2-simplex R in the edge uv, then flipping the clutter over $|\bar{u}\bar{a}|$ would result in that clutter being placed inside the disk D_Q.

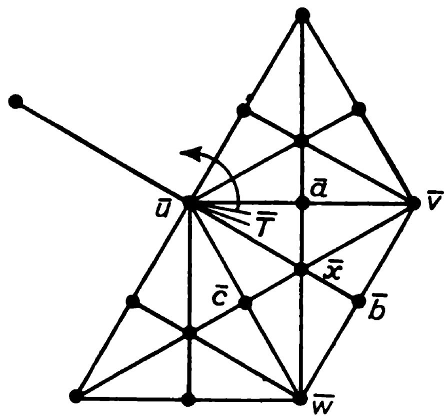

Fig. 1. A component T of the clutter in a maximal disk is attached only at one corner, and it is removed from the disk by rotating it into a free sector

Accordingly, the immediate goal is to prove that $\mathrm{cl}(X) \cap \mathrm{bd}\,|\bar{u}\bar{a}\bar{x}| = \bar{u}$, after which it is to be established that the clutter can be rotated out of the disk D_R into a "free sector", that is, a location not inside any disk D_Q.

Since X is all of the clutter in $|\bar{u}\bar{a}\bar{x}|$, there is a subgraph T of the 1-complex $C'^{(1)}$ whose image $f(T)$ coincides with $\mathrm{cl}(X)$. Since the clutter X lies in the interior of $|\bar{u}\bar{a}\bar{x}|$, we have

$$f(T) \cap \mathrm{bd}\,|\bar{u}\bar{a}\bar{x}| \subseteq \{\bar{u}, \bar{a}, \bar{x}\}.$$

Moreover, it follows that the graph T contains no edge of the subdivided 2-simplex $R'^{(1)}$.

Since the "preclutter" T contains no edge from $R'^{(1)}$, T contains none of the 6 edges incident on the vertex x at the barycenter of the 2-simplex R. Thus $x \notin T$.

Now suppose that $a \in T$. Since the clutter X lies in the interior of $|\bar{u}\bar{a}\bar{x}|$, it is clear that $\bar{a} \in \mathrm{cl}(X) - \{\bar{u}\}$. Thus, there is an edge ad in the preclutter T such that the point image $\bar{d}$ lies in the interior of $|\bar{u}\bar{a}\bar{x}|$. The edge ad of $C'^{(1)}$ does not arise from a 1-simplex of C, because the only two such edges with the endpoint a are ua and va, both in $R'^{(1)}$, so they cannot be in the preclutter T. Nor can ad be any other edge of $R'^{(1)}$, so that the edge ad must come from some $Q'^{(1)}$, where Q is a 2-simplex of the 2-complex

C such that $Q \cap R = uv$. Accordingly, the vertex d must be the barycenter of the 2-simplex Q. Since $\bar{d}$ lies in the interior of $|\overline{uax}|$, it follows that $\text{bd}\,|\overline{uax}|$ separates the point $\bar{d}$ from the point $\bar{v}$ in S^2 and, consequently, that $\text{bd}(uax)$ separates the vertex d from the vertex v in the 1-complex $C'^{(1)}$, which is a contradiction, because there is an edge vd in the subdivided 2-simplex Q' that does not meet $\text{bd}(uax)$. Thus $a \notin T$.

Since $C'^{(1)}$ is connected, it is now evident that $\text{cl}(X) \cap \text{bd}\,|\overline{uax}| = \bar{u}$, exactly as illustrated in Fig. 1. The remaining problem is to find a free sector incident on $\bar{u}$ into which the clutter may be moved.

Introduction of the notion of a *link* is to simplify the remainder of the exposition. First, for any vertex p of a complex K, one defines $\text{star}(p, K)$ to be the subcomplex of K determined by the collection of all the simplexes of K incident on the vertex p. Then, one defines $\text{link}(p, K)$ to be the subcomplex of $\text{star}(p, K)$ consisting of all simplexes q such that $p \cap q$ is the empty set. For example, if p were a vertex of a triangulated closed surface, then its star would be a triangulated disk with p in the middle, and its link would be the simple circuit which is the rim of that disk.

Now suppose ue is an edge of the preclutter T, so that $e \in \text{link}(u, C') \cap T$. In $f(C'^{(1)}) - \{\bar{u}\}$, the point $\bar{e}$ is clearly separated from $\text{bd}\,|\overline{uax}| - \{u\}$. Thus, $\text{star}(u, C') - \{u\}$ has e and $\text{bd}(uax) - \{\bar{u}\}$ in separate components. Thus, the vertex u is a cutpoint of the graph $\text{star}(u, C')^{(1)}$. Since $\text{star}(u, C') - \{u\}$ has $\text{link}(u, C')$ as a deformation retract (or, alternatively, directly from the combinatorial definitions of star and link), $\text{link}(u, C')$ is not connected. From the topological invariance of stars and links under subdivision (or, alternatively, directly from the combinatorial definitions of star and link) it follows that $\text{link}(u, C)$ also is not connected. From Theorem 3 of Harary and Rosen [3] it now follows that $\text{link}(u, C)$ consists of a finite number of mutually disjoint edge-paths and vertices.

Next, let B be the subcomplex of C that contains every 2-simplex Q of C such that the associated disk D_Q is maximal, and that contains every 1-simplex and 0-simplex of C whose image under the imbedding f does not intersect the interior of any maximal disk D_Q. In other words, the subcomplex B is the result of discarding from the complex C the preimage of all of the clutter in all of the maximal disks.

The 2-sphere S^2 may be triangulated so that the restriction of the imbedding f to the subcomplex B is a simplicial map, that is, so that the image under f of every simplex of B is a simplex in the triangulation of S^2. With respect to this triangulation, $\text{link}(\bar{u}, S^2)$ is a closed circuit. It follows that there is an edge J in $\text{link}(\bar{u}, S^2)$ whose interior does not meet $f(C'^{(1)})$. The sought after free sector is the 2-simplex $\bar{u} * J$ in $\text{star}(\bar{u}, S^2)$ containing the vertex $\bar{u}$ and the edge J. The interior of the free sector $\bar{u} * J$ contains no point of the image of the 1-complex $C'^{(1)}$. Moreover, it contains no point of any maximal disk D_Q and, consequently, no point of any other disk D_P. Thus, the clutter X may be rotated out of $|\bar{u}\bar{a}\bar{x}|$ into the free sector, thereby completely decluttering $|\bar{u}\bar{a}\bar{x}|$ without putting clutter into any other disk D_P.

More precisely, there exists an imbedding $g: T \to \mathrm{int}(\bar{u} * J) \cup \{\bar{u}\}$ such that $g(u) = \bar{u}$. Define a new imbedding $f_1: C'^{(1)} \to S^2$ so that f_1 coincides with f at all points of $C'^{(1)} - T$ and f_1 restricted to T coincides with g.

Continuing in this manner for at most 6 steps, one for each 2-simplex of R', it is possible to obtain an imbedding of C' into S^2 such that the disk D_R is completely decluttered, and such that no new clutter has been introduced into any other disk D_Q. Moreover, this decluttering process may then be applied to another maximal disk, and reiterated until every disk D_Q contains only points of $f_n(Q'^{(1)})$, i.e. there is no clutter at all, where f_n is the imbedding ultimately obtained in this iteration. Then f_n may be extended to imbed the entire 2-complex C' into S^2. This completes the proof of Theorem 1.

There is no difficulty in obtaining the following result as a corollary:

THEOREM 2. *Let C be a compact simplicial 2-complex. Then C is planar if and only if C is locally planar, the 1-skeleton $C'^{(1)}$ of its first barycentric subdivision is planar, and C contains no 2-sphere.*

Since Theorem 3 of Harary and Rosen [3] states that a complex C is locally planar if and only if, for every vertex v of C, $\mathrm{link}(v, C)$ imbeds in S^1, it follows that the characterization of planarity just obtained is completely combinatorial. It is used by Gross and Rosen [2] along with a "depth-first search" to construct a planarity algorithm whose execution time is bounded by a quantity proportional to the number of vertices in the 2-complex, thereby generalizing the well-known result of Hopcroft and Tarjan [4] for graphs.

As mentioned in the Introduction, Theorem 1 may be used to obtain a new proof of the old topological characterization of spherical 2-complexes.

THEOREM 1'. *Let C be a compact connected 2-complex. Then C may be imbedded in S^2 if and only if C contains no subset homeomorphic to K_5, to $K_{3,3}$, or to F^2.*

Proof. If C may be imbedded in S^2, the necessity of these restrictions is obvious.

Now suppose, conversely, that C contains none of the forbidden sets. Since C does not contain F^2, it is locally planar. Since $C'^{(1)}$ does not contain K_5 or $K_{3,3}$, it follows that $C'^{(1)}$ is planar. The existence of an imbedding of C in S^2 follows from Theorem 1.

3. On nonsimplicial cell complexes. Some topologically routine extensions of Theorems 1 and 2 may be of interest for applications in combinatorics and in computer science.

THEOREM 3. *Let C be a compact connected, possibly nonsimplicial, cell complex. Then C may be imbedded in the 2-sphere if and only if C is locally planar and the 1-skeleton of its third barycentric subdivision is planar.*

Proof. The $(n+1)$-st barycentric subdivision of a cell complex is the barycentric subdivision of the n-th barycentric subdivision of the complex. The second barycentric subdivision of any cell complex is a simplicial complex that is planar if and only if the original complex is planar. Thus, the result is an immediate consequence of Theorem 1.

THEOREM 4. *Let C be a compact, possibly nonsimplicial, cell 2-complex. Then C is planar if and only if C is locally planar, the 1-skeleton of the third barycentric subdivision of C is planar, and C contains no 2-sphere.*

Proof. This is an immediate corollary to Theorem 2, if one again uses the fact that the second barycentric subdivision of any cell complex is a simplicial complex.

Remark 1. If one wishes to determine whether a given nonsimplicial cell 2-complex C is planar, one need not consider such a possibly clumsy object as the 1-skeleton of the third barycentric subdivision of C. It is sufficient to test the planarity of the 1-skeleton of the barycentric subdivision of any simplicial subdivision of C.

Remark 2. In applying Theorems 1 and 2, one need not take a full barycentric subdivision. It is not necessary to subdivide an edge of the 2-complex unless it is a face of some 2-simplex.

Remark 3. Ion Filotti observes that the genus of a locally planar 2-complex is computable by considering those rotation systems (see [1]) for the 1-skeleton that are consistent with the prescribed 2-cells in the 2-complex.

QUESTION. Does the genus of $C'^{(1)}$ (or of some similar construction) for a locally planar 2-complex C equal the minimum genus of any surface in which C can be imbedded? (**P 1209**)

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Reçu par la Rédaction le 2. 5. 1978;
en version modifiée le 5. 1. 1979