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ON FINITELY BASED VARIETIES OF ALGEBRAS*

BY

BJARNI JÓNSSON (NASHVILLE, TENNESSEE)

1. Introduction. Every class $\mathscr{K}$ of algebras of common similarity type (with operations of finite rank) generates a variety, or equational class, $\mathscr{K}^v = \mathrm{HSP}(\mathscr{K})$, i.e., the class of all homomorphic images of subalgebras of direct products of members of $\mathscr{K}$. By an *equational basis* for $\mathscr{K}$ we mean a set Σ of identities such that $\mathscr{K}^v$ is the class of all models of Σ, and we say that $\mathscr{K}$ is *finitely based* if there exists a finite equational basis for $\mathscr{K}$. Equivalently, $\mathscr{K}$ is finitely based iff $\mathscr{K}^v$ is a strictly elementary class. If $\mathscr{K}$ consists of a single algebra A, then an equational basis for $\mathscr{K}$ is also referred to as an equational basis for A, and A is said to be *finitely based* if $\mathscr{K}$ is finitely based.

Lyndon [6] made the rather surprising discovery that a finite algebra of finite type need not be finitely based, and since then many other such examples have been found. However, there have also been some important positive results. In particular, Oates and Powell showed in [10] that every finite group is finitely based, and McKenzie proved in [8] that every finite lattice with finitely many additional operations is finitely based. Shortly afterwards, Baker made the remarkable discovery that the only property of lattices that is needed is the fact that their congruence lattices are distributive, i.e., he showed that every finite algebra of a finite similarity type that generates a congruence distributive variety is finitely based. He also generalized McKenzie's result in other ways, proving the following theorem:

THEOREM 1.1 (Baker [1]). *Suppose that*

(i) $\mathscr{V}$ *is a congruence distributive variety*;

(ii) $\mathscr{V} = \mathscr{K}^v$ *for some strictly elementary positive universal class* $\mathscr{K}$;

(iii) $\mathscr{V}_{\mathrm{FSI}}$ *is strictly elementary.*

Then $\mathscr{V}$ *is finitely based.*

Here $\mathscr{V}_{\mathrm{FSI}}$ is the class of all finitely subdirectly irreducible members of $\mathscr{V}$. In the particular case where $\mathscr{V}$ is congruence distributive and of

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a finite similarity type, and $\mathscr{V}$ is generated by a finite algebra A, the class $\mathscr{K} = \mathrm{HS}(A)$ satisfies (ii), and (iii) also holds because $\mathscr{V}_{\mathrm{FSI}}$ is equal to $\mathscr{K}_{\mathrm{FSI}}$, and is therefore up to isomorphism just a finite set of finite algebras.

Baker's proof of his theorem is rather involved. After a preliminary version of his results had been made available, Herrmann [4] provided another proof for lattices, and Makkai [7] gave another proof for finite algebras in a congruence distributive variety. Actually, both these results influenced the final version of Baker's proof. Two shorter proofs were later given by Taylor [11] and Burris [2], both for finite algebras of a finite similarity type in a congruence distributive variety. In this note we give a sufficient condition for a variety to be finitely based, that does not involve congruence distributivity, and from this a generalization of Baker's theorem is obtained. In fact, it is shown that hypothesis (ii) in that theorem is redundant. The proof of our first theorem is almost trivial, consisting of a straightforward application of a standard model--theoretic technique. The proof of the generalization of Baker's theorem is obtained by borrowing four lemmas from his paper. In order to make our presentation idependent of Baker's paper, we give detailed proofs of these lemmas.

It has been pointed out to us by Baker that our results can be used to give a new proof of a theorem due to McKenzie [9], which is used in showing that para-primal varieties are finitely based, and with his kind permission this is included. It seems likely that other applications to non-congruence distributive varieties can be found.

2. A sufficient condition. A basic theorem in model theory states that an elementary class of structures is strictly elementary iff its complement is closed under ultraproducts (see, e.g., [3], Theorem 4.1.12, p. 173). This complement can be taken either relative to the class of all structures of the same similarity type or relative to some strictly elementary subclass. We make use of this fact in the sequel.

THEOREM 2.1. *Suppose that $\mathscr{V}$ is a variety of algebras and $\mathscr{B}$ is a strictly elementary class that contains $\mathscr{V}$. If there exists an elementary class $\mathscr{C}$ such that $\mathscr{B}_{\mathrm{SI}} \subseteq \mathscr{C}$ and $\mathscr{V} \cap \mathscr{C}$ is strictly elementary, then $\mathscr{V}$ is finitely based.*

Note that $\mathscr{B}_{\mathrm{SI}}$ is the class of all subdirectly irreducible members of $\mathscr{B}$.

Proof. If $\mathscr{V}$ is not finitely based, then there exists an algebra $A \in \mathscr{V}$ that is an ultraproduct of algebras $B_i \in \mathscr{B} - \mathscr{V}$ $(i \in I)$ modulo some ultrafilter U on I. Each B_i has as a homomorphic image a subdirectly irreducible algebra B'_i that is not in $\mathscr{V}$, and the ultraproduct A' of the algebras B'_i modulo U is in $\mathscr{V}$, since it is a homomorphic image of A. The algebras B'_i need not all be in $\mathscr{B}$, but A' is in $\mathscr{V}$, and therefore certainly in $\mathscr{B}$. Since

$\mathscr{B}$ is strictly elementary, the set $\{i \in I: B'_i \in \mathscr{B}\}$ belongs to U. We may therefore assume that every B'_i belongs to $\mathscr{B}$, and hence to $\mathscr{C}$. Consequently, $A' \in \mathscr{V} \cap \mathscr{C}$. However, this is a contradiction, for $\mathscr{V} \cap \mathscr{C}$ is strictly elementary, and A' is an ultraproduct of algebras that are not in $\mathscr{V} \cap \mathscr{C}$. Thus $\mathscr{V}$ must be finitely based.

COROLLARY 2.1. *Suppose that $\mathscr{V}$ is a variety of algebras in which principal congruences are definable. If either $\mathscr{V}_{\mathrm{FSI}}$ or $\mathscr{V}_{\mathrm{SI}}$ is strictly elementary, then $\mathscr{V}$ is finitely based.*

Proof. To say that principal congruences are definable in $\mathscr{V}$ means that there exists a first order formula $\Phi(x, y, z, u)$ such that, for all $A \in \mathscr{V}$ and $a, b, c, d \in A$,

$$(c, d) \in \operatorname{con}(a, b) \text{ iff } A \models \Phi(a, b, c, d),$$

where $\operatorname{con}(a, b)$ is the smallest congruence relation on A that identifies a and b. It is not hard to see that Φ can always be so chosen that the backward implication holds in every algebra of the same similarity type, i.e., so that the set

$$R_\Phi(a, b) = \{(c, d): A \models \Phi(a, b, c, d)\}$$

is contained in $\operatorname{con}(a, b)$. We then take $\mathscr{B}$ to be the class of all algebras A such that Φ defines congruence relations in A, i.e.,

$A \in \mathscr{B}$ iff, for all $a, b \in A$, $R_\Phi(a, b)$ is a congruence relation on A.

Then $\mathscr{B}$ is strictly elementary, and so are $\mathscr{B}_{\mathrm{FSI}}$ and $\mathscr{B}_{\mathrm{SI}}$, for

$A \in \mathscr{B}_{\mathrm{FSI}}$ iff $A \in \mathscr{B}$ and for all $a, b, a', b' \in A$ with $a \neq b$ and $a' \neq b'$ there exist $c, d \in A$ such that $c \neq d$, $A \models \Phi(a, b, c, d)$, and $A \models \Phi(a', b', c, d)$

and

$A \in \mathscr{B}_{\mathrm{SI}}$ iff $A \in \mathscr{B}$ and there exist $c, d \in A$ with $c \neq d$ such that $A \models \Phi(a, b, c, d)$ whenever $a, b \in A$ and $a \neq b$.

The conclusion now follows from the preceding theorem by taking $\mathscr{C} = \mathscr{B}_{\mathrm{FSI}}$ or $\mathscr{C} = \mathscr{B}_{\mathrm{SI}}$, depending on which one of the classes $\mathscr{V}_{\mathrm{FSI}}$ and $\mathscr{V}_{\mathrm{SI}}$ is strictly elementary.

COROLLARY 2.2 (McKenzie [9]). *Suppose that $\mathscr{V}$ is a variety of algebras of a finite similarity type, in which principal congruences are elementarily definable. If $\mathscr{V}$ has, up to isomorphism, only finitely many subdirectly irreducible algebras all of which are finite, then $\mathscr{V}$ is finitely based.*

3. Projective radius. We recall here the definition and basic properties of the 2-radius of an algebra or of a class of algebras. Since we do not need Baker's more general notion of an n-radius, we refer to the *2-radius* simply as the (projective) radius. Given an algebra A, a map

f from A to A is called a 0-*translation* if f is either constant or the identity map, and f is called a 1-*translation* if it is obtained from one of the basic operations of A by freezing all except one of the variables. A k-*translation*, for $k > 1$, is a composition of k 1-translations, and a map is called a *translation* if it is a k-translation for some natural number k.

For $a, b \in A$ let $\Gamma_k(a, b)$ be the set of all ordered pairs (c, d) such that $\{c, d\} = \{f(a), f(b)\}$ for some k-translation f of A, and let $\Gamma(a, b)$ be the union of the relations $\Gamma_k(a, b)$ for $k = 0, 1, 2, \ldots$ We say that (a, b) and (a', b') are *bounded* if $\Gamma(a, b) \cap \Gamma(a', b') \neq 0$, where 0 is the identity relation on A, and we say that (a, b) and (a', b') are k-*bounded* if $\Gamma_k(a, b) \cap \cap \Gamma_k(a', b') \neq 0$. If there exists a natural number k such that, for all $a, b, a', b' \in A$,

$$\operatorname{con}(a, b) \cap \operatorname{con}(a', b') \neq 0 \Rightarrow \Gamma_k(a, b) \cap \Gamma_k(a', b') \neq 0,$$

then the smallest such k is called the *projective radius* of A (in symbols, $R(A) = k$) but if no such k exists, then we let $R(A) = \infty$. For a class $\mathscr{K}$ of algebras we let $R(\mathscr{K})$ be the supremum of $R(A)$ for $A \in \mathscr{K}$.

Suppose that $\mathscr{V}$ is a congruence distributive variety of a finite similarity type. By [5], there exist ternary polynomials $t_0, t_1, \ldots, t_n$ such that the following identities hold in $\mathscr{V}$:

$$t_0(x, y, z) = x, \quad t_n(x, y, z) = z, \quad t_i(x, y, x) = x \text{ for } i < n,$$

$$t_i(x, x, z) = t_{i+1}(x, x, z) \quad \text{for } i < n, \ i \text{ even},$$

$$t_i(x, z, z) = t_{i+1}(x, z, z) \quad \text{for } i < n, \ i \text{ odd}.$$

Let $\mathscr{V}_0$ be the class of all algebras satisfying these identities. Thus $\mathscr{V}_0$ is a finitely based congruence distributive variety that contains $\mathscr{V}$. This notation will be in effect throughout the next four lemmas. For convenience we assume that the polynomials t_i are among the basic operations of the variety. (Alternatively, we could modify the definition of a 1-translation.)

LEMMA 3.1. *If $A \in \mathscr{V}_0$, $e_0, e_1, \ldots, e_m \in A$ and $e_0 \neq e_m$, then there exists an $i < m$ such that (e_0, e_m) and (e_i, e_{i+1}) are 1-bounded.*

Proof. Consider the matrix with entries $e_{i,j} = t_j(e_0, e_i, e_m)$, $i \leqslant m$, $j \leqslant n$. Let q be the smallest index such that the elements $e_{i,q}$ are not all equal to e_0. Such a q exists because $e_{i,n} = e_m$, and $q > 0$ because $e_{i,0} = e_0$. If q is odd, then $e_{0,q} = e_{0,q-1} = e_0$, and we can therefore choose $p < n$ so that $e_{p,q} = e_0 \neq e_{p+1,q}$. In this case let $c = e_0$ and $d = e_{p+1,q}$, and consider the 1-translations $f(x) = t_q(e_0, e_{p+1}, x)$ and $g(x) = t_q(e_0, x, e_m)$. In the alternative case, where q is even, and therefore $e_{m,q} = e_{m,q-1} = e_0$, choose $p < m$ so that $e_{p,q} \neq e_0 = e_{p+1,q}$, and let $c = e_{p,q}$, $d = e_0$, $f(x) = t_q(e_0, e_p, x)$ and $g(x) = t_q(e_0, x, e_m)$. In either case we have

$$\{c, d\} = \{f(e_0), f(e_m)\} = \{g(e_p), g(e_{p+1})\},$$

and hence $(c, d) \in \Gamma_1(e_0, e_m) \cap \Gamma_1(e_p, e_{p+1})$, so that (e_0, e_m) and (e_p, e_{p+1}) are 1-bounded.

LEMMA 3.2. *For all $A \in \mathscr{V}_0$ and $a, b, a', b' \in \mathscr{V}_0$,*

$$\operatorname{con}(a, b) \cap \operatorname{con}(a', b') \neq 0 \Rightarrow \Gamma(a, b) \cap \Gamma(a', b') \neq 0.$$

Proof. Suppose that $\operatorname{con}(a, b)$ and $\operatorname{con}(a', b')$ identify two distinct elements c and d. Then there exists a sequence $c = e_0, e_1, e_2, \ldots, e_m = d$ such that $(e_i, e_{i+1}) \in \Gamma(a, b)$ for $i < m$. As in the preceding proof, let $e_{i,j} = t_j(e_0, e_i, e_m)$, and choose $p < m$ and $q < n$ so that the elements $c' = e_{p,q}$ and $d' = e_{p+1,q}$ are distinct. Then $(c', d') \in \Gamma(a', b')$, and we also have $(c', d') \in \operatorname{con}(a', b')$, since $\operatorname{con}(a', b')$ identifies every one of the elements $e_{i,j}$ with $t_j(e_0, e_i, e_0) = e_0$. Thus there exists a sequence $c' = e'_0$, $e'_1, \ldots, e'_{m'} = d'$ with $(e'_i, e'_{i+1}) \in \Gamma(a', b')$ for $i < m'$. By the preceding lemma, there exists an $i < m'$ such that (c', d') and (e'_i, e'_{i+1}) are 1-bounded, and we conclude that (a, b) and (a', b') are bounded, i.e., that $\Gamma(a, b) \cap \cap \Gamma(a', b') \neq 0$.

LEMMA 3.3. *For any elementary subclass $\mathscr{C}$ of $\mathscr{V}_0$, $\mathscr{C}_{\mathrm{FSI}}$ is elementary iff $R(\mathscr{C}_{\mathrm{FSI}}) < \infty$.*

Proof. For each $k \in \omega$ we construct a first order formula $\varphi_k(x, y, x', y')$ such that, for all $A \in \mathscr{V}$ and for all $a, b, a', b' \in A$, $A \models \varphi_k(a, b, a', b')$ iff (a, b) and (a', b') are k-bounded. (It is essential here that the similarity type of $\mathscr{V}_0$ is finite.) By Lemma 3.2, an algebra $A \in \mathscr{V}_0$ is FSI iff it satisfies the infinite formula that is the disjunction of the equations $x = y$ and $x' = y'$ and of the formulas $\varphi_k(x, y, x', y')$ with $k \in \omega$. It follows that if $\mathscr{C}_{\mathrm{FSI}}$ is elementary, and therefore closed under ultraproducts, then it must satisfy the disjunction of finitely many of these formulas. In fact, since the formulas φ_k decrease in strength as k grows larger, $\mathscr{C}_{\mathrm{FSI}}$ must satisfy the disjunction of $x = y$, $x' = y'$ and one formula $\varphi_k(x, y, x', y')$. The smallest such k is clearly the radius of $\mathscr{C}_{\mathrm{FSI}}$.

Conversely, if $R(\mathscr{C}_{\mathrm{FSI}}) = k < \infty$, then $\mathscr{C}_{\mathrm{FSI}}$ is precisely the class of those algebras $A \in \mathscr{C}$ which satisfy the disjunction of $x = y$, $x' = y'$ and $\varphi_k(x, y, x', y')$, and therefore $\mathscr{C}_{\mathrm{FSI}}$ is elementary.

LEMMA 3.4. *If $R(\mathscr{V}_{\mathrm{FSI}}) = k < \infty$, then $R(\mathscr{V}) \leqslant k+2$.*

Proof. Consider any $A \in \mathscr{V}$ and $a_0, b_0, a_1, b_1 \in A$, and suppose that

$$(c, d) \in \operatorname{con}(a_0, b_0) \cap \operatorname{con}(a_1, b_1) \quad \text{and} \quad c \neq d.$$

There exists an epimorphism of A onto an SI-algebra A' which maps c and d onto distinct elements. It follows that $a'_0 \neq b'_0$ and $a'_1 \neq b'_1$, where the primes denote images in A'. Therefore, (a'_0, b'_0) and (a'_1, b'_1) are k-bounded, say

$$(u, v) \in \Gamma_k(a'_0, b'_0) \cap \Gamma_k(a'_1, b'_1) \quad \text{and} \quad u \neq v.$$

It is easy to see that there exists $(u_i, v_i) \in \Gamma_k(a_i, b_i)$ such that $u_i' = u$ and $v_i' = v$. In fact, suppose that f' is a k-translation in A' with $f'(a_i') = u$ and $f'(b_i') = v$. The corresponding k-translation of A is then obtained by replacing each element of A' that is used in the construction of f' by one of its counterimages in A, and we let $u_i = f(a_i)$ and $v_i = f(b_i)$.

We now choose a $j < n$ so that the elements $u^* = t_j(u_0, u_1, v_0)$ and $v^* = t_j(u_0, v_1, v_0)$ are distinct. This can be done because in A' we cannot have $t_j(u, u, v) = t_j(u, v, v)$ for all $j \leqslant n$, since this would imply that

$$u = t_0(u, u, v) = t_1(u, u, v) = t_1(u, v, v) = t_2(u, v, v) = t_2(u, u, v)$$
$$= t_3(u, u, v) = \ldots = t_n(u, v, v) = v.$$

Observe that $(u^*, v^*) \in \Gamma_1(u_1, v_1)$ and (u^*, u_0), $(u_0, v^*) \in \Gamma_1(u_0, v_0)$, as is seen by considering the 1-translations $t_j(u_0, x, v_0)$, $t_j(u_0, u_1, x)$ and $t_j(u_0, v_1, x)$. Applying Lemma 3.1 to the sequence u^*, u_0, v^*, we see that either (u^*, v^*) and (u^*, u_0) are 1-bounded or else (u^*, v^*) and (u_0, v^*) are 1-bounded. In either case, (u_0, v_0) and (u_1, v_1) are 2-bounded, and therefore (a_0, b_0) and (a_1, b_1) are $(k+2)$-bounded.

4. Principal theorem. The promised generalization of Baker's theorem is now easily proved.

THEOREM 4.1. *If $\mathscr{V}$ is a congruence distributive variety of a finite similarity type, and if $\mathscr{V}_{\mathrm{FSI}}$ is strictly elementary, then $\mathscr{V}$ is finitely based.*

Proof. By Lemma 3.3, $R(\mathscr{V}_{\mathrm{FSI}}) = k$ is finite, and hence $R(\mathscr{V}) \leqslant k+2$ by Lemma 3.4. We let $\mathscr{B}$ be the class of all $A \in \mathscr{V}_0$ with $R(A) \leqslant k+2$. Since $\mathscr{V}_0$ is strictly elementary and the condition $R(A) \leqslant k+2$ can be expressed by a first order formula, $\mathscr{B}$ is a strictly elementary class. Obviously, $R(\mathscr{B}_{\mathrm{FSI}}) \leqslant k+2$, and $\mathscr{B}_{\mathrm{FSI}}$ is therefore strictly elementary by Lemma 3.3. Since $\mathscr{V} \cap \mathscr{B}_{\mathrm{FSI}} = \mathscr{V}_{\mathrm{FSI}}$ is strictly elementary by hypothesis, we can apply Theorem 2.1 with $\mathscr{C} = \mathscr{B}_{\mathrm{FSI}}$, and we conclude that $\mathscr{V}$ is finitely based.

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