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SOME COVER PROPERTIES OF NONSEPARABLE METRIC SPACES

BY

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1. Introduction. In this paper we investigate metric spaces which admit so-called α-zero-bases (α is a cardinal number cofinal to ω) – a generalization of zero-bases introduced by A. Lelek in [9], p. 12–23, and further discussed by A. Lelek in [10] and by R. Duda and R. Telgársky in [4]. In our consideration we use the results of A. H. Stone on absolutely Borel and α-analytic sets ([11], [12]). We generalize here some theorems of A. H. Stone. We also consider some other cover properties of metrizable spaces.

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2. Preliminaries. Metric spaces are denoted by ordered pairs of the form (X, d). If (X, d) is a metric space, then we use the notation:

$$B(x, \varepsilon) = \{y \in X\colon\ d(x, y) < \varepsilon\} \quad \text{and} \quad \bar{B}(x, \varepsilon) = \{y \in X\colon\ d(x, y) \leqslant \varepsilon\}.$$

If X is a topological space, then $w(X)$ is the weight of X. If β is an ordinal number, then $\mathrm{cf}(\beta)$ denotes its cofinality. If β is a cardinal number, then β^+ denotes the first cardinal number greater than β.

In this paper α always denotes a cardinal number of the cofinality ω. If α is given, then we assume that a sequence $\{\alpha_n\colon\ n < \omega\}$ is also given, in such a way that each α_n is a cardinal number less than α, $\alpha_n < \alpha_{n+1}$ and $\sum \alpha_n = \lim \alpha_n = \alpha$.

3. α-zero-bases. Let (X, d) be a metric space. A family $\mathscr{A}$ of subsets of X is called an *α-zero-family* if

(i) $|\mathscr{A}| \leqslant \alpha$, and

(ii) for each $\varepsilon > 0$ we have $|\{A \in \mathscr{A}\colon\ \operatorname{diam} A > \varepsilon\}| < \alpha$.

If moreover $\mathscr{A}$ is a covering of X (resp. a basis for the topology of X), then we call it an *α-zero-covering* (resp. *α-zero-basis*) of (X, d). Clearly, $\aleph_0$-

zero-basis coincides with a zero-basis. If a metric space (X, d) admits an α-zero-covering (resp. α-zero-basis) and $Y \subset X$, then also (Y, d) admits an α-zero-covering (resp. α-zero-basis).

We say that a metric space (X, d) is *α-totally bounded* if for each $\varepsilon > 0$ there is a set $D \subset X$ such that $|D| < \alpha$ and if $x \in X$ is any point, then there is $y \in D$ such that $d(x, y) < \varepsilon$. In the case $\alpha = \aleph_0$ we get the notion of totally bounded metric spaces.

THEOREM 1 (a generalization of [4], (5.7), p. 79). *If (X, d) is an α-totally bounded metric space, then it admits an α-zero-basis.*

Proof. For each positive integer n let D_n be such a subset of X that $|D_n| < \alpha$ and for each point $x \in X$ there is $y \in D_n$ such that $d(x, y) < 1/n$. Then the family $\{B(y, 1/n): y \in D_n, 0 < n < \omega\}$ is easily seen to be an α-zero-basis of (X, d).

Lemma 1 below is a generalization of [5], Theorem 4.3.3, p. 334, and the proof of it is a simple modification of the proof given in [5] for the case $\alpha = \aleph_0$:

LEMMA 1. *Let $\{(X_n, \varrho_n): n < \omega\}$ be a family of nonempty metric spaces such that each metric ϱ_n is bounded by 1. The Cartesian product $\prod \{X_n: n < \omega\}$ with the metric ϱ defined by the formula*

$$\varrho(x, y) = \sum_{n < \omega} 2^{-n} \varrho_n(x_n, y_n)$$

(where $x = (x_0, x_1, \ldots)$ and $y = (y_0, y_1, \ldots)$) is α-totally bounded if and only if all spaces (X_n, ϱ_n) are α-totally bounded.

Let I denote the closed unit interval $[0, 1]$ of real numbers and let β be any infinite cardinal number. For each $\gamma < \beta$ put $I_\gamma = I \times \{\gamma\}$ and let R be an equivalence relation in the set $\bigcup \{I_\gamma: \gamma < \beta\}$ the only nondegenerate class of which is $\{(0, \gamma): \gamma < \beta\}$. The quotient set $J_\beta = \bigcup \{I_\gamma: \gamma < \beta\}/R$ can be metrized by the formula

$$\delta_\beta([x_1, \gamma_1], [x_2, \gamma_2]) = \begin{cases} \frac{1}{2}|x_1 - x_2| & \text{if} \quad \gamma_1 = \gamma_2, \\ \frac{1}{2}(x_1 + x_2) & \text{if} \quad \gamma_1 \neq \gamma_2. \end{cases}$$

We consider J_β as a topological space, the topology of which is introduced by δ_β. The space J_β is called the *hedgehog of spininess* β (see [5], Example 4.1.5, p. 314–315).

LEMMA 2. *If $\alpha > \aleph_0$ then $J_\alpha \times J_\alpha \times \ldots$ can be embedded into $J_{\alpha_1} \times \times J_{\alpha_2} \times \ldots$*

Proof. It suffices to show that

(*) $\quad J_\alpha$ can be embedded into $J_{\alpha_{k_1}} \times J_{\alpha_{k_2}} \times \ldots$ for any $0 < k_1 < k_2 < \ldots$

Indeed, if we take any partition of $\{1, 2, \ldots\}$ into an infinite family $\{\{k_1^i, k_2^i, \ldots\}: i = 1, 2, \ldots\}$ of infinite sets, then (by (*)) $J_\alpha \times J_\alpha \times \ldots$ can be embedded into

$$\prod \{J_{\alpha_{k^i_1}} \times J_{\alpha_{k^i_2}} \times \ldots : i = 1, 2, \ldots\}.$$

But the last space is homeomorphic to $J_{\alpha_1} \times J_{\alpha_2} \times \ldots$

To show (*) we put $\alpha_{k_0} = \alpha_0$ and

$$f_n([x, \beta]) = \begin{cases} [x, \beta] & \text{if} \quad \beta < \alpha_{k_n}, \\ [x, \alpha_{k_{n-1}}] & \text{if} \quad \beta \geqslant \alpha_{k_n}, \end{cases}$$

for $0 < n < \omega$. It should be clear that mappings $f_n\colon J_\alpha \to J_{\alpha_{k_n}}$ are continuous and, moreover, the family $\{f_n\colon 0 < n < \omega\}$ separates points and closed sets ([5], p. 110). By [5], Theorem 2.3.20, p. 114, the mapping $f\colon J_\alpha \to J_{\alpha_{k_1}} \times \times J_{\alpha_{k_2}} \times \ldots$, $f(t) = (f_1(t), f_2(t), \ldots)$, is an embedding.

Note that $f(J_\alpha)$ is not closed in $J_{\alpha_{k_1}} \times J_{\alpha_{k_2}} \times \ldots$ Indeed, $\{[1, \alpha_{k_n}]\colon 0 < n < \omega\}$ is a sequence which does not converge in J_α and $\{f([1, \alpha_{k_n}])\colon 0 < n < \omega\}$ is a sequence which converges to a point $([1, \alpha_0], [1, \alpha_{k_1}], [1, \alpha_{k_2}], \ldots)$ of $J_{\alpha_{k_1}} \times J_{\alpha_{k_2}} \times \ldots$

Problem 1. Is $J_\alpha \times J_\alpha \times \ldots$ homeomorphic to $J_{\alpha_1} \times J_{\alpha_2} \times \ldots$? **(P 1307)**

Theorem 2. *If X is a metrizable space of the weight $\leqslant \alpha > \aleph_0$, then there is a metric d such that the space (X, d) admits an α-zero-basis. If, moreover, X is topologically complete, we may choose a metric d such that (X, d) is complete and admits an α-zero-basis.*

Proof. By [5], Theorem 4.4.9, p. 353, X is embeddable into $J_\alpha \times \times J_\alpha \times \ldots$, so by Lemma 2, we can consider X as a subset of $J_{\alpha_1} \times J_{\alpha_2} \times \ldots$ Since $w(J_{\alpha_n}) < \alpha$, we see that each $(J_{\alpha_n}, \delta_{\alpha_n})$ is α-totally bounded. By Lemma 1, the space $(J_{\alpha_1} \times J_{\alpha_2} \times \ldots, \delta'_\alpha)$, where

$$\delta'_\alpha\big((x_1, x_2, \ldots), (x'_1, x'_2, \ldots)\big) = \sum_{n=1}^{\infty} 2^{-n} \delta_{\alpha_n}(x_n, x'_n),$$

is also α-totally bounded, so by Theorem 1, it admits an α-zero-basis and therefore so does its subspace (X, δ'_α). To obtain the second statement we repeat the above consideration for the space $J_{\alpha_1} \times J_{\alpha_2} \times \ldots \times R \times R \times \ldots$ (R is the real line), because, if X is a topologically complete subspace of a metrizable space Y, then X is homeomorphic to a closed subset of $Y \times R \times \times R \times \ldots$ (see [5], Lemma 4.3.22, p. 341).

Problem 2. Does $R \times R \times \ldots$ admit a complete metric d such that there exists an $\aleph_0$-zero-basis of $(R \times R \times \ldots, d)$? **(P 1308)**

Theorem 3. *If a metric space (X, d) admits an α-zero-basis, then for any dense subset Y of X and any sequence $\{\varepsilon_n > 0\colon n < \omega\}$ such that $\lim_{n\to\infty} \varepsilon_n = 0$, the basis $\mathscr{B} = \{B(x, \varepsilon_n)\colon x \in Y, n < \omega\}$ contains some α-zero-basis $\mathscr{B}'$ of (X, d).*

Moreover, we can represent $\mathscr{B}'$ *as a union* $\bigcup\{\mathscr{B}'_n\colon n < \omega\}$, *where* $|\mathscr{B}'_n| < \alpha$ *and* $\mathscr{B}'_n \subset \{B(x, \varepsilon_n)\colon x \in Y\}$.

Proof. We may assume that X contains no isolated point (because $w(X) \leqslant \alpha$ and hence $|\{x\colon x$ is an isolated point of $X\}| \leqslant \alpha$) and that $\varepsilon_0 > \varepsilon_1 > \ldots$

Let $\mathscr{A}$ be an α-zero-basis of (X, d). Put

$$\mathscr{A}_0 = \{A \in \mathscr{A}\colon \operatorname{diam} A \geqslant \varepsilon_0\}$$

and

$$\mathscr{A}_n = \{A \in \mathscr{A}\colon \varepsilon_{n-1} > \operatorname{diam} A \geqslant \varepsilon_n\} \quad \text{for} \quad 0 < n < \omega.$$

Since $\mathscr{A}$ is an α-zero-basis, we may index elements of $\mathscr{A}_n$ in such a manner that $\mathscr{A}_n = \{A_n^\beta\colon \beta < \alpha_n\}$ for some cardinal number $\alpha_n < \alpha$. Moreover, for each $n < \omega$ the family $\bigcup\{\mathscr{A}_k\colon k \geqslant n\}$ is still a basis for the topology of X. Let $x_n^\beta \in A_n^\beta \cap Y$ be any point, for $0 < n < \omega$, $\beta < \alpha_n$, so $A_n^\beta \subset B(x_n^\beta, \varepsilon_{n-1})$. Put

$$\mathscr{B}' = \{B(x_n^\beta, \varepsilon_{n-1})\colon 0 < n < \omega, \beta < \alpha_n\},$$

so $\mathscr{B}'$ is an α-zero family. We show that $\mathscr{B}'$ is a basis. Let us take any $x \in X$ and $\varepsilon > 0$. There is $n < \omega$ such that $\varepsilon_n < \varepsilon/2$ and there is $A \in \bigcup\{\mathscr{A}_k\colon k > n\}$ such that $x \in A \subset B(x, \varepsilon_n)$. Hence $A = A_m^\beta$ for some $m > n$, $\beta < \alpha_m$. This means that

$$x \in B(x_m^\beta, \varepsilon_{m-1}) \subset B(x, 2\varepsilon_n) \subset B(x, \varepsilon).$$

THEOREM 4. *Let* (X, d) *be a metric space such that* $X = \bigcup\{X_n\colon n < \omega\}$, *where each* (X_n, d) *admits an* α*-zero-basis. Then* (X, d) *also admits an* α*-zero-basis.*

Proof. Let $n < \omega$ be fixed. By Theorem 3, the space (X_n, d) admits an α-zero-basis $\mathscr{B}^n$ such that

$$\mathscr{B}^n = \bigcup\{\mathscr{B}_k^n\colon 0 < k < \omega\},$$

where

$$\mathscr{B}_k^n = \{B_{X_n}(x_k^n(\beta), 1/k)\colon \beta < \gamma_k^n\} \quad \text{for some } \gamma_k^n < \alpha.$$

The family $\{B_X(x_k^n(\beta), 1/k)\colon k > n, n < \omega, \beta < \gamma_k^n\}$ is an α-zero-basis of (X, d).

Theorem 4 is a generalization of [4], (5.2), p. 77.

PROBLEM 3. Does (X, d) admit an α-zero-basis provided $X = \bigcup\{X_\beta\colon \beta < \alpha\}$, where each (X_β, d) admits an α-zero-basis? **(P 1309)**

THEOREM 5. *Let* (X, d) *be a complete metric space such that for each* $\varepsilon > 0$ *there is* $\eta > 0$ *such that, for each* $x \in X$,

(i) *the ball* $B(x, \varepsilon)$ *contains a set* D, $|D| = \alpha$ *and for any* $y, y' \in D$, *if* $y \neq y'$, *then* $d(y, y') \geqslant \eta$.

Then (X, d) *does not admit an* α*-zero-basis.*

Proof. We may assume that $w(X) = \alpha$. By (i) we get a sequence $\{\varepsilon_n: n < \omega\}$ of positive real numbers such that $\varepsilon_0 = 1$ and for each $n < \omega$ and each $x \in X$ the ball $B(x, \varepsilon_n)$ contains a set D_x^n such that $|D_x^n| = \alpha$ and for any $y, y' \in D_x^n$, if $y \neq y'$, then $d(y, y') \geqslant 5\varepsilon_{n+1}$. Therefore $\varepsilon_n \geqslant 5\varepsilon_{n+1}$ and $\lim_{n\to\infty} \varepsilon_n = 0$.

Now we inductively define sets $A_n = \{a_{\beta_0 \ldots \beta_n} \in X: \beta_0, \ldots, \beta_n < \alpha\}$ for each $n < \omega$. If a is any fixed point of X, then put $A_0 = D_a^0$. Suppose that A_n is already defined for some $n < \omega$ and put

$$A_{n+1} = \bigcup \{D_x^n: x \in A_n\},$$

where

$$D^n_{a_{\beta_0 \ldots \beta_n}} = \{a_{\beta_0 \ldots \beta_n \beta}: \beta < \alpha\} \quad \text{for any } \beta_0, \ldots, \beta_n < \alpha.$$

For any $n < \omega$, $\beta_0, \ldots, \beta_n < \alpha$ put

$$B_{\beta_0 \ldots \beta_n} = \mathrm{cl}\big(\bigcup \{B(a_{\beta_0 \ldots \beta_n \ldots \beta_{n+m}}, \varepsilon_{n+m+1}): m < \omega,\ \beta_{n+1}, \ldots, \beta_{n+m} < \alpha\}\big).$$

We have

$$B_{\beta_0 \ldots \beta_n} \subset \bar{B}(a_{\beta_0 \ldots \beta_n}, 5\varepsilon_{n+1}/4)$$

by the triangle inequality for the metric d. Hence if $x \in B_{\beta_0 \ldots \beta_n}$, $y \in B_{\beta'_0 \ldots \beta'_n}$ and $\beta_m \neq \beta'_m$ for some m, $0 \leqslant m \leqslant n$, then

$$d(x, y) \geqslant (5 - 2 \cdot \tfrac{5}{4})\varepsilon_{m+1} > 2\varepsilon_{m+1} \geqslant 2\varepsilon_{n+1}.$$

In this way we have obtained the following statement:
there is no ball $B(z, \varepsilon_{n+1})$, $z \in X$, such that both

(ii) $B_{\beta_0 \ldots \beta_n} \cap B(z, \varepsilon_{n+1}) \neq \emptyset$ and $B_{\beta'_0 \ldots \beta'_n} \cap B(z, \varepsilon_{n+1}) \neq \emptyset$ for any $n < \omega$, $\beta_0, \beta'_0, \ldots, \beta_n, \beta'_n < \alpha$, $\beta_m \neq \beta'_m$ for some $0 \leqslant m \leqslant n$.

Let us suppose that the basis $\mathscr{B} = \{B(x, \varepsilon_n): x \in X, 0 < n < \omega\}$ of X contains some α-zero-basis $\mathscr{B}'$ of (X, d). We have $\mathscr{B}' = \bigcup \{\mathscr{B}'_n: 0 < n < \omega\}$, where $\mathscr{B}'_n \subset \{B(x, \varepsilon_n): x \in X\}$ and hence $|\mathscr{B}'_n| < \alpha$. Therefore by (ii) there is $\gamma_0 < \alpha$ such that $B_{\gamma_0} \cap \bigcup \mathscr{B}'_1 = \emptyset$. Let us suppose that we have already defined all $\gamma_0, \ldots, \gamma_n$ for some $n < \omega$. By (ii) there is $\gamma_{n+1} < \alpha$ such that

$$B_{\gamma_0 \ldots \gamma_n \gamma_{n+1}} \cap \bigcup \mathscr{B}'_{n+2} = \emptyset.$$

We have $B_{\gamma_0} \supset B_{\gamma_0 \gamma_1} \supset \ldots$; $B_{\gamma_0 \ldots \gamma_n}$ is closed for $n < \omega$, and

$$\lim_{n\to\infty} (\operatorname{diam} B_{\gamma_0 \ldots \gamma_n}) \leqslant \lim_{n\to\infty} \varepsilon_{n+1} 5/4 = 0,$$

so by the completeness of (X, d), the intersection $\bigcap \{B_{\gamma_0 \ldots \gamma_n}: n < \omega\}$ is a single point x_0. Since $x_0 \in B_{\gamma_0 \ldots \gamma_n}$, we see that $x_0 \notin \bigcup \mathscr{B}'_{n+1}$ for each $n < \omega$, so $x_0 \notin \bigcup \mathscr{B}'$. This means that $\mathscr{B}'$ is not even a covering of X. By Theorem 3 (X, d) does not admit an α-zero-basis.

Corollary 1. *Let X be a Banach space and let d be a metric on X induced by its norm. Then* $\dim X < \infty$ *if and only if* (X, d) *admits a zero-basis.*

Proof. If $\dim X < \infty$, then X is locally compact and separable. Thus X is σ-compact, so by [4], (5.8), p. 79, (X, d) admits a zero-basis.

If $\dim X = \infty$, then X is not locally compact at any point, so if $x \in X$, then there is no totally bounded neighbourhood of x (since (X, d) is complete). This shows that (X, d) fulfils the assumptions of Theorem 5 (because each Banach space is metrically homogeneous).

Remark 1. Corollary 1 can be obtained with the use of [3]. Theorem on p. 143, [3], Remarks on p. 145, and the fact that each open $\aleph_0$-zero-covering of a metric space contains a locally finite subcovering ([10], p. 211).

Example. Let D_β denote a discrete space of the cardinality $\beta \geqslant \aleph_0$. Then (B_β, d_β) is a metric space, where

$$B_\beta = \{(d_0, d_1, \ldots): d_n \in D_\beta, n < \omega\}$$

and

$$d_\beta((d_0, d_1, \ldots), (d'_0, d'_1, \ldots)) = (\min\{n: d_n \neq d'_n\} + 1)^{-1}$$

for distinct points of B_β. It is easy to verify that B_β is homeomorphic to the countable product of D_β with itself and that (B_β, d_β) is a complete space. For the properties of (B_β, d_β) see [11], p. 5–8.

(a) From Theorem 5 it follows that (B_α, d_α) does not admit an α-zero-basis.

(b) Suppose that $\alpha > \aleph_0$. By [11], 2.4 (1), p. 7, the product space $C_\alpha = \prod\{D_{\alpha_n}: n < \omega\}$ with the metric σ_α is a complete metric space homeomorphic to B_α, where

$$\sigma_\alpha((d_0, d_1, \ldots), (d'_0, d'_1, \ldots)) = (\min\{n: d_n \neq d'_n\} + 1)^{-1}$$

for distinct points of C_α. The family

$$\{\{(d_0, \ldots, d_n, d_{n+1}, \ldots): d_{n+1} \in D_{\alpha_{n+1}}, d_{n+2} \in D_{\alpha_{n+2}}, \ldots\}: \\ n < \omega, d_0 \in D_{\alpha_0}, \ldots, d_n \in D_{\alpha_n}\}$$

is an α-zero-basis of $(C_\alpha, \sigma_\alpha)$.

Corollary 2. *If a metrizable space X contains a closed subset Y homeomorphic to B_α, then there is a metric d on X such that (X, d) does not admit an α-zero-basis. If X is topologically complete, we may assume that (X, d) is complete.*

Proof. It follows from the part (a) of our Example and [6] (see also [5], Exercise 4.5.20 (c), p. 369). The second statement follows in the same way from Example and [2] (see also [5], Exercise 4.5.20 (f), p. 369).

4. α-nucleus. Suppose that $\alpha > \aleph_0$ and let X be a metrizable space. We say that

(i) X is *α-scarce*, if $X = \bigcup\{X_n: n < \omega\}$, where $w(X_n) < \alpha$;

(ii) X is *α-scarce at the point* x, if there is an open set U, $x \in U \subset X$, such that U is α-scarce.

If, moreover, $w(X) \leqslant \alpha$, then we define an *α-nucleus* $n_\alpha(X)$ of X as a set of all points of X at which X is not α-scarce. Therefore $n_\alpha(X)$ is a closed subset of X and if $Y \subset X$, then $n_\alpha(Y) \subset n_\alpha(X)$.

The notion of an α-scarce space is quite similar to the notion of a space σ-locally of weight $< \alpha$ introduced by A. H. Stone ([12], p. 251) and an α-nucleus of a space is the same as a "nowhere $\sigma\,\mathrm{lw}(<\alpha)$ kernel" (if $w(X) \leqslant \alpha$) ([12], p. 254).

THEOREM 6. *If X is a metrizable space such that $w(X) \leqslant \alpha > \aleph_0$, then*

(a) *if $X = \bigcup\{X_\beta: \beta < \alpha\}$ and $n_\alpha(X_\beta) = \emptyset$, for each $\beta < \alpha$, then X is α-scarce;*

(b) *if $\emptyset \neq U \subset n_\alpha(X)$ is open in $n_\alpha(X)$ and $U = \bigcup\{U_\beta: \beta < \alpha\}$, then there is $\beta_0 < \alpha$ such that $n_\alpha(U_{\beta_0}) \neq \emptyset$;*

(c) *$n_\alpha(X) = \emptyset$ if and only if X is α-scarce;*

(d) *$n_\alpha(n_\alpha(X)) = n_\alpha(X)$;*

(e) *if, moreover, X is topologically complete, then $X = n_\alpha(X)$ if and only if for each nonempty open subset U of X we have $w(U) = \alpha$.*

Proof. (a) Since $w(X_\beta) \leqslant w(X) \leqslant \alpha$, we see (by [5], Theorem 1.1.14, p. 34) that there is an open (in X_β) covering $\{U_\beta^\gamma: \gamma < \alpha\}$ of X_β such that $U_\beta^\gamma = \bigcup\{U_\beta^{\gamma n}: n < \omega\}$ and $w(U_\beta^{\gamma n}) \leqslant \alpha_n < \alpha$ for each γ, $\beta < \alpha$. Hence

$$X = \bigcup\{U_\beta^{\gamma n}: \gamma, \beta < \alpha;\ n < \omega\}.$$

Put

$$Y_k = \bigcup\{U_\beta^{\gamma n}: n \leqslant k;\ \beta, \gamma < \alpha_k\} \quad \text{for} \quad k < \omega,$$

so $w(Y_k) \leqslant \alpha_k \cdot \alpha_k \cdot k$ and $X = \bigcup\{Y_k: k < \omega\}$ because $\lim \alpha_n = \alpha$.

(b) If such β_0 does not exist, then by (a) U is α-scarce, i.e.,

$$U = \bigcup\{U_n: n < \omega\}, \quad \text{where} \quad w(U_n) < \alpha.$$

Since U is open in $n_\alpha(X)$, there is a set V open in X such that $U = n_\alpha(X) \cap V$. Since $V \setminus U \subset X \setminus n_\alpha(X)$ is open and $w(V \setminus U) \leqslant \alpha$, we can find (by [5], Theorem 1.1.14, p. 34) an open (in X) covering $\{V_\beta: \beta < \alpha\}$ of $V \setminus U$ by α-scarce sets. Therefore $n_\alpha(V_\beta) = \emptyset$ for $\beta < \alpha$, $n_\alpha(U_n) = \emptyset$ for $n < \omega$. By (a) V is an α-scarce set, so $U \subset V \subset X \setminus n_\alpha(X)$ – a contradiction.

(c) follows from (a). (d) follows from (b) and (c). (e) is an easy corollary to [12], 2.2 (7), p. 255.

Remark 2. The part (a) of Theorem 6 shows that [12], Theorem 3, p. 260, is true for uncountable cardinals of cofinality ω, without the assumption of Generalized Continuum Hypothesis (compare it with [12], Remark, p. 261).

From Theorem 6 (e) we obtain

COROLLARY 3. (a) $n_\alpha(B_\alpha) = B_\alpha$ *for* $\alpha > \aleph_0$.

(b) *If* X *is a Banach space,* $w(X) = \alpha > \aleph_0$, *then* $n_\alpha(X) = X$.

We say that a metrizable space X is *absolutely α-analytic* ($\alpha > \aleph_0$) if X is a continuous image of B_α. It follows that each absolutely Borel metrizable space of the weight $\leqslant \alpha$ is absolutely α-analytic (see [11], Corollary 3.6, p. 15). For other properties of absolutely α-analytic spaces see [11].

THEOREM 7. *If* $\alpha > \aleph_0$ *and* X *is an absolutely α-analytic space such that* $n_\alpha(X) \neq \emptyset$, *then* X *contains a closed subset* T *homeomorphic to* B_α.

Proof. Put $Y = n_\alpha(X)$, so Y is an absolutely α-analytic set (as a closed subset of X). Therefore there is a continuous map f from B_α onto Y. Let ϱ be a fixed metric on Y. Let P be the set of all finite nonempty sequences of points of D_α. If $\sigma = (d_1, \ldots, d_n) \in P$, then put $l(\sigma) = n$ and

$$B_\sigma = \{(\sigma, d_{n+1}, d_{n+2}, \ldots) = (d_1, \ldots, d_n, d_{n+1}, \ldots) \in B_\alpha\colon d_{n+1}, d_{n+2}, \ldots \in D_\alpha\}.$$

Let R be the set of all finite sequences (including the empty sequence $\emptyset$) $r = (e_1, \ldots, e_n)$ such that $e_k \in D_{\alpha_k}$, for $1 \leqslant k \leqslant n$, and let Q be the set $\prod \{D_{\alpha_k}\colon 0 < k < \omega\}$. If $r \in R$, then let $l(r)$ be its length.

If u and v are sequences such that either $u, v \in P$ or $u \in R$, $v \in R \cup Q$, then $u \prec v$ denotes the fact that v is an extension of u (so, in particular, $l(u) \leqslant l(v)$).

We inductively construct families $\{\varepsilon_r > 0\colon r \in R\}$ and $\{\sigma_r \in P\colon r \in R\setminus\{\emptyset\}\}$ such that:

(i) $\varepsilon_r < 1/(l(r)+1)$;

(ii) if $r_1, r_2 \in R\setminus\{\emptyset\}$, $r_1 \prec r_2$, $l(r_1) < l(r_2)$, then

$$\sigma_{r_1} \prec \sigma_{r_2} \quad \text{and} \quad l(\sigma_{r_1}) < l(\sigma_{r_2});$$

(iii) if $r, r_1, r_2 \in R$, $r_1 \neq r_2$, $r \prec r_1$, $r \prec r_2$ and $l(r_1) = l(r_2) = l(r)+1$, then

$$\varrho\big(f(B(\sigma_{r_1})), f(B(\sigma_{r_2}))\big) > \varepsilon_r;$$

in particular,

(iii)′ $B(\sigma_{r_1}) \cap B(\sigma_{r_2}) = \emptyset$;

(iv) if $r \in R\setminus\{\emptyset\}$, then $n_\alpha\big(f(B(\sigma_r))\big) \neq \emptyset$.

Let us suppose that for some $n < \omega$ we have already defined ε_r and σ_{r_1}, for each $r, r_1 \in R$, $l(r) = n = l(r_1)-1$, such that (i)–(iv) hold. Let $r_1 \in R$, $l(r_1) = n+1$, be fixed and let $r \in R$ be the unique sequence from R such that $r \prec r_1$ and $l(r) = n$. By (iv) the set $A = n_\alpha\big(f(B(\sigma_{r_1}))\big)$ is nonempty, so $w(A) = \alpha$ and hence we can find $0 < \varepsilon_{r_1} < 1/(n+2)$ and a set

$$D = \{a_d\colon d \in D_{\alpha_{n+2}}\} \subset A$$

such that $\varrho(a, a') > 3\varepsilon_{r_1}$ for $a, a' \in D$, $a \neq a'$. Let $a_d \in D$ be fixed. Since f is continuous and $a_d \in f(B(\sigma_{r_1}))$, we can find $\sigma_{(r_1,d)} \in P$ (if $r_1 = (d_1, \ldots, d_{n+1})$, then $(r_1, d) = (d_1, \ldots, d_{n+1}, d))$ such that

$$\sigma_{(r_1,d)} \prec \sigma_{r_1}, \quad l(\sigma_{(r_1,d)}) > l(\sigma_{r_1})$$

and

$$a_d \in f\big(B(\sigma_{(r_1,d)})\big) \subset B(a_d, \varepsilon_{r_1}).$$

Moreover, by Theorem 6 (b), we can take $\sigma_{(r_1,d)}$ such that $n_\alpha\big(f\big(B(\sigma_{(r_1,d)})\big)\big) \neq \emptyset$. This finishes our induction.

Put

$$C_k = \bigcup \{B(\sigma_r)\colon r \in R, \ l(r) = k\}, \quad \text{for each } 0 < k < \omega,$$

and

$$C = \bigcap \{C_k\colon 0 < k < \omega\},$$

so C is a G_δ-subset of B_α. By (ii) and (iii)′, it is not difficult to see that C is homeomorphic to B_α — this follows from [11], Theorem 1, p. 6 (the argument is similar to the consideration in [11], p. 7).

Put $\mathscr{A}_n = \{\mathrm{cl}\big(f\big(B(\sigma_r)\big)\big)\colon r \in R, \ l(r) = n\}$, so by (iii) $\mathscr{A}_n$ is a discrete family, for each $0 < n < \omega$. Put $T_n = \bigcup \mathscr{A}_n$ and $T = \bigcap \{T_n\colon 0 < n < \omega\}$, so T is a closed subset of X (becuase Y is closed in X). It suffices to show that the map $g\colon C \to T$, $g(x) = f(x)$ for $x \in C$, is a homeomorphism.

By (ii) the intersection $\bigcap \{B(\sigma_r)\colon r \in R \setminus \{\emptyset\}, \ r \prec q\}$ is a single point $b_q \in C$, for each $q \in Q$. Moreover $C = \{b_q\colon q \in Q\}$. For each $q \in Q$ we have

$$g(b_q) = f(b_q) \in \bigcap \{\mathrm{cl}\big(f\big(B(\sigma_r)\big)\big)\colon r \in R \setminus \{\emptyset\}, \ r \prec q\} \subset T$$

and by (i)

$$\lim_{l(r) \to \infty} \mathrm{diam}\Big(\mathrm{cl}\big(f\big(B(\sigma_r)\big)\big)\Big) = 0.$$

This shows that g is well-defined. Moreover,

$$T \subset \bigcup \Big\{\bigcap \{\mathrm{cl}\big(f\big(B(\sigma_r)\big)\big)\colon r \in R \setminus \{\emptyset\}; \ r \prec q\}\colon q \in Q\Big\} = g(C),$$

i.e., g maps C onto T. By (iii) g is a one-to-one map and it is continuous as a restriction of the continuous map f. The inverse map $g^{-1}\colon T \to C$ is continuous because the family $\{B(\sigma_r) \cap C\colon r \in R \setminus \{\emptyset\}\}$ is a basis of C and $g\big(B(\sigma_r) \cap C\big) = f\big(B(\sigma_r) \cap C\big) \subset f\big(B(\sigma_r)\big)$ is a closed-open subset of T (since each family $\mathscr{A}_k$ is discrete).

Remark 3. Theorem 7 generalizes [11], Theorem 22, p. 37, and [12], Theorem 2, p. 259, which were stated in the case of an absolutely Borel space X. Compare Theorem 7 also with a discussion in [12], Remark on p. 259.

COROLLARY 4. *If $\alpha > \aleph_0$ and X is an absolutely α-analytic space, then the following conditions are equivalent*:

(a) *X contains a subset homeomorphic to B_α*;

(b) $n_\alpha(X) \neq \emptyset$;

(c) *X contains a closed subset homeomorphic to B_α.*

Proof. Indeed, if X contains a subset Y homeomorphic to B_α, then by Corollary 3 (a) we have $0 \neq n_\alpha(Y) \subset n_\alpha(X)$, so (a) $\Rightarrow$ (b).

Remark 4. (a) Using Theorem 7 one can show the following generalization of [12], Theorem 5, p. 262: if $\alpha > \aleph_0$ and X is an absolutely α-analytic space, then $n_\alpha(X) = \bigcup\{Y \subset X\colon Y \text{ is homeomorphic to } B_\alpha\}$ (the proof is similar to the proof of [12], Theorem 5, p. 262).

(b) The implication (a) $\Rightarrow$ (c) of Corollary 4 is rather surprising but using some ideas of the proof of Theorem 7, the reader can show that if a metrizable space X contains a subset Y homeomorphic to B_α, then X contains a closed subset Z homeomorphic to B_α such that $Z \subset Y$.

5. Hurewicz spaces. Let β, γ be cardinal numbers and let X be a topological space. We say that X is a *(β, γ)-Hurewicz space*, if for any family $\mathscr{A} = \{\mathscr{A}_\delta\colon \delta < \gamma\}$ of open coverings of X there exists a covering $\mathscr{C} = \bigcup\{\mathscr{C}_\delta\colon \delta < \gamma\}$ of X such that $\mathscr{C}_\delta \subset \mathscr{A}_\delta$ and $|\mathscr{C}_\delta| < \beta$ for each $\delta < \gamma$. Therefore if X is a (β, γ)-Hurewicz space and β', γ' are cardinal numbers, $\beta' \geqslant \beta$, $\gamma' \geqslant \gamma$, then X is also a (β', γ')-Hurewicz space. Moreover, if a topological space Y is either a continuous image or a closed subset of a (β, γ)-Hurewicz space, then Y is again a (β, γ)-Hurewicz space. It is clear that each compact space is an $(\aleph_0, 1)$-Hurewicz space, moreover, each σ-compact space is an $(\aleph_0, \aleph_0)$-Hurewicz space.

$(\aleph_0, \aleph_0)$-Hurewicz spaces were introduced by W. Hurewicz in [7] (property E*). Then they were investigated by A. Lelek in [10] (the name "Hurewicz space" was introduced there). In [10] the reader can find basic properties of $(\aleph_0, \aleph_0)$-Hurewicz spaces and more complete references than given here.

If X is a topological space, then we define $H(X)$ as the least cardinal number γ such that X is a $(w(X), \gamma)$-Hurewicz space, so $1 \leqslant H(X) \leqslant |X|$.

LEMMA 3. *If X is a topological space, then each well-ordered by inclusion (either increasing or decreasing) sequence of open subsets of X has at most $w(X)$ distinct elements.*

Proof. It is a slight modification of [1], IV, §7, Theorem 31 (the Baire–Hausdorff Theorem), p. 161.

THEOREM 8. *Each topological space X is a $(2, w(X)^+)$-Hurewicz space.*

Proof. Let $\{\mathscr{A}_\beta\colon \beta < w(X)^+\}$ be any family of open coverings of X. Let us suppose that for some cardinal number $\gamma < w(X)^+$ we have already defined the sequence $\{A_\beta \in \mathscr{A}_\beta\colon \beta < \gamma\}$. If $X \setminus \bigcup\{A_\beta\colon \beta < \gamma\} = \emptyset$, then let $A_\gamma \in \mathscr{A}_\gamma$ be any set and in the opposite case let x be any fixed point of $X \setminus \bigcup\{A_\beta\colon \beta < \gamma\} \neq \emptyset$ and let A_γ be any set from $\mathscr{A}_\gamma$ such that $x \in A_\gamma$.

Put $C_\gamma = \bigcup\{A_\beta\colon \beta < \gamma\}$ for each $\gamma < w(X)^+$. The family $\{C_\gamma\colon \gamma < w(X)^+\}$ of open sets is well-ordered by inclusion, so by Lemma 3, it has at most $w(X)$ distinct elements. This means that for some $\gamma_0 < w(X)^+$, the set $X \setminus C_{\gamma_0}$ is empty, i.e., the family $\{A_\beta\colon \beta < \gamma_0\}$ is a covering of X.

COROLLARY 5. *If X is a topological space, then $H(X) \leqslant w(X)^+$.*

THEOREM 9. *If X is a topological space, $X = \bigcup \{X_\varepsilon : \varepsilon < \gamma\}$ and each X_ε is a (β, γ)-Hurewicz space and $\gamma \geqslant \aleph_0$, then also X is a (β, γ)-Hurewicz space.*

Proof. Let $f: \gamma \times \gamma \to \gamma$ be a fixed one-to-one and onto map (f does exist because $\gamma \geqslant \aleph_0$). Let $\{\mathscr{A}_\varepsilon : \varepsilon < \gamma\}$ be any family of open coverings of X. Therefore

$$\mathscr{B}^\delta = \{\{X_\delta \cap A : A \in \mathscr{A}_\sigma\} : \sigma = f(\delta, \eta) \text{ for } \eta < \gamma\}$$

is a family of open coverings of X_δ for each $\delta < \gamma$. Since X_δ is a (β, γ)-Hurewicz space, we see that there is a covering $\mathscr{C}^\delta$ of X_δ such that

$$\mathscr{C}^\delta = \bigcup \{\mathscr{C}^\delta_\eta : \eta < \gamma\},$$

$$\mathscr{C}^\delta_\eta \subset \{X_\delta \cap A : A \in \mathscr{A}_{f(\delta,\eta)}\} \quad \text{and} \quad |\mathscr{C}^\delta_\eta| < \beta \quad \text{for} \quad \eta < \gamma.$$

If $C \in \mathscr{C}^\delta_\eta$ for some $\delta, \eta < \gamma$, then let C' be such a fixed element of $\mathscr{A}_{f(\delta,\eta)}$ that $C = C' \cap X_\delta$. If $\varepsilon < \gamma$, $\varepsilon = f(\delta, \eta)$ for some δ, $\eta < \gamma$, then put $\mathscr{A}'_\varepsilon = \{C' \in \mathscr{A}_\varepsilon : C \in \mathscr{C}^\delta_\eta\}$. It is clear that the family $\bigcup \{\mathscr{A}'_\varepsilon : \varepsilon < \gamma\}$ is an open covering of X such that $|\mathscr{A}'_\varepsilon| < \beta$ and $\mathscr{A}'_\varepsilon \subset \mathscr{A}_\varepsilon$ for $\varepsilon < \gamma$, i.e., X is a (β, γ)-Hurewicz space.

The following Theorem 10 is a generalization of [10], Theorem 1, p. 213.

THEOREM 10. *Let X be a metrizable space such that $w(X) \leqslant \alpha$. The following conditions are equivalent:*

(a) *X is an $(\alpha, \aleph_0)$-Hurewicz space;*

(b) *for every metric d on X there exists an open α-zero-covering $\mathscr{A}$ of X such that $\operatorname{diam}_d(A) < 1$, for $A \in \mathscr{A}$;*

(c) *for every metric d on X there exists an α-zero-basis $\mathscr{B}$ of (X, d);*

(d) *there exists a metric d on X such that every basis $\mathscr{B}$ of X contains an α-zero-covering $\mathscr{A}$ of (X, d);*

(e) *for every metric d on X each basis $\mathscr{B}$ of X contains an α-zero-covering $\mathscr{A}$ of (X, d);*

(f) *for every metric d on X each basis $\mathscr{B}$ of X contains an α-zero-basis $\mathscr{B}'$ of (X, d).*

Proof. It suffices to show that (f) $\Rightarrow$ (e) $\Rightarrow$ (d) $\Rightarrow$ (a) $\Rightarrow$ (f) and (f) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a) $\Rightarrow$ (f). The only nontrivial implications are: (b) $\Rightarrow$ (a), (d) $\Rightarrow$ (a) and (a) $\Rightarrow$ (f). The proofs of them are quite similar to those of [10], p. 213–215, given in the case $\alpha = \aleph_0$. In the proof of (b) $\Rightarrow$ (a) we should use the following Lemma 4, analogous to [10], Lemma 1.10, p. 213:

LEMMA 4. *Let X be a paracompact space, $w(X) \leqslant \alpha$ and let $\mathscr{A}_0, \mathscr{A}_1, \ldots$ be open coverings of X. Then there is a pseudo-metric p on X and open coverings $\mathscr{C}_0, \mathscr{C}_1, \ldots$ of X such that $\mathscr{C}_n$ refines $\mathscr{A}_n$ and for each $Y \subset X$ satisfying $\operatorname{diam}_p(Y) < 2^{-n}$, Y intersects only finitely many elements of $\mathscr{C}_n$, for $n < \omega$.*

THEOREM 11. $H(B_{\aleph_0}) = \aleph_1$.

Proof. For each $n < \omega$ put

$$\mathscr{A}_n = \{\{(d_0, d_1, \ldots) \in B_{\aleph_0}\colon d_{n+1}, d_{n+2}, \ldots \in D_{\aleph_0}\}\colon d_0, \ldots, d_n \in D_{\aleph_0}\},$$

so each $\mathscr{A}_n$ is an open covering of $B_{\aleph_0}$. The family $\{\mathscr{A}_n\colon n < \omega\}$ shows that $B_{\aleph_0}$ is not an $(\aleph_0, \aleph_0)$-Hurewicz space, i.e., $H(B_{\aleph_0}) > \aleph_0$.

THEOREM 12. *Each metrizable space X such that $w(X) \leqslant \alpha$ and $n_\alpha(X) = \emptyset$, for some $\alpha > \aleph_0$, is an $(\alpha, \aleph_0)$-Hurewicz space.*

Proof. Since X is α-scarce (by Theorem 6(c)), we see that $X = \bigcup\{X_n\colon n < \omega\}$, where $w(X_n) < \alpha$. Therefore each X_n is an $(\alpha, \aleph_0)$-Hurewicz space, so by Theorem 9, also X is such a space.

THEOREM 13. *Let X be an absolutely α-analytic space for some $\alpha > \aleph_0$.*

(a) *X is an $(\alpha, \aleph_0)$-Hurewicz space if and only if $n_\alpha(X) = \emptyset$.*

(b) *If $n_\alpha(X) \neq \emptyset$, then $H(X) = H(B_\alpha)$.*

Proof. (a) By Theorem 12 it suffices to show that if $n_\alpha(X) \neq \emptyset$, then X is not an $(\alpha, \aleph_0)$-Hurewicz space. If $n_\alpha(X) \neq \emptyset$, then by Theorem 7, X contains a closed subset homeomorphic to B_α, so by Corollary 2, there is a metric d on X such that (X, d) does not admit an α-zero-basis. By Theorem 10, X is not an $(\alpha, \aleph_0)$-Hurewicz space.

(b) Since X contains a closed subset homeomorphic to B_α, we see that $H(X) \geqslant H(B_\alpha)$. But X is a continuous image of B_α, so $H(X) \leqslant H(B_\alpha)$.

COROLLARY 6. *If Generalized Continuum Hypothesis holds and X is an absolutely α-analytic space for some $\alpha > \aleph_0$, then X is an $(\alpha, \aleph_0)$-Hurewicz space if and only if $|X| \leqslant \alpha$.*

Proof. By [11], Theorem 22, p. 37, if $|X| > \alpha$, then X contains a closed subset Y homeomorphic to B_β for some β such that $|B_\beta| = \beta^{\aleph_0} > \alpha$. By Generalized Continuum Hypothesis, if $\beta < \alpha$, then $\beta^{\aleph_0} \leqslant \beta^+ < \alpha$. Therefore Y is homeomorphic to B_α. By Theorem 13, X is not an $(\alpha, \aleph_0)$-Hurewicz space.

Remark 5. If we do not assume Generalized Continuum Hypothesis, then it may happen that $2^{\aleph_0} = \aleph_{\omega+1}$ (see [8], Theorem 37 (the Easton theorem), p. 63). Therefore $B_{\aleph_0}$ will be an absolutely $\aleph_\omega$-analytic space which is an $(\aleph_\omega, \aleph_0)$-Hurewicz space and $|B_{\aleph_0}| > \aleph_\omega$.

PROBLEM 4. What is $H(B_\alpha)$ for $\alpha > \aleph_0$? **(P 1310)**

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