

**THE MAXIMAL SUBSEMIGROUPS  
OF THE IDEALS OF SOME SEMIGROUPS  
OF PARTIAL INJECTIONS**

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**Abstract**

We study the structure of the ideals of the semigroup  $IO_n$  of all isotone (order-preserving) partial injections as well as of the semigroup  $IM_n$  of all monotone (order-preserving or order-reversing) partial injections on an  $n$ -element set. The main result is the characterization of the maximal subsemigroups of the ideals of  $IO_n$  and  $IM_n$ .

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## 1. INTRODUCTION

Let  $X_n = \{1, 2, \dots, n\}$  be an  $n$ -element set ordered in the usual way. The monoid  $PT_n$  of all partial transformations of  $X_n$  is a very interesting object. In this paper we will multiply transformations from the right to the left and use the corresponding notation for the right to the left composition of transformations:  $x(\alpha\beta) = (x\alpha)\beta$ , for  $x \in X_n$ . We say that a transformation  $\alpha \in PT_n$  is *isotone* (*order-preserving*) if  $x \leq y \implies x\alpha \leq y\alpha$  for all  $x, y$  from the domain of  $\alpha$ , *antitone* (*order-reversing*) if  $x \leq y \implies y\alpha \leq x\alpha$  for all  $x, y$  from the domain of  $\alpha$  and *monotone* if it is isotone or antitone.

In the present paper, we study the structure of the semigroups  $IO_n$  of all isotone partial injections and  $IM_n$  of all monotone partial injections of  $X_n$ . From the definition of monotone transformations, it is clear that  $IO_n \subseteq IM_n$ .

Some semigroups of transformations have been studied since the sixties. In fact, presentations of the semigroup  $O_n$  of all isotone transformations and of the semigroup  $PO_n$  of all isotone partial transformations (excluding the permutation in both cases) were established by Aizenštat ([1]) in 1962 and by Popova ([16]), respectively, in the same year. Some years later (1971), Howie ([14]) studied some combinatorial and algebraic properties of  $O_n$  and, in 1992, Gomes and Howie ([13]) established some more properties of  $O_n$ , namely its rank and idempotent rank. In recent years it has been studied in different aspects by several authors (for example [4, 15, 17, 18]). The monoid  $IO_n$  of all isotone partial injections of  $X_n$  has been the object of study since 1997 by Fernandes in various papers ([7, 8, 9]). Some basic properties of  $IO_n$ , in particular, a description of Green's relations, congruences and a presentation, were obtained in [2]. Ganyushkin and Mazorchuk ([12]) studied some properties of  $IO_n$  as describe ideals, systems of generators, maximal subsemigroups and maximal inverse subsemigroups of  $IO_n$ .

In [10], Fernandes, Gomes and Jesus gave a presentation of both the semigroups  $M_n$  of all monotone transformations of  $X_n$  and the semigroup  $PM_n$  of all monotone partial transformations. Dimitrova and Koppitz ([4]) considered the maximal subsemigroups of  $M_n$  and its ideals. Delgado and Fernandes ([3]) have computed the abelian kernels of the semigroup  $IM_n$ . Fernandes, Gomes and Jesus ([11]) exhibited some properties as well as a presentation for the semigroup  $IM_n$ . Dimitrova and Koppitz ([5]) characterized the maximal subsemigroups of  $IM_n$ .

In this paper we consider the ideals of the semigroups  $IO_n$  and  $IM_n$ . In Section 2 we describe the maximal subsemigroups of the ideals of the semigroup  $IO_n$ . Each of the considered ideals has exactly  $2^{\binom{n}{r}} - 2$  maximal subsemigroups. In Section 3 we characterize the maximal subsemigroups of the ideals of the semigroup  $IM_n$ . It happens that each of the considered ideals has exactly  $2^{\binom{n}{r}+1} - 3$  maximal subsemigroups.

We will try to keep the standard notation. For every partial transformation  $\alpha$  by  $\text{dom } \alpha$  and  $\text{im } \alpha$  we denote the domain and the image of  $\alpha$ , respectively. If  $\alpha$  is injective, the number  $\text{rank } \alpha := |\text{dom } \alpha| = |\text{im } \alpha|$  is called the rank of  $\alpha$ . Clearly,  $\text{rank } \alpha\beta \leq \min\{\text{rank } \alpha, \text{rank } \beta\}$  and  $\text{im } \beta = \text{im } \alpha\beta$  as well as  $\text{dom } \alpha = \text{dom } \alpha\beta$  if  $\text{im } \alpha = \text{dom } \beta$ . From the definition of isotone and antitone transformation, it follows that every element  $\alpha \in IM_n$  is uniquely determined by  $\text{dom } \alpha$  and  $\text{im } \alpha$  satisfying  $|\text{dom } \alpha| = |\text{im } \alpha|$ . Moreover, for every  $A, B \subset X_n$  of the same cardinality there exists one isotone transformation  $\alpha \in IO_n \subseteq IM_n$  and one antitone transformation  $\beta \in IM_n$  such that  $\text{dom } \alpha = \text{dom } \beta = A$  and  $\text{im } \alpha = \text{im } \beta = B$ . We will denote by  $\alpha_{A,B}$  the unique isotone element  $\alpha \in IM_n$  for which  $A = \text{dom } \alpha$  and  $B = \text{im } \alpha$ , and by  $\beta_{A,B}$  the unique antitone element  $\beta \in IM_n$  for which  $A = \text{dom } \beta$  and  $B = \text{im } \beta$ . The elements  $\alpha_{A,A}$ ,  $A \in X_n$ , exhaust all idempotents in  $IO_n$  as well as in  $IM_n$ . For the elements  $\beta_{A,A}$ , we have  $\beta_{A,A}^2 = \alpha_{A,A}$ . In case  $A = B = X_n$ , we will use the notations  $\alpha_n$  and  $\beta_n$  instead of  $\alpha_{X_n, X_n}$  and  $\beta_{X_n, X_n}$ .

The Green's relations  $\mathcal{L}$ ,  $\mathcal{R}$ ,  $\mathcal{J}$  and  $\mathcal{H}$  on  $IO_n$  as well as on  $IM_n$  are characterized as follows:

$$\alpha \mathcal{L} \beta \iff \text{im } \alpha = \text{im } \beta$$

$$\alpha \mathcal{R} \beta \iff \text{dom } \alpha = \text{dom } \beta$$

$$\alpha \mathcal{J} \beta \iff \text{rank } \alpha = \text{rank } \beta$$

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

Obviously, every  $\mathcal{H}$ -class in  $IO_n$  contains exactly one element and every  $\mathcal{H}$ -class in  $IM_n \setminus \{\alpha \in IM_n : \text{rank } \alpha \leq 1\}$  contains exactly two elements. In the set  $\{\alpha \in IM_n : \text{rank } \alpha \leq 1\}$ , every  $\mathcal{H}$ -class contains exactly one element.

2. MAXIMAL SUBSEMIGROUPS OF THE IDEALS OF  $IO_n$ 

The semigroup  $IO_n$  is the union of the  $\mathcal{J}$ -classes  $J_0, J_1, \dots, J_n$ , where

$$J_r := \{\alpha \in IO_n : \text{rank } \alpha = r\} \text{ for } r = 0, \dots, n.$$

It is well known that the ideal  $I(n, r)$  ( $r = 0, \dots, n$ ) of the semigroup  $IO_n$  is the union of  $\mathcal{J}$ -classes  $J_0, J_1, \dots, J_r$ , i.e.

$$I(n, r) = \{\alpha \in IO_n : \text{rank } \alpha \leq r\}.$$

Every principal factor on  $IO_n$  is a Rees quotient  $I(n, r)/I(n, r-1)$  ( $1 \leq r \leq n$ ) of which we think as  $J_r \cup \{0\}$  (as it is usually convenient), where the product of two elements of  $J_r$  is taken to be zero if it falls into  $I(n, r-1)$ .

Let us denote by  $\Lambda_r$  the collection of all subsets of  $X_n$  of cardinality  $r$ .

The  $\mathcal{R}$  -,  $\mathcal{L}$  - and  $\mathcal{H}$  - classes in  $J_r$  have the following form:

$$R_A := \{\alpha \in I(n, r) : \text{dom } \alpha = A\}, \quad A \in \Lambda_r;$$

$$L_B := \{\alpha \in I(n, r) : \text{im } \alpha = B\}, \quad B \in \Lambda_r;$$

$$H_{A,B} := \{\alpha_{A,B}\} = R_A \cap L_B, \quad A, B \in \Lambda_r.$$

Clearly, each  $R_A$  - class ( $L_A$  - class),  $A \in \Lambda_r$  contains exactly one idempotent  $\alpha_{A,A}$ . Thus if  $E_r$  is the set of all idempotents in the class  $J_r$ , then  $|E_r| = \binom{n}{r}$ .

Since the product  $\alpha\beta$  for all  $\alpha, \beta \in J_r$  belongs to the class  $J_r$  if and only if  $\text{im } \alpha = \text{dom } \beta$ , it is obvious that

**Lemma 1.**

$$1. \quad L_B R_A = \begin{cases} J_r, & \text{if } A = B, \\ 0, & \text{if } A \neq B. \end{cases}$$

$$2. \quad \alpha_{A,B} \alpha_{C,D} = \begin{cases} \alpha_{A,D}, & \text{if } B = C, \\ 0, & \text{if } B \neq C. \end{cases}$$

**Proposition 1** [7].  $\langle J_r \rangle = I(n, r)$ , for  $0 \leq r \leq n-1$ .

Now we begin with the description of the maximal subsemigroups of the ideals of the semigroup  $IO_n$ .

Let us denote by  $Dec(\Lambda_r)$  the set of all decompositions  $(N_1, N_2)$  of  $\Lambda_r$ , i.e.  $N_1 \cup N_2 = \Lambda_r$  and  $N_1 \cap N_2 = \emptyset$  where  $N_1, N_2 \neq \emptyset$ .

**Definition 1.** Let  $(N_1, N_2) \in Dec(\Lambda_r)$  ( $r = 1, \dots, n-1$ ). Then we put

$$S_{(N_1, N_2)} := I(n, r-1) \cup \{\alpha_{A,B} : A \in N_1 \text{ or } B \in N_2\}.$$

The maximal subsemigroups of the ideal  $I(n, n) = IO_n$  were described by Ganyushkin and Mazorchuk:

**Theorem 1** [12]. *A subsemigroup  $S$  of  $IO_n$  is maximal if and only if  $S = I(n, n-1)$  or  $S = \{\alpha_n\} \cup S_{(N_1, N_2)}$ , where  $(N_1, N_2) \in Dec(\Lambda_{n-1})$ .*

In the following, we will consider the maximal subsemigroups of the ideals  $I(n, r)$  for  $r = 1, \dots, n-1$ .

**Lemma 2.** *Every maximal subsemigroup in  $I(n, r)$  contains the ideal  $I(n, r-1)$ .*

**Proof.** Let  $S$  be a maximal subsemigroup of  $I(n, r)$ . Assume that  $J_r \subseteq S$ , then according to Proposition 1 it follows that  $I(n, r) = \langle J_r \rangle \subseteq S$ , i.e.  $S = I(n, r)$ , a contradiction. Thus  $J_r \not\subseteq S$ . Then  $S \cup I(n, r-1)$  is a proper subsemigroup of  $I(n, r)$  since  $I(n, r-1)$  is an ideal, and hence  $S \cup I(n, r-1) = S$  by maximality of  $S$ . This implies  $I(n, r-1) \subseteq S$ . ■

**Theorem 2.** *Let  $1 \leq r \leq n-1$ . Then a subsemigroup  $S$  of  $I(n, r)$  is maximal if and only if there is an element  $(N_1, N_2) \in Dec(\Lambda_r)$  with  $S = S_{(N_1, N_2)}$ .*

**Proof.** Let  $S = S_{(N_1, N_2)}$  for some  $(N_1, N_2) \in Dec(\Lambda_r)$ . Then

$$S = I(n, r-1) \cup \{\alpha_{A,B} : A \in N_1 \text{ or } B \in N_2\}.$$

Therefore, if  $\alpha_{A,B} \notin S$  then  $A \in N_2$  and  $B \in N_1$ , and thus  $\alpha_{B,A} \in S$ .

From Lemma 1 it follows that  $S$  is a semigroup. Really, let  $\alpha_{A,B}, \alpha_{C,D} \in S$ , i.e.  $A, C \in N_1$  or  $B, D \in N_2$  or  $A \in N_1, D \in N_2$ . Then we have  $\alpha_{A,B}\alpha_{C,D} = \alpha_{A,D} \in S$  for  $B = C$  and  $\alpha_{A,B}\alpha_{C,D} = 0 \in I(n, r-1) \subseteq S$  for  $B \neq C$ .

Now we will show that  $S$  is maximal. Let  $\alpha_{C,D} \in I(n, r) \setminus S$ , i.e.  $C \notin N_1$  and  $D \notin N_2$ . Then  $D \in N_1$ , since  $N_1 \cup N_2 = \Lambda_r$  and so  $\alpha_{D,P} \in S$  for all  $P \in \Lambda_r$  and thus  $R_D = \{\alpha_{D,P} : P \in \Lambda_r\} \subseteq S$ . Moreover, we have  $\alpha_{C,P} = \alpha_{C,D}\alpha_{D,P}$ , for all  $P \in \Lambda_r$ , by Lemma 1. Thus we obtain the  $\mathcal{R}$ -class  $R_C = \{\alpha_{C,P} : P \in \Lambda_r\} \subseteq \langle S \cup \{\alpha_{C,D}\} \rangle$ . Moreover,  $C \in N_2$  and so  $L_C = \{\alpha_{P,C} : P \in \Lambda_r\} \subseteq S$ . Using Lemma 1, we have  $L_C R_C = J_r \subseteq \langle S \cup \{\alpha_{C,D}\} \rangle$ . Thus, we obtain that  $\langle S \cup \{\alpha_{C,D}\} \rangle = I(n, r)$ . Therefore,  $S$  is a maximal subsemigroup of the ideal  $I(n, r)$ .

For the converse part let  $S$  be a maximal subsemigroup of the ideal  $I(n, r)$ . From Lemma 2, we have that  $I(n, r-1) \subseteq S$ . Then  $S = I(n, r-1) \cup T$ , where  $T \subseteq J_r$ .

Let  $\alpha_{A,B} \notin S$ . Then  $\langle S \cup \{\alpha_{A,B}\} \rangle = I(n, r)$ . Let now  $P, Q \in \Lambda_r$ . Suppose that  $\alpha_{P,Q} \notin S$ . Then  $\alpha_{P,Q} \in \langle S \cup \{\alpha_{A,B}\} \rangle$  and  $\alpha_{P,Q} = \alpha_{P,A}\alpha_{A,B}\alpha_{B,Q}$ . Moreover,  $\alpha_{P,A} = \alpha_{P,A}\alpha_{A,B}\alpha_{B,A}$  and  $\alpha_{B,Q} = \alpha_{B,A}\alpha_{A,B}\alpha_{B,Q}$ . This shows that we need  $\alpha_{P,A}$  and  $\alpha_{B,Q}$  to generate  $\alpha_{P,A}$  and  $\alpha_{B,Q}$ , respectively, with elements of  $S \cup \{\alpha_{A,B}\}$ . Hence  $\alpha_{P,A}, \alpha_{B,Q} \in S$ .

Assume that  $\alpha_{Q,P} \notin S$ . Then  $\alpha_{Q,P} = \alpha_{Q,A}\alpha_{A,B}\alpha_{B,P}$  and by the same arguments, we obtain that  $\alpha_{Q,A}, \alpha_{B,P} \in S$ .

Further, from  $\alpha_{Q,P} = \alpha_{Q,A}\alpha_{A,P}$  it follows that  $\alpha_{A,P} \notin S$ . But  $\alpha_{P,Q} \notin \langle S \cup \{\alpha_{A,P}\} \rangle$  since  $\alpha_{P,Q} = \alpha_{P,A}\alpha_{A,P}\alpha_{P,Q}$ . This contradicts the maximality of  $S$  and thus  $\alpha_{Q,P} \in S$ . Hence if  $\alpha_{P,Q} \notin S$  then  $\alpha_{Q,P} \in S$  for any  $P, Q \in \Lambda_r$ . Therefore, for  $N_1 = \{B : \alpha_{A,B} \notin S\}$  and  $N_2 = \{A : \alpha_{A,B} \notin S\}$  we have that  $S = S_{(N_1, N_2)}$ . ■

There are exactly  $2^{\binom{n}{r}} - 2$  maximal subsemigroups of the ideal  $I(n, r)$ , for  $r = 1, \dots, n-1$  and  $2^n - 1$  maximal subsemigroups of  $I(n, n)$ .

### 3. MAXIMAL SUBSEMIGROUPS OF THE IDEALS OF $IM_n$

The semigroup  $IM_n$  is the union of the  $\mathcal{J}$ -classes  $J_0, J_1, \dots, J_n$ , where

$$J_r := \{\alpha \in IM_n : \text{rank } \alpha = r\} \text{ for } r = 0, \dots, n.$$

It is well known that the ideal  $I(n, r)$  ( $r = 0, \dots, n$ ) of the semigroup  $IM_n$  is the union of  $\mathcal{J}$ -classes  $J_0, J_1, \dots, J_r$ , i.e.

$$I(n, r) = \{\alpha \in IM_n : \text{rank } \alpha \leq r\}.$$

Every principal factor on  $IM_n$  is a Rees quotient  $I(n, r)/I(n, r-1)$  ( $1 \leq r \leq n$ ) of which we think as  $J_r \cup \{0\}$ , where the product of two elements of  $J_r$  is taken to be zero if it falls into  $I(n, r-1)$ .

The  $\mathcal{R}$  -,  $\mathcal{L}$  - and  $\mathcal{H}$  - classes in  $J_r$  have the following form:

$$R_A := \{\alpha \in I(n, r) : \text{dom } \alpha = A\}, \quad A \in \Lambda_r;$$

$$L_B := \{\alpha \in I(n, r) : \text{im } \alpha = B\}, \quad B \in \Lambda_r;$$

$$H_{A,B} := \{\alpha_{A,B}, \beta_{A,B}\} = R_A \cap L_B, \quad A, B \in \Lambda_r.$$

The  $\mathcal{L}$ -class,  $\mathcal{R}$ -class and  $\mathcal{H}$ -class, respectively, containing the element  $\alpha \in IM_n$  will be denoted by  $L_\alpha$ ,  $R_\alpha$ , and  $H_\alpha$ , respectively.

Since the product  $\alpha\beta$  for all  $\alpha, \beta \in J_r$  belongs to the class  $J_r$  if and only if  $\text{im } \alpha = \text{dom } \beta$ , it is easy to show that

**Lemma 3.**

1.  $L_B R_A = \begin{cases} J_r, & \text{if } A = B, \\ 0, & \text{if } A \neq B. \end{cases}$
2.  $H_{A,B} H_{C,D} = \begin{cases} H_{A,D}, & \text{if } B = C, \\ 0, & \text{if } B \neq C. \end{cases}$

Let  $U$  be a subset of the semigroup  $IM_n$ . We denote by  $U^i$  (respectively  $U^a$ ) the set of all isotone (respectively antitone) transformations in the set  $U$ . An immediate but important property is that the product of two isotone transformations or two antitone transformations is an isotone, and the product of an isotone transformation with an antitone transformation, or vice versa, is an antitone one.

**Proposition 2.**  $J_r \subseteq \langle J_r^a \rangle$  and  $J_r \subseteq \langle J_r^i \cup \{\beta_{A,B}\} \rangle$ , for all  $A, B \in \Lambda_r$ .

**Proof.** Let  $A, B \in \Lambda_r$ . Then for all  $C \in \Lambda_r$ , we have  $\alpha_{A,B} = \beta_{A,C}\beta_{C,B}$ . Therefore,  $J_r \subseteq \langle J_r^a \rangle$ .

From  $L_A^i \beta_{A,B} = L_B^a$  and  $L_B^a R_B^i = J_r^a$ , we have  $J_r \subseteq \langle J_r^i \cup \{\beta_{A,B}\} \rangle$ . ■

**Proposition 3.**  $\langle J_r \rangle = I(n, r)$ , for  $0 \leq r \leq n - 1$ .

**Proof.** Clearly  $\langle J_0 \rangle = I(n, 0)$ . In [5], it was shown that  $J_{r-1}^i \subseteq J_r^i J_r^i$  and  $J_{r-1}^a \subseteq J_{r-1}^i J_r^a J_{r-1}^i$  for  $1 \leq r \leq n - 1$ . Since  $I(n, r) = J_0 \cup J_1 \cup \dots \cup J_r$ , we have  $\langle J_r \rangle = I(n, r)$ . ■

From Proposition 2 and Proposition 3 we have

**Corollary 1.** Let  $1 \leq r \leq n - 1$ . Then  $\langle J_r^a \rangle = \langle J_r^i \cup \{\beta_{A,B}\} \rangle = I(n, r)$ , for all  $A, B \in \Lambda_r$ .

Now we begin with the description of the maximal subsemigroups of the ideals of the semigroup  $IM_n$ .

Clearly, the ideal  $I(n, 1)$  of  $IM_n$  coincides with the ideal  $I(n, 1)$  of  $IO_n$ . Thus the maximal subsemigroups of this ideal are characterized in Theorem 2 and there are exactly  $2^n - 2$  such semigroups.

Now we will consider the maximal subsemigroups of the ideals  $I(n, r)$  for  $r = 2, \dots, n - 1$ .

**Lemma 4.** Every maximal subsemigroup in  $I(n, r)$  contains the ideal  $I(n, r - 1)$ .

The proof is similar as that in Lemma 2.

**Theorem 3** Let  $2 \leq r \leq n - 1$ . Then a subsemigroup  $S$  of  $I(n, r)$  is maximal if and only if it belongs to one of the following three types:

- (1)  $S^{(1)} := I(n, r - 1) \cup J_r^i$ ;
- (2)  $S_{(N_1, N_2)}^{(2)} := \bigcup \{H_\alpha : \alpha \in S_{(N_1, N_2)}\}$ , for  $(N_1, N_2) \in \text{Dec}(\Lambda_r)$ ;
- (3)  $S_{(N_1, N_2)}^{(3)} := I(n, r - 1) \cup \{\alpha_{A,B} : A, B \in N_1 \text{ or } A, B \in N_2\} \cup$   
 $\cup \{\beta_{A,B} : A \in N_1, B \in N_2 \text{ or } A \in N_2, B \in N_1\}$  for  $(N_1, N_2) \in \text{Dec}(\Lambda_r)$ .



**Proof.**

(1) It is obvious that  $S^{(1)} = I(n, r-1) \cup J_r^i$  is a semigroup, since  $I(n, r-1)$  is an ideal and  $(J_r^i)^2 \subseteq I^i(n, r) \subseteq I(n, r-1) \cup J_r^i$ . From Proposition 2, we have that  $J_r \subseteq \langle J_r^i \cup \{\beta_{A,B}\} \rangle$  for all  $\beta_{A,B} \in J_r^a$ . Since  $I(n, r) \setminus S^{(1)} = J_r^a$ , we obtain  $\langle S^{(1)} \cup \{\beta_{A,B}\} \rangle = I(n, r)$  for all  $\beta_{A,B} \in J_r^a$ . Therefore,  $S^{(1)}$  is maximal in  $I(n, r)$ .

(2) Let  $S = S_{(N_1, N_2)}^{(2)}$  for some  $(N_1, N_2) \in Dec(\Lambda_r)$ . Then

$$S = I(n, r-1) \cup \{H_{A,B} : A \in N_1 \text{ or } B \in N_2\}.$$

From Lemma 3 it follows that  $S$  is a semigroup. Really, let  $H_{A,B}, H_{C,D} \subseteq S$ , i.e.  $A, C \in N_1$  or  $B, D \in N_2$  or  $A \in N_1, D \in N_2$ . Then we have  $H_{A,B}H_{C,D} = H_{A,D} \subseteq S$  for  $B = C$  and  $H_{A,B}H_{C,D} \subseteq I(n, r-1) \subseteq S$  for  $B \neq C$ .

Now we will show that  $S$  is maximal. Let  $H_{C,D} = \{\alpha_{C,D}, \beta_{C,D}\} \subseteq I(n, r) \setminus S$ , i.e.  $C \notin N_1$  and  $D \notin N_2$ . Then  $D \in N_1$ , since  $N_1 \cup N_2 = \Lambda_r$  and so  $H_{D,P} \in S$  for all  $P \in \Lambda_r$  and thus  $R_D = \bigcup_{P \in \Lambda_r} H_{D,P} \subseteq S$ . Moreover, we have

$$H_{C,P} = H_{C,D}H_{D,P}, \text{ for } P \in \Lambda_r,$$

by Lemma 3. Thus we obtain the  $\mathcal{R}$ -class  $R_C = \bigcup_{P \in \Lambda_r} H_{C,P} \subseteq \langle S \cup H_{C,D} \rangle$ . Moreover,  $C \in N_2$  and so  $L_C = \bigcup_{P \in \Lambda_r} H_{P,C} \subseteq S$ . Using Lemma 3, we have  $L_C R_C = J_r \subseteq \langle S \cup H_{C,D} \rangle$ . Since  $\alpha_{C,D} = \beta_{C,D}\beta_{D,D}$  and  $\beta_{C,D} = \alpha_{C,D}\beta_{D,D}$ , where  $\beta_{D,D} \in R_D \subseteq S$ , we obtain that  $\langle S \cup \{\alpha_{C,D}\} \rangle = I(n, r)$  and  $\langle S \cup \{\beta_{C,D}\} \rangle = I(n, r)$ . Therefore,  $S$  is a maximal subsemigroup of the ideal  $I(n, r)$ .

(3) Let  $S = S_{(N_1, N_2)}^{(3)}$  for some  $(N_1, N_2) \in Dec(\Lambda_r)$ . From Lemma 3, it follows that  $S$  is a semigroup. We will show that  $S$  is maximal. Let

$$\begin{aligned} V := I(n, r) \setminus S = & \{\beta_{A,B} : A, B \in N_1 \text{ or } A, B \in N_2\} \cup \\ & \cup \{\alpha_{A,B} : A \in N_1, B \in N_2 \text{ or } A \in N_2, B \in N_1\} \end{aligned}$$

and let  $\gamma \in V$ . Then for the transformation  $\gamma$  we have four possibilities:

Let  $\gamma \in \{\beta_{A,B} : A, B \in N_1\}$ . Then  $\alpha_{C,A} \in S$  (since  $A \in N_1$ ) and so  $\alpha_{C,A}\beta_{A,B} = \beta_{C,B} \in \langle S \cup \{\gamma\} \rangle$  for all  $C \in N_1$ . Also, we have  $\beta_{C,A} \in S$  and thus  $\beta_{C,A}\beta_{A,B} = \alpha_{C,B} \in \langle S \cup \{\gamma\} \rangle$  for all  $C \in N_2$ . Since  $\alpha_{C,B} \in S$  for all  $C \in N_1$  and  $\beta_{C,B} \in S$  for all  $C \in N_2$ , we obtain  $L_B = \bigcup_{C \in \Lambda_r} H_{C,B} \subseteq \langle S \cup \{\gamma\} \rangle$ . Further,  $\beta_{B,B} \in L_B$  and  $\beta_{B,B}\beta_{B,D} = \alpha_{B,D}$  for all  $D \in N_2$  as well as  $\beta_{B,B}\alpha_{B,D} = \beta_{B,D}$  for all  $D \in N_1$ . Thus since  $\alpha_{B,D} \in S$  for all  $D \in N_1$  and  $\beta_{B,D} \in S$  for all  $D \in N_2$ , we obtain  $R_B = \bigcup_{D \in \Lambda_r} H_{B,D} \subseteq \langle S \cup \{\gamma\} \rangle$ . From Lemma 3, we have  $L_B R_B = J_r$  and therefore  $\langle S \cup \{\gamma\} \rangle = I(n, r)$ .

– For  $\gamma \in \{\beta_{A,B} : A, B \in N_2\}$ , the proof is similar.

– Let  $\gamma \in \{\alpha_{A,B} : A \in N_1, B \in N_2\}$ . Then  $\alpha_{C,A} \in S$  (since  $A \in N_1$ ) and so  $\alpha_{C,A}\alpha_{A,B} = \alpha_{C,B} \in \langle S \cup \{\gamma\} \rangle$  for all  $C \in N_1$ . Also, we have  $\beta_{C,A} \in S$  and thus  $\beta_{C,A}\alpha_{A,B} = \beta_{C,B} \in \langle S \cup \{\gamma\} \rangle$  for all  $C \in N_2$ . Since  $\alpha_{C,B} \in S$  for all  $C \in N_2$  and  $\beta_{C,B} \in S$  for all  $C \in N_1$ , we obtain  $L_B = \bigcup_{C \in \Lambda_r} H_{C,B} \subseteq \langle S \cup \{\gamma\} \rangle$ . Further,  $\beta_{B,B} \in L_B$  and  $\beta_{B,B}\alpha_{B,D} = \beta_{B,D}$  for all  $D \in N_2$  as well as  $\beta_{B,B}\beta_{B,D} = \alpha_{B,D}$  for all  $D \in N_1$ . Thus since  $\alpha_{B,D} \in S$  for all  $D \in N_2$  and  $\beta_{B,D} \in S$  for all  $D \in N_1$ , we obtain  $R_B = \bigcup_{D \in \Lambda_r} H_{B,D} \subseteq \langle S \cup \{\gamma\} \rangle$ . From Lemma 3, we have  $L_B R_B = J_r$  and therefore  $\langle S \cup \{\gamma\} \rangle = I(n, r)$ .

– For  $\gamma \in \{\alpha_{A,B} : A \in N_2, B \in N_1\}$ , the proof is similar.

Altogether, this shows that  $S$  is maximal.

For the converse part let  $S$  be a maximal subsemigroup of the ideal  $I(n, r)$ . From Lemma 4, we have that  $I(n, r-1) \subseteq S$ . Then  $S = I(n, r-1) \cup T$ , where  $T \subseteq J_r$ . We consider two cases for the set  $T$ .

1. Let  $T = J_r^i$ . Then  $S = I(n, r-1) \cup J_r^i = S^{(1)}$ .
2. Let now  $T \neq J_r^i$ . Assume that  $J_r^i \subseteq T$ . Then  $T = J_r^i \cup T'$  where  $\emptyset \neq T' \subseteq J_r^a$ . From Corollary 1, we have  $S = I(n, r)$ , a contradiction. Thus  $J_r^i \not\subseteq T$ . We also have that  $J_r^a \not\subseteq T$  since  $\langle J_r^a \rangle = I(n, r)$ .

Admit that  $H_{A,B} \subseteq S$  or  $H_{A,B} \cap S = \emptyset$ , for all  $A, B \in \Lambda_r$ . Assume that  $S^i = S \cap I^i(n, r)$  is not a maximal subsemigroup of  $I^i(n, r)$ . Then there is an isotone transformation  $\alpha_{A,B} \in I(n, r) \setminus S$  such that  $\langle S^i \cup \{\alpha_{A,B}\} \rangle$  is a proper subset of  $I^i(n, r)$ . Therefore, there exists an  $\alpha_{C,D} \in I(n, r) \setminus S$  such that  $\alpha_{C,D} \notin \langle S^i \cup \{\alpha_{A,B}\} \rangle$ . But  $\langle S \cup \{\alpha_{A,B}\} \rangle = I(n, r)$  since  $S$  is maximal and  $\alpha_{C,D} = \beta_{C,A}\alpha_{A,B}\beta_{B,D}$ . Moreover,  $\beta_{C,A} = \beta_{C,A}\alpha_{A,B}\alpha_{B,A} = \alpha_{C,A}\alpha_{A,B}\beta_{B,A}$  and  $\beta_{B,D} = \beta_{B,A}\alpha_{A,B}\alpha_{B,D} = \alpha_{B,A}\alpha_{A,B}\beta_{B,D}$ . This shows that we need

$\beta_{C,A}$  or  $\alpha_{C,A}$  and  $\beta_{B,D}$  or  $\alpha_{B,D}$  to generate  $\beta_{C,A}$  and  $\beta_{B,D}$ , respectively, with elements of  $S \cup \{\alpha_{A,B}\}$ . This implies that  $\beta_{C,A}, \alpha_{C,A}, \beta_{B,D}, \alpha_{B,D} \in S$ , since we assume that  $H_{A,B} \subseteq S$  or  $H_{A,B} \cap S = \emptyset$ , for all  $A, B \in \Lambda_r$ . Hence  $\alpha_{C,D} = \alpha_{C,A}\alpha_{A,B}\alpha_{B,D} \in \langle S^i \cup \{\alpha_{A,B}\} \rangle$ , a contradiction. Therefore, we obtain that  $S^i$  is maximal in  $I^i(n, r)$ . Since all maximal subsemigroups of the ideal  $I^i(n, r)$  are of type  $S_{(N_1, N_2)}$  we have  $S = \cup\{H_\alpha : \alpha \in S^i\} = S_{(N_1, N_2)}^{(2)}$ , for some  $(N_1, N_2) \in Dec(\Lambda_r)$ .

Now, admit that  $|H_{A,B} \cap S| = 1$ , for some  $A, B \in \Lambda_r$ . Suppose that  $\alpha_{A,B} \notin S$  and  $\beta_{A,B} \in S$ . Then from  $\alpha_{A,B} = \beta_{A,B}\beta_{B,B}$  and  $\alpha_{A,B} = \beta_{A,A}\beta_{A,B}$ , it follows that  $\beta_{A,A}, \beta_{B,B} \notin S$ . Moreover, from  $\beta_{A,B}\alpha_{B,A} = \beta_{A,A} \notin S$ , we get  $\alpha_{B,A} \notin S$ . Assume that  $\beta_{B,A} \notin S$ . Then  $\beta_{B,A} \in \langle S \cup \{\alpha_{B,A}\} \rangle$ , because of the maximality of  $S$ , and since  $\beta_{B,A} = \beta_{B,B}\alpha_{B,A}\alpha_{A,A} = \alpha_{B,B}\alpha_{B,A}\beta_{A,A}$ , we obtain  $\beta_{A,A} \in S$  or  $\beta_{B,B} \in S$ , a contradiction, and thus  $\beta_{B,A} \in S$ .

Further, let  $P, Q \in \Lambda_r$ . Suppose that  $\alpha_{P,Q} \notin S$ . Then from  $\alpha_{P,Q} = \alpha_{P,A}\beta_{A,B}\beta_{B,Q}$ , it follows that if  $\alpha_{P,A} \in S$  then  $\beta_{B,Q} \notin S$  and vice versa. Also from  $\alpha_{P,Q} = \beta_{P,A}\beta_{A,B}\alpha_{B,Q}$ , it follows that if  $\beta_{P,A} \in S$  then  $\alpha_{B,Q} \notin S$  and vice versa. Moreover,  $\alpha_{P,Q} \in \langle S \cup \{\alpha_{A,B}\} \rangle$  since  $S$  is maximal. Hence  $\alpha_{P,Q} = \alpha_{P,A}\alpha_{A,B}\alpha_{B,Q} = \beta_{P,A}\alpha_{A,B}\beta_{B,Q}$ . Therefore, we have  $\alpha_{P,A}, \alpha_{B,Q} \in S$  and  $\beta_{P,A}, \beta_{B,Q} \notin S$  or vice versa.

Assume that  $\beta_{P,Q} \notin S$ . Then  $\beta_{P,Q} \in \langle S \cup \{\alpha_{A,B}\} \rangle$  and so  $\beta_{P,Q} = \alpha_{P,A}\alpha_{A,B}\beta_{B,Q} = \beta_{P,A}\alpha_{A,B}\alpha_{B,Q}$ . But we obtain already that if  $\alpha_{P,A}, \alpha_{B,Q} \in S$  then  $\beta_{P,A}, \beta_{B,Q} \notin S$  or vice versa. Therefore,  $\beta_{P,Q} \notin \langle S \cup \{\alpha_{A,B}\} \rangle$ . This contradicts the maximality of  $S$  and thus  $\beta_{P,Q} \in S$ .

Further, from  $\alpha_{P,Q} = \beta_{P,Q}\beta_{Q,Q}$  and  $\alpha_{P,Q} = \beta_{P,P}\beta_{P,Q}$ , it follows that  $\beta_{P,P}, \beta_{Q,Q} \notin S$ . Moreover, from  $\beta_{P,Q}\alpha_{Q,P} = \beta_{P,P} \notin S$ , we get  $\alpha_{Q,P} \notin S$ . Assume that  $\beta_{Q,P} \notin S$ . Then  $\beta_{Q,P} \in \langle S \cup \{\alpha_{Q,P}\} \rangle$ , because of the maximality of  $S$ , and since  $\beta_{Q,P} = \beta_{Q,Q}\alpha_{Q,P}\alpha_{P,P} = \alpha_{Q,Q}\alpha_{Q,P}\beta_{P,P}$ , we obtain  $\beta_{P,P} \in S$  or  $\beta_{Q,Q} \in S$ , a contradiction, and thus  $\beta_{Q,P} \in S$ .

Analogously, if  $\beta_{P,Q} \notin S$  we have that  $\beta_{Q,P} \notin S$  and  $\alpha_{P,Q}, \alpha_{Q,P} \in S$ .

Suppose that  $\alpha_{P,Q} \in S$  for some  $P, Q \in \Lambda_r$ . Then  $\beta_{P,Q} \notin S$ . Otherwise, from  $\alpha_{A,B} = \alpha_{A,P}\beta_{P,Q}\beta_{Q,B} \notin S$  it follows

- i)  $\alpha_{A,P} \notin S$  and  $\beta_{Q,B} \in S$ , i.e.  $\beta_{A,P} \in S$  and  $\beta_{Q,B} \in S$ ;
- ii)  $\alpha_{A,P} \in S$  and  $\beta_{Q,B} \notin S$ , i.e.  $\alpha_{A,P} \in S$  and  $\alpha_{Q,B} \in S$ ;
- iii)  $\alpha_{A,P} \notin S$  and  $\beta_{Q,B} \notin S$ , i.e.  $\beta_{A,P} \in S$  and  $\alpha_{Q,B} \in S$ .

Then  $\alpha_{A,B} = \beta_{A,P}\alpha_{P,Q}\beta_{Q,B} = \alpha_{A,P}\alpha_{P,Q}\alpha_{Q,B} = \beta_{A,P}\beta_{P,Q}\alpha_{Q,B}$ , which contradicts that  $\alpha_{A,B} \notin S$ .

The proof when  $\alpha_{A,B} \in S$  and  $\beta_{A,B} \notin S$  is similar.

Finally, we obtain that

$$(1) \quad \alpha_{P,Q} \in S \iff \beta_{P,Q} \notin S$$

for  $P, Q \in \Lambda_r$ .

Let  $\rho_r := \{(P, Q) : \alpha_{P,Q} \in S\}$ . Obviously,  $\rho_r$  is an equivalence relation on  $\Lambda_r$  with  $\Lambda_r/\rho_r = \{N_1, N_2, \dots, N_m\}$  ( $m \geq 2$ ). Indeed,  $\rho_r$  is reflexive since  $E_r \subseteq S$ , symmetric because of the previous considerations and transitive since  $\alpha_{P,Q}\alpha_{Q,R} = \alpha_{P,R} \in S$  for  $\alpha_{P,Q}, \alpha_{Q,R} \in S$ . Moreover,  $m \geq 2$  becomes clear by  $J_r^i \not\subseteq T$ . Assume that the decomposition contains more than two elements, i.e.  $m > 2$ . Then there are  $N_1, N_2, N_3$  in our decomposition such that  $A \in N_1, B \in N_2$  and  $C \in N_3$ . Thus  $\alpha_{A,B} = \beta_{A,C}\beta_{C,B} \in S$ , a contradiction. Therefore,  $\Lambda_r/\rho_r = \{N_1, N_2\}$  and  $S = S_{(N_1, N_2)}^{(3)}$ , because of (1).  $\blacksquare$

There are exactly  $2^{\binom{n}{r}} - 2$  maximal subsemigroups of the ideal  $I^i(n, r)$  and exactly  $2^{\binom{n}{r}} - 2$  maximal subsemigroups of type (3). Taking  $I(n, r-1) \cup J_r^i$  into account, we get  $2^{\binom{n}{r}+1} - 3$  maximal subsemigroups of the ideal  $I(n, r)$ , for  $r = 2, \dots, n-1$ .

Finally, we characterize the maximal subsemigroups of the ideal  $I(n, n) = IM_n$ .

For  $A \in \Lambda_{n-1}$  we put  $\bar{A} := \{n+1-i : i \in A\}$  and for  $N \subseteq \mathcal{P}(X_n)$  we set  $\bar{N} := \{\bar{A} : A \in N\}$ . Then we have

$$(2) \quad \begin{aligned} \beta_{A, \bar{A}} \alpha_{\bar{A}, B} &= \beta_n \alpha_{\bar{A}, B} = \beta_{A, B}, \\ \beta_{A, \bar{A}} \beta_{\bar{A}, B} &= \beta_n \beta_{\bar{A}, B} = \alpha_{A, B}, \\ \alpha_{B, A} \beta_{A, \bar{A}} &= \alpha_{B, A} \beta_n = \beta_{B, \bar{A}}, \\ \beta_{B, A} \beta_{A, \bar{A}} &= \beta_{B, A} \beta_n = \alpha_{B, \bar{A}}. \end{aligned}$$

**Theorem 4.** *A subsemigroup  $S$  of  $IM_n$  is maximal if and only if it belongs to one of the following three types:*

- (1)  $T := I(n, n-1) \cup \{\alpha_n\}$ ;
- (2)  $T_{(N_1, N_2)} := J_n \cup \{H_\alpha : \alpha \in S_{(N_1, N_2)}\}$ , for  $(N_1, N_2) \in Dec(\Lambda_{n-1})$   
with  $\bar{N}_1 = N_1$  and  $\bar{N}_2 = N_2$ ;
- (3)  $T_{(N, \bar{N})} := J_n \cup I(n, n-2) \cup \{\alpha_{A, B} : A, B \in N \text{ or } A, B \in \bar{N}\}$   
 $\cup \{\beta_{A, B} : A \in N, B \in \bar{N} \text{ or } A \in \bar{N}, B \in N\}$  for  $(N, \bar{N}) \in Dec(\Lambda_{n-1})$ .

**Proof.** It is clear that  $T$  is a maximal subsemigroup of  $IM_n$ . Further, we put

$$Inv := \{\beta_{A, \bar{A}} : A \in \Lambda_{n-1}\}.$$

Let  $(N_1, N_2) \in Dec(\Lambda_{n-1})$  be a decomposition with the required properties. Since  $Inv \subseteq T_{(N_1, N_2)}$  and by (2) it is easy to verify that  $T_{(N_1, N_2)}$  is a subsemigroup of  $IM_n$ . Since  $T_{(N_1, N_2)} \setminus J_n$  is a maximal subsemigroup of  $I(n, n-1)$  by Theorem 3 and  $J_n \subseteq T_{(N_1, N_2)}$ , it follows that  $T_{(N_1, N_2)}$  is a maximal subsemigroup of  $IM_n$ . Analogously, one can show that  $T_{(N, \bar{N})}$  is a maximal subsemigroup of  $IM_n$ .

For the converse part, let  $S$  be maximal in  $IM_n$ . Admit that  $J_n \not\subseteq S$ . Then it is easy to see that  $S = T$ . Now suppose that  $J_n \subseteq S$ . Assume that  $Inv \not\subseteq S$ . Then there is an  $A \in \Lambda_{n-1}$  with  $\beta_{A, \bar{A}} \notin S$ . Since  $S$  is maximal, we have  $IM_n = \langle S \cup \{\beta_{A, \bar{A}}\} \rangle = S \cup \{\beta_{A, \bar{A}}\}$  by (2). Thus  $S = IM_n \setminus \{\beta_{A, \bar{A}}\}$ . But  $\beta_{A, \bar{A}} = \alpha_{A, B} \beta_{B, \bar{A}}$  for some  $B \in \Lambda_{n-1}$  with  $B \neq A$ . Since  $\alpha_{A, B}, \beta_{B, \bar{A}} \in S$ , we have  $S = IM_n$ , a contradiction. Hence  $Inv \subseteq S$ . Let  $S_{n-1} := S \cap I(n, n-1)$ . Assume that  $S_{n-1}$  is not a maximal subsemigroup of  $I(n, n-1)$ . Clearly,  $S_{n-1} \neq I(n, n-1)$ . Let  $\gamma \in I(n, n-1) \setminus S_{n-1}$ . Then for all  $\delta \in I(n, n-1)$ , we have  $\delta \in \langle S \cup \{\gamma\} \rangle = \langle S_{n-1} \cup \{\gamma\} \rangle \cup J_n$  by (2) and since  $Inv \subseteq S$ . This shows that  $\delta \in \langle S_{n-1} \cup \{\gamma\} \rangle$  and thus  $\langle S_{n-1} \cup \{\gamma\} \rangle = I(n, n-1)$ . Consequently,  $S_{n-1}$  is a maximal subsemigroup of  $I(n, n-1)$ . Using Theorem 3 we choose all decompositions  $(N_1, N_2) \in Dec(\Lambda_{n-1})$  such that  $Inv \subseteq S_{(N_1, N_2)}^{(2)}$  and  $Inv \subseteq S_{(N_1, N_2)}^{(3)}$ , respectively. In this way we obtain the semigroups  $T_{(N_1, N_2)}$  and  $T_{(N, \bar{N})}$ . ■

It is straightforward to calculate that there are exactly  $2^{\frac{n+1}{2}} - 1$  maximal subsemigroups of  $IM_n$  if  $n$  is odd and exactly  $2^{\frac{n}{2}} - 1$  maximal subsemigroups of  $IM_n$  if  $n$  is even.

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