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ON THE SPECIAL CONTEXT OF INDEPENDENT SETS*

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Abstract

In this paper the context of independent sets $\mathcal{J}_L^p$ is assigned to the complete lattice $(\mathcal{P}(M), \subseteq)$ of all subsets of a non-empty set M. Some properties of this context, especially the irreducibility and the span, are investigated.

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Let us denote by $(L, \leq)$ a complete lattice in which $\vee, \wedge$ mean the supremum and the infimum of any subset of L, respectively. The least and the greatest elements in $(L, \leq)$ are denoted by 0, 1, respectively. If $a, b \in L$, then $a \| b$ means that a, b are incomparable in $(L, \leq)$.

For a subset $A \subseteq L$ we put $U(A) = \{x \in L \mid (\forall a \in A)[a \leq x]\}$ and $L(A) = \{x \in L \mid (\forall a \in A)[x \leq a]\}$. Obviously, $U(A) = U(\vee A)$ and $L(A) = L(\wedge A)$. Moreover, let us put $|A| := \operatorname{card} A$ and $A_a := A \setminus \{a\}$, $s_a := \vee A_a, i_a := \wedge A_a$ for all $a \in A$.

Definition 1 (F. Machala, [6]). A subset $A \subseteq L$ is said to be *join-independent* if and only if $a \nleq s_a$ for all $a \in A$. A subset $B \subseteq L$ is said to be *meet-independent* if and only if $i_b \nleq b$ for all $b \in B$.

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Remark 1. The concepts of join-independent and meet-independent sets are the special cases of the definition of independent sets in a context † (an incidence structure) or, more precisely, in two closure spaces associated to each context. Any complete lattice $(L, \leq)$ (and even any partially ordered set) can be understood as the context $(L, L, \leq)$, where (under the denotation established for contexts) $A^{\uparrow} = U(A), A^{\uparrow\downarrow} = LU(A)$ and $B^{\downarrow} = L(B), B^{\downarrow\uparrow} = UL(B)$ for $A, B \subseteq L$. The closure operators are given by $A \mapsto A^{\uparrow\downarrow}$, $B \mapsto B^{\downarrow\uparrow}$ for $A, B \subseteq L$.

The notion of an independent set in a lattice appears in various approaches in literature (see [1], [3], [4], [7], [9] and [10]). In fact, in this paper irredundant sets in complete lattices are discussed, but we prefer to use the terms "join-independent" and "meet-independent" with respect to connections with closure systems and incidence structures.

Remark 2. The notions of join- and meet-independent sets are dual in complete lattices. In the following we will only investigate join-independent sets. Analogous results for meet-independent sets can be obtained dually.

Propositions 1– 7 are easy consequences of the definitions of join- and meet-independencies. Thus, the proofs of them are omitted.

Proposition 1. *Every singleton* $A = \{a\}, a \neq 0, a \in L$*, is join-independent.*

Proposition 2. *A subset* $A \subseteq L, |A| \geq 2$*, is join-independent if and only if* $a||s_a$ *for all* $a \in A$.

Proposition 3. *If a subset* $A \subseteq L$ *is join-independent, then* $a||b$ *for all* $a, b \in A, a \neq b$.

Let us introduce one more denotation: If $A \subseteq L$, then for $a \in A$ we put $X^A(a) := U(s_a) \setminus U(a)$, $Y^A(a) := L(i_a) \setminus L(a)$.

Proposition 4. *If* $A \subseteq L$ *is join-independent, then* $X^A(a) \cap X^A(b) = \emptyset$ *for any* $a, b \in A, a \neq b$.

Proposition 5. *The subset* $A \subseteq L$ *is join-independent if and only if* $X^A(a) \neq \emptyset$ *for all* $a \in A$.

†A context is the triple (G, H, I), where G and H are sets and $I \subseteq G \times H$ (see [2]).

Proposition 6. *If the subset $A \subseteq L$ is join-independent, then every choice $Q^A = \{m_a \in X^A(a) \mid a \in A\}$ is a meet-independent set.*

Remark 3. Let $A \subseteq L$ be join-independent. Then for any choice $Q^A = \{m_a \in X^A(a) \mid a \in A\}$ the mapping $\alpha : a \mapsto m_a$ is a one-to-one mapping of the join-independent set A onto the meet-independent set Q^A. Analogously for a meet- independent subset. This is called *a norming mapping* of the set A (see [5]). If we denote by L_j^p (L_m^p) the set of all p-element join-independent (meet-independent) sets of $(L, \leq)$ (p is any cardinal number), then it is possible to define the *context of independent sets* $\mathcal{J}_L^p = (L_j^p, L_m^p, I^p)$, where the relation I^p is given by the following: For $A \in L_j^p, B \in L_m^p$ we put AI^pB if and only if there exists a norming mapping $\alpha : A \to B$. (If $L_j^p = \emptyset$, then $L_m^p = \emptyset$ and $\mathcal{J}_L^p = (\emptyset, \emptyset, \emptyset)$.) If $A \in L_j^p$, then obviously AI^pS_A where $S_A = \{s_a \mid a \in A\}$.

Proposition 7. *If a set $A \subseteq L$ is join-independent, then every subset of A is join-independent.*

Now we recall some basic notions from the general theory of contexts (see [8]):

Definition 2. Let $\mathcal{J} = (G, H, I)$ be a context. A sequence $(g_0, m_0, g_1, m_1, \ldots, g_{r-1}, m_{r-1}, g_r)$, where $g_i \in G$ for $i \in \{0, \ldots, r\}$, $m_j \in H$ for $j \in \{0, \ldots, r-1\}$ and $g_j I m_j, g_{j+1} I m_j$ for all $j \in \{0, \ldots, r-1\}$, is called a *path* between elements g_0 and g_r. In a similar way we can define a *path* between two elements of H.

A positive integer r is said to be a *length of a path* between elements g_0, g_r. We suppose that the path (g, m, g) has a length 0. If a path between two elements of G exists, then we say that they are *joinable*. The context $\mathcal{J}$ is said to be *irreducible* if every two elements of G are joinable. The minimal length of all paths between elements $g, h \in G$ we call a *distance* of these elements and denote by $v(g, h)$. The maximal distance of any two elements of G in an irreducible context $\mathcal{J}$ is said to be a *span* of G and denoted by $d(G)$. Similarly for the set H.

We will investigate the contexts of independent sets (their joinability, distances, irreducibility, spans) associated to the lattice $(\mathcal{P}(M), \subseteq)$ where $\mathcal{P}(M)$ denotes the power set of a non-empty set M. Thus $(\mathcal{P}(M), \subseteq)$ is the complete (boolean) lattice of all subsets of M.

Let us denote by $\mathcal{M} = \{\{a\} \mid a \in M\} \subseteq \mathcal{P}(M)$ the set of all atoms of $(\mathcal{P}(M), \subseteq)$. This set (and every its subset) is obviously join-independent.

Further we put $\mathcal{N} = \{s_a \mid a \in M\}$ where $s_a = \vee\mathcal{M}_{\{a\}} = \vee(\mathcal{M} \smallsetminus \{\{a\}\})$. Then $\mathcal{N}$ is the set of all coatoms of $(\mathcal{P}(M), \subseteq)$ and it is meet-independent (also every its subset).

In what follows, $\mathcal{J}_L^p = (L_j^p, L_m^p, I^p)$ denotes the context of the p-element independent sets associated to the lattice $(\mathcal{P}(M), \subseteq)$, where M is a non-empty set and p is any cardinal number.

Proposition 8. *The following statements are equivalent:*

1. $|M| < p$,
2. $L_j^p = \emptyset$.

Proof. $1 \Longrightarrow 2$: Let $A = \{A_i \mid i \in J\} \in L_j^p$ where $A_i \subseteq M$ and $|J| = p$, $|M| < p$. If we put $J_i := J \smallsetminus \{i\}$, then $A_i \nsubseteq \bigcup_{j \in J_i} A_j$ for all $i \in J$. This implies $(A_i \smallsetminus \bigcup_{j \in J_i} A_j) = A^i \neq \emptyset$. For each $a \in A^i$ we have $a \notin A_j$ for all $j \in J_i$. Then we can make a choice $M' = \{a^i \in A^i \mid i \in J\} \subseteq M$ and $\alpha : a^i \mapsto i$ is a one-to-one mapping of the subset M' of M onto J. However, this is a contradiction to $|M| < p$.

$2 \Longrightarrow 1$: Let us assume that $L_j^p = \emptyset$ and $p \leq |M|$. Then there exists a subset $\mathcal{M}' \subseteq \mathcal{M}$ such that $|\mathcal{M}'| = p$. Since every subset of $\mathcal{M}$ is join-independent, we get $\mathcal{M}' \in L_j^p$ and $L_j^p \neq \emptyset$. Thus $|M| < p$. ∎

Proposition 9. *Let p be a finite cardinal number. Then the following statements are equivalent:*

1. $|M| = p$,
2. $L_j^p = \{\mathcal{M}\}$.

Proof. $1 \Longrightarrow 2$: Let $A = \{A_i \mid i \in J\} \in L_j^p$, $A_i \subseteq M$ and $|J| = p = |M|$. Then $A_i \nsubseteq \bigcup_{j \in J_i} A_j$ for all $i \in J$, where $J_i = J \smallsetminus \{i\}$ again. Hence $A_i \smallsetminus \bigcup_{j \in J_i} A_j = A^i \neq \emptyset$ for all $i \in J$.

Assume that $x \in A^r \cap A^s$ for some $r, s \in J, r \neq s$. Then $x \in A^r, x \notin A_j$ for all $j \in J, j \neq r$. Thus $x \notin A^s$ which is a contradiction to $x \in A^s \subseteq A_s$. We have obtained $A^i \cap A^j = \emptyset$ for all $i, j \in J, i \neq j$.

If we make a choice $M' = \{a^i \in A^i \mid i \in J\} \subseteq M$, then $\alpha : a^i \mapsto i$ is a bijection of M' onto J and because of $|M| = |J|$ we have $M' = M$. Therefore $|A^i| = 1$ for all $i \in J$. Let $A^t = \{a\}$ for a certain $t \in J$. Then $a \in A_t$. If $b \in A_t, b \neq a$, then at the same time $b \in A^u$ for a certain $u \neq t$.

This yields $b \notin A_t$ which is a contradiction. Hence, $|A_i| = 1$ for all $i \in J$. We have proved that $A_i = \{a_i\}$ for all $i \in J$. It means that the only p-element join-independent set is $\mathcal{M}$.

$2 \implies 1$: According to the previous proposition, $p \leq |M|$. Every p-element set of atoms $\{\{a_i\} \mid i \in I\} \subseteq \mathcal{P}(M)$, $|I| = p$, is join-independent. If $p < |M|$, then there exist at least two distinct p-element sets of atoms. Thus $|L_j^p| > 1$. ∎

Example. If $|M| = 3$, then $|L_j^2| = 9$ and $|L_j^3| = 1$. If $|M| = 4$, then $|L_j^2| = 55$, $|L_j^3| = 26$ and $|L_j^4| = 1$.

Proposition 10. *The set $\{A_i \mid i \in J\}$ is join-independent in $(\mathcal{P}(M), \subseteq)$ if and only if the set $\{M \setminus A_i \mid i \in J\}$ is meet-independent in $(\mathcal{P}(M), \subseteq)$.*

Proof. For all $i \in J$ we put $J_i = J \setminus \{i\}$. Then it is easy to see that

$$A_i \not\subseteq \bigcup_{j \in J_i} A_j \Leftrightarrow M \setminus \bigcup_{j \in J_i} A_j \not\subseteq M \setminus A_i \Leftrightarrow \bigcap_{j \in J_i} (M \setminus A_j) \not\subseteq M \setminus A_i.$$

∎

Remark 4. It follows from Propositions 8 – 10 that $p > |M|$ if and only if $\mathcal{J}_L^p = (\emptyset, \emptyset, \emptyset)$, and $p = |M|$ if and only if $|L_j^p| = |L_m^p| = 1$. Also in the case $p < |M|$ we get $|L_j^p| = |L_m^p|$.

Proposition 11. *Let $A, B \in L_j^p$, $A = \{A_i \mid i \in J\}$, $B = \{B_i \mid i \in J\}$, $|J| = p$. If we denote $C = \{M \setminus A_i \mid i \in J\}$, $D = \{M \setminus B_i \mid i \in J\}$, then $v(A, B) = 1$ if and only if $v(C, D) = 1$.*

Proof. Assume that $v(A, B) = 1$. Then there exists $\bar{A} \in L_m^p$ such that $AI^p\bar{A}$, $BI^p\bar{A}$. Let us put $\bar{A} = \{\bar{A}_i \mid i \in J\}$ and $J_i = J \setminus \{i\}$. Under a suitable enumeration we get $\bar{A}_i \in X^A(A_i) \cap X^B(B_i)$ for all $i \in J$. Thus $\bigcup_{j \in J_i} A_j \subseteq \bar{A}_i, A_i \not\subseteq \bar{A}_i$ and $\bigcup_{j \in J_i} B_j \subseteq \bar{A}_i, B_i \not\subseteq \bar{A}_i$ for all $i \in J$. Let us put $\bar{C}_i = M - \setminus \bar{A}_i$. Then we have $\bar{C}_i = M \setminus \bar{A}_i \subseteq M \setminus \bigcup_{j \in J_i} A_j$, $\bar{C}_i \not\subseteq M \setminus A_i$, $\bar{C}_i \subseteq M \setminus \bigcup_{j \in J_i} B_j$, $\bar{C}_i \not\subseteq M \setminus B_i$. This yields $\bar{C}_i \in Y^C(M \setminus A_i)$ and $\bar{C}_i \in Y^D(M \setminus B_i)$, thus $\bar{C}_i \in Y^C(M \setminus A_i) \cap Y^D(M \setminus B_i)$ for all $i \in J$. If we denote $\bar{C} = \{\bar{C}_i \mid i \in J\}$, then $\bar{C}I^pC$, $\bar{C}I^pD$ and $v(C, D) = 1$. Similarly for the converse assertion. ∎

Proposition 12. *The sets $A = \{A_i \mid i \in J\}$, $B = \{B_i \mid i \in J\} \in L_j^p$ are joinable in $\mathcal{J}_L^p$ if and only if the sets $C = \{M \setminus A_i \mid i \in J\}$, $D = \{M \setminus B_i \mid i \in J\}$ are joinable in $\mathcal{J}_L^p$.*

Proof. The sets $A, B \in L_j^p$ are joinable if and only if there exist sets $A'_1, \ldots, A'_r \in L_j^p$, $A''_1, \ldots, A''_{r+1} \in L_m^p$ such that $AI^pA''_1, A'_1I^pA''_1, A'_1I^pA''_2, A'_2I^pA''_2, \ldots, A'_rI^pA''_r, A'_rI^pA''_{r+1}, BI^pA''_{r+1}$. Thus, $v(A, A'_1) = v(A'_1, A'_2) = \ldots = v(A'_r, B) = 1$. It follows from propositions 10 and 11 that there exist meet-independent sets $\bar{A}'_1, \bar{A}'_2, \ldots, \bar{A}'_r$ such that $v(C, \bar{A}'_1) = v(\bar{A}'_1, \bar{A}'_2) = \ldots = v(\bar{A}'_r, D) = 1$. Hence, the sets C, D are joinable. Similarly for the converse assertion. ■

Remark 5. If $A \subseteq \mathcal{M}$ (the subset of atoms), then for $\{a\} \in A$ we will write just $X^A(a), A_a, U(a)$ etc. instead of (more correct) $X^A(\{a\}), A_{\{a\}}, U(\{a\})$ etc. Then $X^A(a) = U(\vee A_a) \setminus U(a)$ and hence, $Y_a \in X^A(a)$ if and only if $A_a \subseteq Y_a, a \notin Y_a$.

Proposition 13. *If $A, B \subseteq \mathcal{M}$, $A \neq B, |A| = |B| = p$, then $v(A, B) = 1$.*

Proof. Let us denote $C = A \cap B$. There exists a bijective mapping $\varphi\colon A \to B$ such that $\varphi(c) = c$ for all $c \in C$. Further we put $Y_a = A_a \cup B_{\varphi(a)}$ for all $a \in A$. If $a \in C$, then $a = \varphi(a)$ and $a \notin A_a, B_a$. Thus $a \notin Y_a$. If $a \notin C$, then $a \notin B$ and $a \notin Y_a$. Similarly, $\varphi(a) \notin A$ and $\varphi(a) \notin Y_a$. It follows that $Y_a \in X^A(a) \cap X^B(\varphi(a))$. If we put $Y = \{Y_a \mid a \in A\}$, then $A \to Y : a \mapsto Y_a$ and $B \to Y : \varphi(a) \mapsto Y_a$ are norming mappings. Thus, AI^pY, BI^pY and $v(A, B) = 1$. ■

Proposition 14. *If $A, B \subseteq \mathcal{N}$, $A \neq B, |A| = |B| = p$, then $v(A, B) = 1$.*

Proof. Dual to the previous one. ■

Theorem 1. *Let $\mathcal{J}_L^p$ be a context of independent sets associated to the complete lattice $(\mathcal{P}(M), \subseteq)$, where M is a non-empty set and p is a cardinal number with the property $3 \leq p < |M|$. Then $\mathcal{J}_L^p$ is irreducible and (the span) $d(L_j^p) = 2$.*

Proof. Consider join-independent sets $A = \{A_i \mid i \in J\}$, $B = \{B_i \mid i \in J\}$, where $A_i, B_i \subseteq M$ for all $i \in J$, $|J| = p$. For each $i \in J$, we put $J_i = J \setminus \{i\}$ and $A^i = \bigcup_{j \in J_i} A_j$. Then $Y \in X^A(A_i)$ if and only if $A^i \subseteq Y, A_i \not\subseteq Y$. Since A is join-independent, we have $A_i \not\subseteq A^i$ for all $i \in J$. It follows that there always exists an element $a_i \in A_i$ such that $a_i \notin A^i$. Then $A^i \subseteq M_{a_i} = s_{a_i}$. From $a_i \notin s_{a_i}$, we get $A_i \not\subseteq s_{a_i}$ and hence $s_{a_i} \in X^A(A_i)$. We can make a choice $Y_1 = \{s_{a_i} \mid i \in J\}$. The set $Y_1 \subseteq \mathcal{N}$ is meet-independent and AI^pY_1. In a similar way, we can proceed in the case of the set B and we obtain

BI^pY_2 for a certain set $Y_2 \subseteq \mathcal{N}$. According to Proposition 14, there exists a set $C \subseteq \mathcal{M}$, $|C| = p$ such that CI^pY_1, CI^pY_2. Thus $v(A,B) \leq 2$.

It remains to find join-independent sets $A = \{A_i \mid i \in J\}$, $B = \{B_i \mid i \in J\}$, $|J| = p$, such that $v(A,B) = 2$. We determine them in the following way: Consider three distinct elements $a, b, c \in M$. Let us put $A_1 = \{a,b\} = B_1$, $A_2 = \{a,c\}$, $B_2 = \{b,c\}$ and $A_i = B_i = \{x_i\}$ for the other sets where $x_i \in M$ are pairwise distinct elements not equal to a, b, c. Moreover, we denote $C = \{a,b,c\}$ and $X = \{x_i \mid i \in J'\}$.

It is easy to verify that the sets A, B defined above are join-independent. Obviously, $X^A(x_i) = X^B(x_i)$ for all $i \in J'$ and $X^A(A_2) = X^B(B_2)$. It is also clear that $Y \subseteq X^A(A_1)$ if and only if $\{a,c\} \cup X \subseteq Y$, $A_1 \not\subseteq Y$, and $Y \subseteq X^B(B_1)$ if and only if $\{b,c\} \cup X \subseteq Y$, $B_1 \not\subseteq Y$. Let $Y \in X^A(A_1) \cap X^B(B_1)$. Then $C \cup X \subseteq Y$ which is a contradiction to $A_1, B_1 \subseteq Y$. Therefore, there is no meet-independent set Z such that AI^pZ, BI^pZ. Thus $v(A,B) = 2$. ■

Remark 6. Dually we can prove that also every two meet-independent sets are joinable and $d(L^p_m) = 2$.

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