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ON DISTRIBUTIVE TRICES

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Abstract

A triple-semilattice is an algebra with three binary operations, which is a semilattice in respect of each of them. A trice is a triple-semilattice, satisfying so called roundabout absorption laws. In this paper we investigate distributive trices. We prove that the only subdirectly irreducible distributive trices are the trivial one and a two element one. We also discuss finitely generated free distributive trices and prove that a free distributive trice with two generators has 18 elements.

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1. Introduction

An algebra $(T; \nearrow_1, \nwarrow_2, \downarrow_3)$ of a type with three binary operations is a *triple semilattice* if it is a semilattice in respect of each of the operations. We denote orders on T by

(1) $$a \leq_1 b \text{ if and only if } a \nearrow_1 b = b,$$

(2) $$a \leq_2 b \text{ if and only if } a \nwarrow_2 b = b,$$

(3) $$a \leq_3 b \text{ if and only if } a \downarrow_3 b = b.$$

A triple semilattice T is a *trice* if it satisfies the *roundabout absorption laws*:

$$((a \nearrow_1 b) \nwarrow_2 b) \ \downarrow_3 b = b, \tag{4}$$

$$((a \nearrow_1 b) \ \downarrow_3 b) \nwarrow_2 b = b, \tag{5}$$

$$((a \nwarrow_2 b) \nearrow_1 b) \ \downarrow_3 b = b, \tag{6}$$

$$((a \nwarrow_2 b) \ \downarrow_3 b) \nearrow_1 b = b, \tag{7}$$

$$((a \ \downarrow_3 b) \nearrow_1 b) \nwarrow_2 b = b, \tag{8}$$

and

$$((a \ \downarrow_3 b) \nwarrow_2 b) \nearrow_1 b = b \tag{9}$$

for all $a, b \in T$.
Trices are introduced and investigated in [2] as a generalization of lattices.
A *distributive trice* is a trice satisfying the following six distributive laws:

$$a \nearrow_1 (b \nwarrow_2 c) = (a \nearrow_1 b) \nwarrow_2 (a \nearrow_1 c), \tag{10}$$

$$a \nwarrow_2 (b \nearrow_1 c) = (a \nwarrow_2 b) \nearrow_1 (a \nwarrow_2 c), \tag{11}$$

$$a \nearrow_1 (b \ \downarrow_3 c) = (a \nearrow_1 b) \ \downarrow_3 (a \nearrow_1 c), \tag{12}$$

$$a \ \downarrow_3 (b \nearrow_1 c) = (a \ \downarrow_3 b) \nearrow_1 (a \ \downarrow_3 c), \tag{13}$$

$$a \nwarrow_2 (b \ \downarrow_3 c) = (a \nwarrow_2 b) \ \downarrow_3 (a \nwarrow_2 c), \tag{14}$$

and

$$a \ \downarrow_3 (b \nwarrow_2 c) = (a \ \downarrow_3 b) \nwarrow_2 (a \ \downarrow_3 c) \tag{15}$$

for all $a, b, c \in T$.

2. Subdirect decomposition of distributive trices

Lemma 1. *A triple semilattice T having all three semilattices as chains is a trice if and only if for all $x, y \in T$, there are $\leq_i$ and $\leq_j$, for $i, j \in \{1, 2, 3\}$, such that $x \leq_i y$ and $y \leq_j x$.*

Proof. By contraposition, if for all orderings $x \leq_i y$ $i \in \{1, 2, 3\}$ is satisfied, than $x \nearrow_1 (x \nwarrow_2 (x \ \downarrow_3 y))) = y$, i.e., roundabout absorption law (9) is not satisfied. On the other hand, if, say, $x \leq_1 y$ and $y \leq_2 x$, then it is easy to prove that all roundabout absorption laws for x and y are satisfied. ■

Lemma 2. *Let* $(T; \nearrow_1, \nwarrow_2, \downarrow_3)$ *be a distributive trice. Let* $x, y, t \in T$. *If* $x \nearrow_1 t = y \nearrow_1 t$, $x \nwarrow_2 t = y \nwarrow_2 t$ *and* $x \downarrow_3 t = y \downarrow_3 t$, *then* $x = y$.

Proof. Using repeatedly the hypotheses, we have

$$
\begin{aligned}
x &= x \nearrow_1 (x \nwarrow_2 (x \downarrow_3 t))) = x \nearrow_1 (x \nwarrow_2 (y \downarrow_3 t))) \\
&= x \nearrow_1 ((x \nwarrow_2 y) \downarrow_3 (x \nwarrow_2 t)) = x \nearrow_1 ((x \nwarrow_2 y) \downarrow_3 (y \nwarrow_2 t)) \\
&= x \nearrow_1 (y \nwarrow_2 (x \downarrow_3 t)) = x \nearrow_1 (y \nwarrow_2 (y \downarrow_3 t)) \\
&= (x \nearrow_1 y) \nwarrow_2 (x \nearrow_1 (y \downarrow_3 t)) = (x \nearrow_1 y) \nwarrow_2 ((x \nearrow_1 y) \downarrow_3 (x \nearrow_1 t)) \\
&= (x \nearrow_1 y) \nwarrow_2 ((x \nearrow_1 y) \downarrow_3 (y \nearrow_1 t)) = (x \nearrow_1 y) \nwarrow_2 (y \nearrow_1 (x \downarrow_3 t)) \\
&= (x \nearrow_1 y) \nwarrow_2 (y \nearrow_1 (y \downarrow_3 t)) = y \nearrow_1 (x \nwarrow_2 (y \downarrow_3 t)) \\
&= y \nearrow_1 ((x \nwarrow_2 y) \downarrow_3 (x \nwarrow_2 t)) = y \nearrow_1 ((x \nwarrow_2 y) \downarrow_3 (y \nwarrow_2 t)) \\
&= y \nearrow_1 (y \nwarrow_2 (x \downarrow_3 t)) = y \nearrow_1 (y \nwarrow_2 (y \downarrow_3 t)) \\
&= y.
\end{aligned}
$$

■

Let $(T; \nearrow_1, \nwarrow_2, \downarrow_3)$ be a distributive trice, and let $p \in T$ be a fixed element. We define relations on T by

$$x \,\theta_1\, y \text{ if and only if } x \nearrow_1 p = y \nearrow_1 p, \tag{16}$$

$$x \,\theta_2\, y \text{ if and only if } x \nwarrow_2 p = y \nwarrow_2 p, \tag{17}$$

and

$$x \,\theta_3\, y \text{ if and only if } x \downarrow_3 p = y \downarrow_3 p. \tag{18}$$

Lemma 3. *The relations* θ_1, θ_2 *and* θ_3 *defined by* (16)–(18) *are congruences on the distributive trice.*

Proof. It is obvious that every θ_i, for $i \in \{1, 2, 3\}$ is an equivalence relation. Moreover, it is compatible with all operations. Let $x \ \theta_1 \ y$ and $z \ \theta_1 \ t$. Then $x \nearrow_1 p = y \nearrow_1 p$ and $z \nearrow_1 p = t \nearrow_1 p$. And then $(x \nearrow_1 p) \nwarrow_2 (z \nearrow_1 p) = (y \nearrow_1 p) \nwarrow_2 (t \nearrow_1 p)$. By distributivity, $(x \nwarrow_2 z) \nearrow_1 p = (y \nwarrow_2 t) \nearrow_1 p$, i.e., $(x \nwarrow_2 z) \ \theta_1 \ (y \nwarrow_2 t)$. Similary, we get $(x \downarrow_3 z) \ \theta_1 \ (y \downarrow_3 t)$. Hence, θ_1 is a congruence on the trice. For θ_2 and θ_3 , we can prove it in a similar way. ■

Lemma 4. *The relation* θ_i *is the identity relation if and only if* p *is the bottom element in the* $(T; \leq_i)$, *for all* $i \in \{1, 2, 3\}$.

Proof. If p is the bottom element in $(T, \leq_1)$, then $p \leq_1 x$ for all $x \in T$. Hence, $x \, \theta_1 \, y$ if and only if $x = x \nearrow_1 p = y \nearrow_1 p = y$. That is, $\theta_1 = \Delta$.

On the other hand, if there exists $x \in T$ such that $\neg(p \leq_1 x)$, then $(p \nearrow_1 x) \neq x$. As $(p \nearrow_1 x) \nearrow_1 p = x \nearrow_1 p$, we get $(p \nearrow_1 x) \, \theta_1 \, x$. Then, $\theta_1 \neq \Delta$. For θ_2 and θ_3, we can prove the statements in a similar way. ∎

Lemma 5. *If* p *is not the bottom element of any of semilattices of the distributive trice* T, *then not all of* θ_1, θ_2 *and* θ_3 *are equal.*

Proof. Suppose that all the congruences are equal. Let $x <_1 p$. Then, $x \, \theta_1 \, p$. As congruences are the same, $x \, \theta_2 \, p$ and $x \, \theta_3 \, p$. Hence $x \leq_2 p$ and $x \leq_3 p$, and thus $((x \nearrow_1 p) \nwarrow_2 p) \downarrow_3 p = p$, and finally from our assumption we obtain $p = x$, a contradiction. ∎

Lemma 6. *There are no subdirectly irreducible distributive trices with more than three elements.*

Proof. Suppose that T is a subdirectly irreducible distributive trice with four or more elements. Then, there is an element, say $p \in T$, which is not the bottom element in any of the semilattices. This element determines three congruences θ_1, θ_2 and θ_3, defined by formulas (16) – (18). Those relations are all distinct from the identity relation by Lemma 4, and at least two of them are not equal by Lemma 5. Using Lemma 2 we easily prove that

$$\theta_1 \cap \theta_2 \cap \theta_3 = \Delta.$$

By the well known theorem on congruence lattice of subdirectly irreducible algebras (see e.g. [1], p. 57. Thm. 8.4), we have that T is not subdirectly irreducible. ∎

Lemma 7. *There are no subdirectly irreducible distributive trices with three elements.*

Proof. There is only one (up to the isomorphism and the order of operations) distributive trice with three elements $(T; \nearrow_1, \nwarrow_2, \downarrow_3)$, diagrams of its semilattices given in Figure 1. It is not subdirectly irreducible. Indeed, congruences of this trice, besides Δ and ∇, are $\{\{a, b\}, \{c\}\}$, and $\{\{a\}, \{b, c\}\}$, that is, congruence lattice is the four element boolean algebra. Thus, this trice is not subdirectly irreducible. ∎

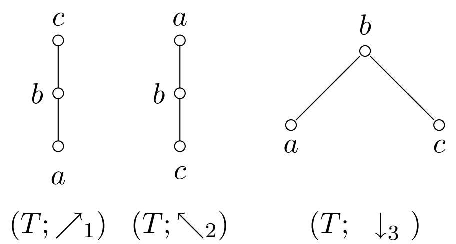

Figure 1

By lemmas 1 – 7, we have:

Theorem 1. *The only subdirectly irreducible distributive trices are, up to the isomorphism and the order of operations, the two element one, given in Figure* 2*, and the trivial one.*

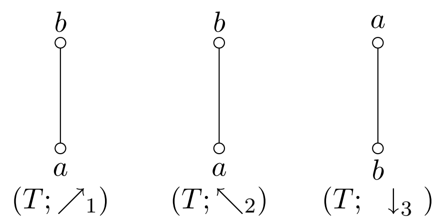

Figure 2

∎

Theorem 2. *Every non-trivial distributive trice is isomorphic to a subdirect product of two element trices.*

Proof. This is a consequence of previous theorem and the Birkhoff theorem on subdirect products. ∎

An obvious corrolary is that every distributive trice is a subtrice of the direct product of two element trices.

Example 1. In the sequel, we give representation of the three element distributive trice in Figure 1, as a subdirect product of two element trices.

Proof. Let $T_1 = \{a, b\}$ and $T_2 = \{c, d\}$, with $a \leq_1 b$, $a \leq_2 b$, $b \leq_3 a$, $c \leq_1 d$, $d \leq_2 c$ and $d \leq_3 c$. The direct product has four elements $\{ac, bc, ad, bd\}$. The mentioned three element trice is isomorphic with the subtrice $\{ac, bc, bd\}$.

3. Free distributive trices

In the sequel we consider free distributive trices.

Obviously, free distributive trice with one generator is the one element trivial trice. Now, consider n generators, x_1,... x_n. Every element of a free distributive trice can be written in the form $F_1 \downarrow_3 F_2 \downarrow_3 ... \downarrow_3 F_m$, where every $F_i (i \in \{1, 2, ...m\})$ is of the form: $g_1 \nwarrow_2 g_2 \nwarrow_2 ... \nwarrow_2 g_k$, and every $g_j (j \in \{1, 2, ...k\})$ is of the form: $x_{i_1} \nearrow_1 x_{i_2} \nearrow_1 ... \nearrow_1 x_{i_l}$, where all x_s appearing in the mentioned expression, are generators. We can easily prove, by using distributive laws, that every element of a free distributive trice have a representation of that form. And obviously, some elements have several different representations.

By the previous considerations, the following theorem is evident:

Theorem 3. *Every free distributive trice with a finite set of generators is finite.*

Proof. Let n be the number of generators. Let G be the set of all elements of the form: $x_{i_1} \nearrow_1 x_{i_2} \nearrow_1 ... \nearrow_1 x_{i_l}$, where all x_s appearing in the mentioned expressions are generators. Then, the cardinality of G is not greater than $2^n - 1$. Let F be the set of all elements of the form: $g_1 \nwarrow_2 g_2 \nwarrow_2 ... \nwarrow_2 g_k$, where $g_i \in G$, for all $i \in \{1, ..., k\}$. Then, the cardinality of F is not greater than $2^{2^n-1} - 1$. As every element of a free distributive trice can be written in the form $F_1 \downarrow_3 F_2 \downarrow_3 ... \downarrow_3 F_m$, with $F_i \in F$, the order of free distributive trice with n generators is not greater than $2^{2^{2^n-1}-1} - 1$. There is some possibility of overlapping. But, this completes the proof. ∎

Example 2. Free distributive trice with two generators has 18 elements.

We effectively construct a free distributive trice with two generators x and y. The notations in the sequel is taken from the proof of the previous theorem. Now, the set G is $\{x,\ y,\ x \nearrow_1 y\}$. From $x \nwarrow_2 y \nwarrow_2 (x \nearrow_1 y) = (x \nwarrow_2 y \nwarrow_2 x) \nearrow_1 (x \nwarrow_2 y \nwarrow_2 y) = x \nwarrow_2 y$, it follows that the set F is $\{x,\ y,\ x \nearrow_1 y,\ x \nwarrow_2 y,\ x \nwarrow_2 (x \nearrow_1 y),\ y \nwarrow_2 (x \nearrow_1 y)\}$. In a similar way, we can deduce that the free distributive trice with two generators has 18 elements.

All different elements of that trice are represented by the following terms:

$1 = x$, $\quad 2 = y$, $\quad 3 = x \nearrow_1 y$,

$4 = x \nwarrow_2 y = x \nwarrow_2 y \nwarrow_2 (x \nearrow_1 y)$, $\quad 5 = x \nwarrow_2 (x \nearrow_1 y)$,

$6 = y \nwarrow_2 (x \nearrow_1 y)$, $\quad 7 = x \downarrow_3 y$, $\quad 8 = x \downarrow_3 (x \nearrow_1 y)$,

$9 = x \downarrow_3 (x \nwarrow_2 y)$, $\quad 10 = y \downarrow_3 (x \nearrow_1 y)$, $\quad 11 = y \downarrow_3 (x \nwarrow_2 y)$,

$12 = x \downarrow_3 (y \nwarrow_2 (x \nearrow_1 y))$, $\quad 13 = y \downarrow_3 (x \nwarrow_2 (x \nearrow_1 y))$,

$14 = (x \nwarrow_2 y) \downarrow_3 (x \nearrow_1 y) = (x \nwarrow_2 (x \nearrow_1 y)) \downarrow_3 (y \nwarrow_2 (x \nearrow_1 y))$,

$15 = (x \nearrow_1 y) \downarrow_3 (x \nwarrow_2 (x \nearrow_1 y))$, $\quad 16 = (x \nearrow_1 y) \downarrow_3 (y \nwarrow_2 (x \nearrow_1 y))$,

$17 = (x \nwarrow_2 y) \downarrow_3 (x \nwarrow_2 (x \nearrow_1 y))$, $\quad 18 = (x \nwarrow_2 y) \downarrow_3 (y \nwarrow_2 (x \nearrow_1 y))$.

Diagrams of the free distributive trice with two generators are presented by Figures 3.1–3.3. The orders in each of the semilattices of the trice are represented by arrows. 3, 4 and 7 are the top elements in the orders $\leq_1$, $\leq_2$ and $\leq_3$, respectively.

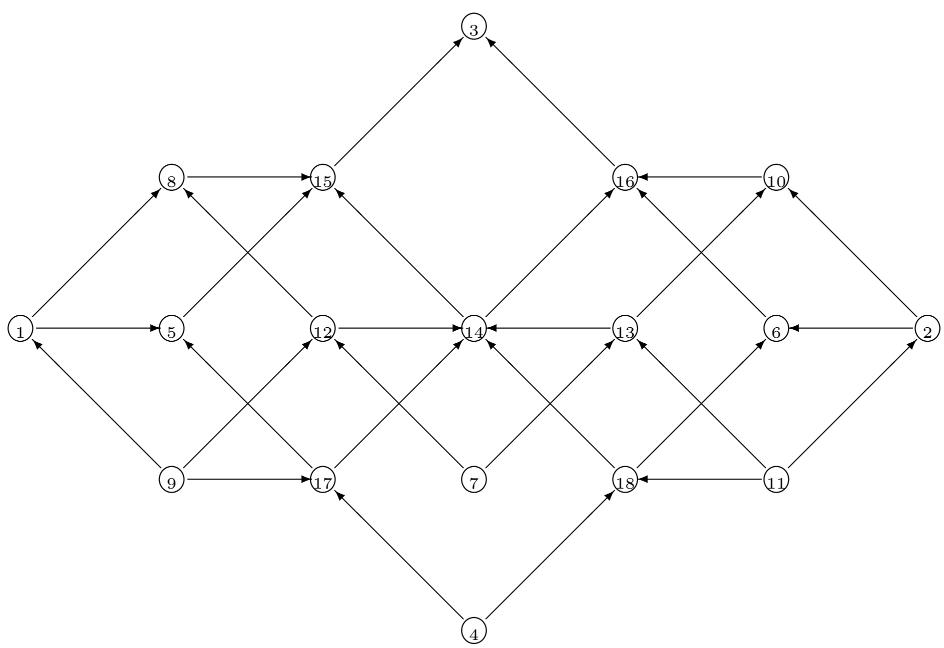

Figure 3 – 1 The order $\leq_1$

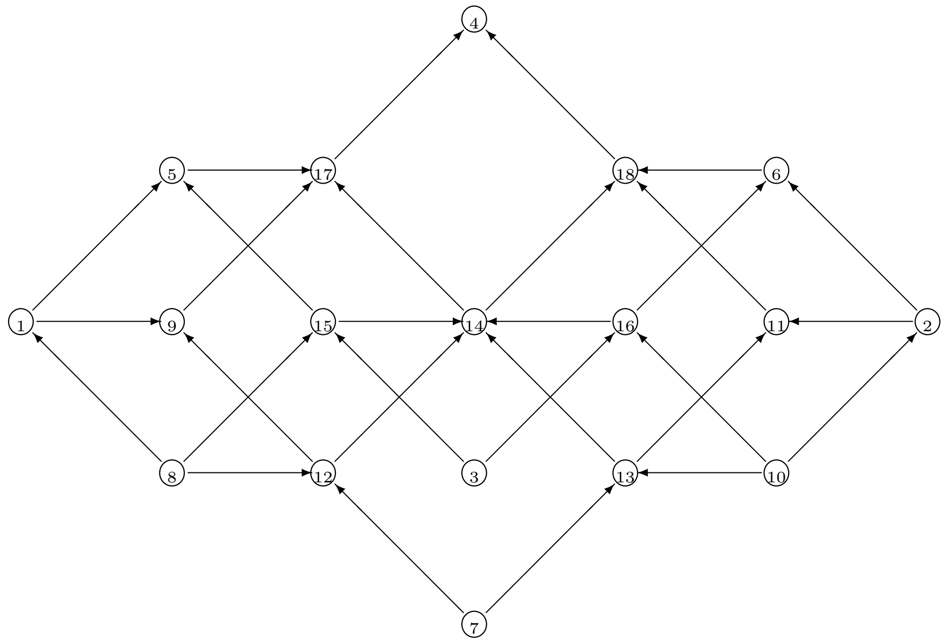

Figure 3 – 2 The order $\leq_2$

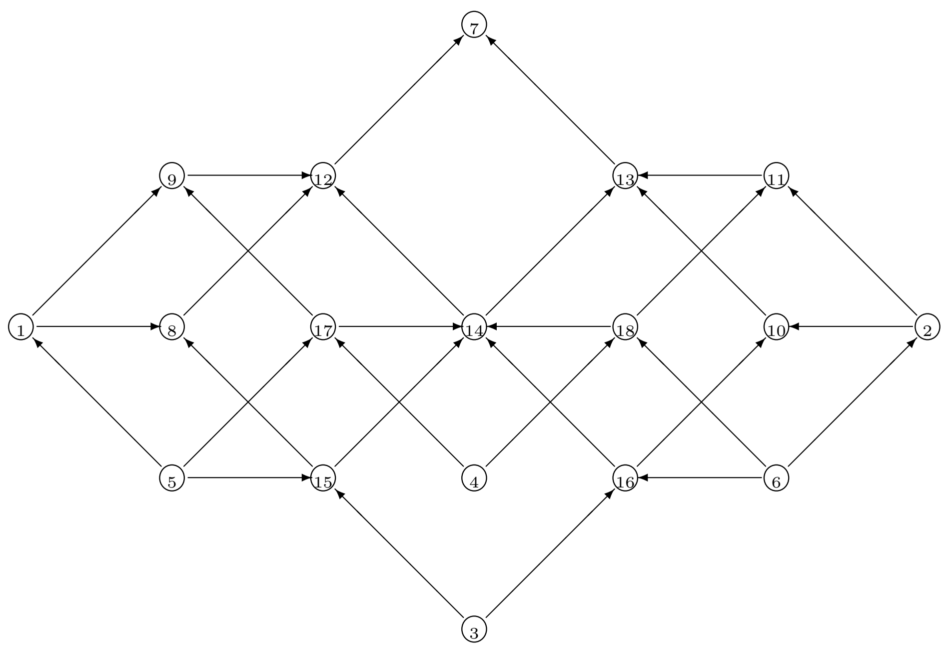

Figure 3 – 3 The order $\leq_3$

References

[1] S. Burris, and H.P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, New York 1981 (new, electronic version, 1999: is available at the address: www.thoralf.uwaterloo.ca).

[2] K. Horiuchi, *Trice and Two delegates operation*, Sci. Math. **2** (1999), 373–384.

[3] J.A. Kalman, *Subdirect decomposition of distributive quasi-lattices*, Fund. Math. **71** (1971), 161–163.

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