

ON COVARIETY LATTICES

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Abstract

This paper shows basic properties of covariety lattices. Such lattices are shown to be infinitely distributive. The covariety lattice $L_{CV}(\mathbf{K})$ of subcovarieties of a covariety \mathbf{K} of F -coalgebras, where $F : \mathbf{Set} \rightarrow \mathbf{Set}$ preserves arbitrary intersections is isomorphic to the lattice of subcoalgebras of a \mathcal{P}_κ -coalgebra for some cardinal κ . A full description of the covariety lattice of \mathcal{Id} -coalgebras is given. For any topology τ there exist a bounded functor $F : \mathbf{Set} \rightarrow \mathbf{Set}$ and a covariety \mathbf{K} of F -coalgebras, such that $L_{CV}(\mathbf{K})$ is isomorphic to the lattice (τ, \cup, \cap) of open sets of τ .

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1. INTRODUCTION

Many mathematicians and computer scientists have been recently studying the universal theory of coalgebras - objects dual to algebras. Many interesting properties of coalgebras have been shown. E.g. an analogue of the Birkhoff Variety Theorem was developed, which describes syntactically the classes of coalgebras called *covarieties*.

This paper studies the basic properties of covariety lattices. We show that the covariety lattices are infinitely distributive. Corollary 3.9 shows

that given any F -coalgebra \mathbb{A} there is a covariety \mathbf{K} of $A \times F$ -coalgebras such that the lattice $L_{\mathcal{CV}}(\mathbf{K})$ of subcovarieties of \mathbf{K} is isomorphic to the lattice $\mathbf{S}(\mathbb{A})$ of subcoalgebras of \mathbb{A} .

Next, Theorem 4.1 shows that, whenever F preserves arbitrary intersections, the covariety lattice is isomorphic to the lattice $\mathcal{D}(\mathfrak{R}_{\mathbf{Set}_F})$ of subsets of all rooted coalgebras closed under taking rooted subcoalgebras of homomorphic images. As an example, the covariety lattice of \mathcal{Id} -coalgebras is described.

Finally, the covariety lattice $L_{\mathcal{CV}}(\mathbf{K})$ of subcovarieties of a covariety \mathbf{K} of F -coalgebras, where $F : \mathbf{Set} \rightarrow \mathbf{Set}$ preserves arbitrary intersections, is characterized in Theorem 4.5 as the lattice of subcoalgebras of some \mathcal{P}_κ -coalgebra.

2. BASIC DEFINITIONS AND PROPERTIES

Let \mathbf{Set} be the category of all sets and mappings between them. Let $F : \mathbf{Set} \rightarrow \mathbf{Set}$ be a functor. An F -coalgebra \mathbb{A} is a pair (A, α) , where A is a set and α is a mapping $\alpha : A \rightarrow F(A)$. The set A is called the *carrier* of the coalgebra (A, α) and the mapping α is called the *structure*.

Let $\mathbb{A} = (A, \alpha)$ and $\mathbb{B} = (B, \beta)$ be two F -coalgebras. A *homomorphism* from the coalgebra \mathbb{A} to the coalgebra \mathbb{B} is a mapping $h : A \rightarrow B$, such that $F(h) \circ \alpha = \beta \circ h$.

The class of all F -coalgebras together with homomorphisms as morphisms forms a category denoted by \mathbf{Set}_F . An F -coalgebra \mathbb{B} is said to be a *homomorphic image* of an F -coalgebra \mathbb{A} if there exists a surjective homomorphism from \mathbb{A} onto \mathbb{B} . An F -coalgebra \mathbb{S} is said to be a *subcoalgebra* of an F -coalgebra \mathbb{A} if there exists an injective homomorphism from \mathbb{S} into \mathbb{A} . This is denoted by $\mathbb{S} \leq \mathbb{A}$.

Theorem 2.1 [2]. *Let $F : \mathbf{Set} \rightarrow \mathbf{Set}$ be a functor. Let $\{\mathbb{S}_i\}_{i \in I}$ be a family of subcoalgebras of an F -coalgebra \mathbb{A} . Then*

- *there exists a unique structure $\alpha : \bigcup_{i \in I} S_i \rightarrow F(\bigcup_{i \in I} S_i)$ such that the coalgebra $\bigcup_{i \in I} \mathbb{S}_i := (\bigcup_{i \in I} S_i, \alpha)$ is a subcoalgebra of \mathbb{A} ;*
- *if I is a finite set of indices, then there exists a unique structure $\beta : \bigcap_{i \in I} S_i \rightarrow F(\bigcap_{i \in I} S_i)$ such that $\bigcap_{i \in I} \mathbb{S}_i := (\bigcap_{i \in I} S_i, \beta)$ is a subcoalgebra of \mathbb{A} .*

In other words, subcoalgebras of a given coalgebra form a topology.

Theorem 2.2 [2]. *Let $F : \mathbf{Set} \rightarrow \mathbf{Set}$ be a functor and \mathbb{A} be an F -coalgebra. If $S \subseteq A$, then there exists at most one structure $\sigma : S \rightarrow F(S)$ such that $(S, \sigma) \leq \mathbb{A}$.*

The disjoint union of a family $\{X_j\}_{j \in J}$ of sets is denoted by $\Sigma_{j \in J} X_j$. Now let $\{\mathbb{A}_i\}_{i \in I}$ be a family of F -coalgebras. The *disjoint sum* $\Sigma_{i \in I} \mathbb{A}_i$ of the family $\{\mathbb{A}_i\}_{i \in I}$ of F -coalgebras is an F -coalgebra defined as follows. The carrier set of the disjoint sum $\mathbb{A} = \Sigma_{i \in I} \mathbb{A}_i$ is the disjoint union of the carriers of \mathbb{A}_i , i.e.

$$A := \Sigma_{i \in I} A_i.$$

The structure $\alpha : A \rightarrow F(A)$ of the disjoint sum $\mathbb{A} = \Sigma_{i \in I} \mathbb{A}_i$ is defined as follows

$$\alpha : A \rightarrow F(A); A_i \ni a \mapsto F(e_i) \circ \alpha_i(a),$$

where the mapping α_i denotes the structure of the coalgebra \mathbb{A}_i and

$$e_i : A_i \rightarrow A; a \mapsto (a, i),$$

for every $i \in I$. We say that an F -coalgebra \mathbb{A} is a *conjunct sum of the family* $\{\mathbb{G}_i\}_{i \in I}$ of F -coalgebras if there exists a family $\{e_i : \mathbb{G}_i \rightarrow \mathbb{A}\}_{i \in I}$ of injective homomorphisms such that $A = \bigcup_{i \in I} e_i(G_i)$. We denote it by $\mathbb{A} \in \Sigma^C(\{\mathbb{G}_i\}_{i \in I})$.

A functor $F : \mathbf{Set} \rightarrow \mathbf{Set}$ is said to *preserve arbitrary intersections* if for any family of subcoalgebras $\{\mathbb{A}_i\}_{i \in I}$ of an F -coalgebra \mathbb{A} , there exists a structure $\alpha : \bigcap A_i \rightarrow F(\bigcap A_i)$ such that the F -coalgebra $\bigcap \mathbb{A}_i := (\bigcap A_i, \alpha)$ is a subcoalgebra of \mathbb{A} .

A functor $F : \mathbf{Set} \rightarrow \mathbf{Set}$ is said to be *bounded by κ* , if κ is the cardinal number such that for every F -coalgebra \mathbb{A} and for every $a \in A$ there exists an F -coalgebra \mathbb{U}_a , such that $|U_a| \leq \kappa$, $a \in U_a$ and $\mathbb{U}_a \leq \mathbb{A}$. We say that F is *bounded* if it is bounded by κ for some cardinal κ .

Example 2.3. Let κ be a cardinal number. Let $\mathcal{P}_\kappa : \mathbf{Set} \rightarrow \mathbf{Set}$ be the functor given by $\mathcal{P}_\kappa(X) = \{S \subseteq X \mid |S| \leq \kappa\}$ for a set X and

$$\mathcal{P}_\kappa(f) : \mathcal{P}_\kappa(X) \rightarrow \mathcal{P}_\kappa(Y); S \mapsto f(S)$$

for a mapping $f : X \rightarrow Y$. The functor \mathcal{P}_κ is an example of a bounded functor which preserves arbitrary intersections (see [5]).

Example 2.4. The filter functor $\mathcal{F} : \mathbf{Set} \rightarrow \mathbf{Set}$ assigns to every set X the set of filters $\mathcal{F}(X)$ on X and to every mapping $f : X \rightarrow Y$ the mapping

$$\mathcal{F}(f) : \mathcal{F}(X) \rightarrow \mathcal{F}(Y); F \mapsto \uparrow \{f(W) \mid W \in F\},$$

where $\uparrow \{f(W) \mid W \in F\}$ denotes the filter generated by the set $\{f(W) \mid W \in F\}$. This functor is an example of a functor which does not preserve arbitrary intersections.

It is important to mention that any topological space can be turned into an \mathcal{F} -coalgebra. Let (X, τ) be a topological space. Define the mapping

$$\sigma : X \rightarrow \mathcal{F}(X); x \mapsto \{W \subseteq X \mid \exists O \in \tau \text{ such that } x \in O \subseteq W\}.$$

The subcoalgebras of the \mathcal{F} -coalgebra (X, σ) are precisely the open subsets of (X, τ) (see [3]). Since the intersection of an arbitrary family of open sets in a given topological space may not exist, it is clear that \mathcal{F} does not preserve arbitrary intersections. The filter functor \mathcal{F} is not bounded.

Let \mathbf{K} be a class of F -coalgebras. We define the following classes of F -coalgebras:

$$\mathcal{S}(\mathbf{K}) := \{\mathbb{S} \mid \exists \mathbb{A} \in \mathbf{K} \text{ such that } \mathbb{S} \leq \mathbb{A}\},$$

$$\mathcal{H}(\mathbf{K}) := \{\mathbb{B} \mid \exists \mathbb{A} \in \mathbf{K} \text{ such that } \mathbb{A} \twoheadrightarrow \mathbb{B}\},$$

$$\Sigma(\mathbf{K}) := \{\Sigma_{i \in I} \mathbb{A}_i \mid \{\mathbb{A}_i\}_{i \in I} \subseteq \mathbf{K}\}.$$

A class \mathbf{K} of F -coalgebras is called a *covariety* if it is closed under \mathcal{S}, \mathcal{H} and Σ , i.e., $\mathcal{S}(\mathbf{K}) \subseteq \mathbf{K}$, $\mathcal{H}(\mathbf{K}) \subseteq \mathbf{K}$ and $\Sigma(\mathbf{K}) \subseteq \mathbf{K}$.

Theorem 2.5 [2]. *Let \mathbf{K} be a class of F -coalgebras. The class $\mathcal{SH}\Sigma(\mathbf{K})$ is the smallest covariety containing \mathbf{K} .*

We say that a class \mathbf{K}' of F -coalgebras is a *subcovariety* of a covariety \mathbf{K} whenever \mathbf{K}' is a covariety and $\mathbf{K}' \subseteq \mathbf{K}$.

The assumption of boundedness of a functor F guarantees that the collection of all subcovarieties of the covariety \mathbf{Set}_F is a set (see [2]). Since we do not want to focus only on coalgebras for bounded functors we need to allow *class based lattices*, i.e., partially ordered classes in which each pair of elements has a supremum and an infimum. Obviously, any lattice is a class based lattice. We may easily generalize the notion of completeness to the

class based lattices. Namely, a partially ordered class (C, \leq) is a complete class based lattice if all its subclasses have a supremum and infimum. We see that whenever (C, \leq) is a complete class based lattice and C is a proper set then (C, \leq) is simply a complete lattice. The following holds.

Theorem 2.6. *The collection of all subcovarieties of a given covariety \mathbf{K} of F -coalgebras ordered by inclusion is a complete class based lattice.*

We denote the class based lattice of all subcovarieties of \mathbf{K} by $L_{CV}(\mathbf{K})$. Let $\{\mathbf{K}_i\}_{i \in I}$ be a collection of subcovarieties of the covariety \mathbf{K} of F -coalgebras. Note that the collection $\{\mathbf{K}_i\}_{i \in I}$ and hence I may be a proper class. The infimum and supremum of $\{\mathbf{K}_i\}_{i \in I}$ in $L_{CV}(\mathbf{K})$ are of the following form.

$$\prod_{i \in I} \mathbf{K}_i := \bigcap_{i \in I} \mathbf{K}_i,$$

$$\sum_{i \in I} \mathbf{K}_i := \mathcal{SH}\Sigma \left(\bigcup_{i \in I} \mathbf{K}_i \right).$$

We will clearly distinguish between the class based lattices whose carrier is a proper class and lattices with a set carrier. We will use the term *proper lattice* to emphasize the fact that the latter holds, i.e. a class based lattice is simply a lattice.

3. COVARIETY LATTICES

In this section we discuss the distributivity of covariety class based lattices. Then we describe the lattices $L_{CV}(\mathcal{SH}\Sigma(\mathbb{A}))$ for certain coalgebras \mathbb{A} and show that the lattice of open sets of any topological space is isomorphic to some covariety lattice $L_{CV}(\mathbf{K})$.

Suppose F is bounded by $|X|$ for some set X . Then the cofree F -coalgebra \mathbb{C}_X over the set X exists. In this case there is a one-to-one correspondence between the so-called invariant subcoalgebras of \mathbb{C}_X and covarieties of F -coalgebras. This correspondence is given by the following formula:

$$\mathbf{K} = \mathcal{Q}(\mathbb{C}_X, \mathbb{U}) := \{\mathbb{A} \mid \forall \phi : \mathbb{A} \rightarrow \mathbb{C}_X, \phi(A) \subseteq U\},$$

where $\mathbb{U} := \bigcup \{\phi(\mathbb{A}) \mid \phi : \mathbb{A} \rightarrow \mathbb{C}_X \text{ and } \mathbb{A} \in \mathbf{K}\}$ (see [2]). Therefore, the lattice $L_{CV}(\text{Set}_F)$ of all covarieties of F -coalgebras is isomorphic to the lattice

of invariant subcoalgebras of \mathbb{C}_X ordered by inclusion. Because it is clear that the invariant subcoalgebras are closed under infinite unions and finite intersections the lattice $L_{\mathcal{CV}}(\mathbf{Set}_F)$ is infinitely distributive. If we do not assume boundedness of F then we cannot speak of the above correspondence. Yet, we are able to derive the following result directly.

Theorem 3.1. *The class based lattice $L_{\mathcal{CV}}(\mathbf{Set}_F)$ of covarieties of F -coalgebras is distributive.*

Proof. Let $\{K_i\}_{i \in I}$ be a collection of covarieties of F -coalgebras and let K be a covariety. Note that I may be a proper class. To show that the covariety class based lattice $L_{\mathcal{CV}}(\mathbf{Set}_F)$ is distributive it is enough to verify that the following inequality is true:

$$K \cdot \left(\sum_{i \in I} K_i \right) \leq \sum_{i \in I} K \cdot K_i.$$

Let $\mathbb{A} \in K \cdot (\sum_{i \in I} K_i)$. This means that $\mathbb{A} \in K$ and $\mathbb{A} \in \sum_{i \in I} K_i$. Since $\sum_{i \in I} K_i = \mathcal{SH}\Sigma(\bigcup_{i \in I} K_i)$, it follows that $\mathbb{A} \leq h(\sum_{j \in J} \mathbb{B}_j)$, where $\mathbb{B}_j \in \bigcup_{i \in I} K_i$ for any j coming from the set of indices J and h is a homomorphism. Let $e_k : \mathbb{B}_k \rightarrow \sum_{j \in J} \mathbb{B}_j$ for $k \in J$ denote the canonical embeddings. Then $\mathbb{A} \leq \bigcup_{j \in J} h(e_j(\mathbb{B}_j))$. By Theorem 2.1 we have

$$\mathbb{A} = \bigcup_{j \in J} h(e_j(\mathbb{B}_j)) \cap \mathbb{A}.$$

Since all K_i 's are covarieties and $h(e_j(\mathbb{B}_j)) \cap \mathbb{A} \leq h(e_j(\mathbb{B}_j))$, it follows that $h(e_j(\mathbb{B}_j)) \cap \mathbb{A} \in K_{i_j}$ for some $i_j \in I$. Moreover, because $h(e_j(\mathbb{B}_j)) \cap \mathbb{A} \leq \mathbb{A}$ and $\mathbb{A} \in K$, we have $h(e_j(\mathbb{B}_j)) \cap \mathbb{A} \in K$. Hence $h(e_j(\mathbb{B}_j)) \cap \mathbb{A} \in K \cdot K_{i_j}$ and therefore

$$\mathbb{A} \in \sum_{i \in I} K \cdot K_i.$$

■

Definition 3.2 ([2]). An F -coalgebra \mathbb{A} is called *strongly simple* whenever it does not possess any nontrivial homomorphic images.

We will now show some properties of strongly simple coalgebras, necessary for characterisation of $L_{\mathcal{CV}}(\mathcal{SH}\Sigma(\mathbb{A}))$.

Lemma 3.3 ([2]). *Let \mathbb{A} be a strongly simple F -coalgebra. If \mathbb{B} is an F -coalgebra, then there exists at most one homomorphism $h : \mathbb{B} \rightarrow \mathbb{A}$.*

Lemma 3.4. *Let $\mathbb{A} = (A, \alpha)$ be a strongly simple F -coalgebra. Let $\mathbb{S} \leq \mathbb{A}$ and $\mathbb{T} \leq \mathbb{A}$ be such that $\mathbb{S} \cong \mathbb{T}$. Then $\mathbb{S} = \mathbb{T}$.*

Lemma 3.5. *Let \mathbb{A} be a strongly simple F -coalgebra. If $\mathbb{B} \in \mathcal{SH}\Sigma(\mathbb{A})$, then $\mathbb{B} \in \Sigma^C\mathcal{S}(\mathbb{A})$.*

Proof. If $\mathbb{B} \in \mathcal{SH}\Sigma(\mathbb{A})$ then $\mathbb{B} \leq h(\Sigma_{i \in I} \mathbb{A})$, where h is a homomorphism. Let $e_i : \mathbb{A} \rightarrow \Sigma_{i \in I} \mathbb{A}$ denote the canonical embeddings. Since \mathbb{A} is strongly simple, it follows that the image coalgebra $h(e_i(\mathbb{A}))$ is isomorphic to \mathbb{A} for each $i \in I$. Since $h(\Sigma_{i \in I} \mathbb{A}) = \bigcup_{i \in I} h(e_i(\mathbb{A}))$, it follows that $\mathbb{B} = \bigcup_{i \in I} h(e_i(\mathbb{A})) \cap \mathbb{B}$. Because $h(e_i(\mathbb{A})) \cap \mathbb{B} \leq h(e_i(\mathbb{A})) \cong \mathbb{A}$, we have $\mathbb{B} \in \Sigma^C\mathcal{S}(\mathbb{A})$. ■

Let \mathbb{A} be an F -coalgebra. Let $\mathcal{S}(\mathbb{A})$ denote the set of carriers of subcoalgebras of \mathbb{A} , i.e.

$$\mathcal{S}(\mathbb{A}) := \{B \mid B \leq \mathbb{A}\}.$$

By Theorem 2.1, the set $\mathcal{S}(\mathbb{A})$ together with the operations of union and intersection forms a lattice.

What we now want to do is to show without any additional assumptions that $L_{\mathcal{CV}}(\mathcal{SH}\Sigma(\mathbb{A}))$ is isomorphic to the proper lattice $(\mathcal{S}(\mathbb{A}), \cup, \cap)$ for any strongly simple coalgebra \mathbb{A} . If we assume that F is bounded then the cofree F -coalgebra \mathbb{C}_1 over the one-element set 1 exists. The coalgebra \mathbb{C}_1 is the terminal object in the category \mathbf{Set}_F . Therefore, it is strongly simple. Moreover, strongly simple F -coalgebras are precisely subcoalgebras of \mathbb{C}_1 and all subcoalgebras of \mathbb{C}_1 are invariant. Hence, $L_{\mathcal{CV}}(\mathcal{SH}\Sigma(\mathbb{C}_1))$ is isomorphic to $(\mathcal{S}(\mathbb{C}_1), \cup, \cap)$. The same thing is clearly true for any subcoalgebra of \mathbb{C}_1 . If we do not assume that F is bounded the terminal object in \mathbf{Set}_F may not exist. Yet, we can expand our category \mathbf{Set}_F to class based coalgebras, where the terminal object always exists (see [1]). Using a similar argument and working with class based coalgebras and we get a general result. At the same time if one does not prefer to work with classes then the direct proof of the following theorem is an alternative.

Theorem 3.6. *Let \mathbb{A} be a strongly simple F -coalgebra. Then $L_{\mathcal{CV}}(\mathcal{SH}\Sigma(\mathbb{A}))$ is a proper lattice and*

$$L_{\mathcal{CV}}(\mathcal{SH}\Sigma(\mathbb{A})) \cong (\mathcal{S}(\mathbb{A}), \cup, \cap).$$

Proof. Let \mathbf{K} be a subcovariety of the covariety $\mathcal{SH}\Sigma(\mathbb{A})$. Define

$$\mathbb{S}_{\mathbf{K}} := \bigcup \{ \mathbb{S} \mid \mathbb{S} \leq \mathbb{A} \text{ and } \mathbb{S} \in \mathbf{K} \}.$$

In other words, the F -coalgebra $\mathbb{S}_{\mathbf{K}}$ is the union of subcoalgebras of \mathbb{A} which are elements of the covariety \mathbf{K} .

It is clear that $\mathbb{S}_{\mathbf{K}}$ is the greatest subcoalgebra of \mathbb{A} contained in \mathbf{K} .

Let $\mathbb{B} \in \mathbf{K}$. We have $\mathbb{B} \leq f(\Sigma_{i \in I} \mathbb{A})$ for a homomorphism f . By Lemma 3.5, $\mathbb{B} \in \Sigma^C(\{\mathbb{C}_i\}_{i \in I})$, where $\mathbb{C}_i \leq \mathbb{A}$ for $i \in I$. Since $\mathbb{C}_i \leq \mathbb{B}$, it follows that $\mathbb{C}_i \in \mathbf{K}$. Hence $\mathbb{C}_i \leq \mathbb{S}_{\mathbf{K}}$ for $i \in I$ and $\mathbb{B} \in \Sigma^C \mathcal{S}(\mathbb{S}_{\mathbf{K}})$. Therefore any coalgebra $\mathbb{B} \in \mathbf{K}$ is a conjunct sum of subcoalgebras of $\mathbb{S}_{\mathbf{K}}$, i.e. $\mathbf{K} = \Sigma^C \mathcal{S}(\mathbb{S}_{\mathbf{K}})$.

We will now prove that the mapping

$$\mathbb{S}_{(-)} : L_{CV}(\mathcal{SH}\Sigma(\mathbb{A})) \rightarrow \mathcal{S}(\mathbb{A}); \mathbf{K} \mapsto \mathbb{S}_{\mathbf{K}}$$

is a lattice isomorphism. To show that it is injective, let \mathbf{K}_1 and \mathbf{K}_2 be subcovarieties of the covariety $\mathcal{SH}\Sigma(\mathbb{A})$ such that $\mathbb{S}_{\mathbf{K}_1} = \mathbb{S}_{\mathbf{K}_2}$. Then

$$\mathbf{K}_1 = \Sigma^C \mathcal{S}(\mathbb{S}_{\mathbf{K}_1}) = \Sigma^C \mathcal{S}(\mathbb{S}_{\mathbf{K}_2}) = \mathbf{K}_2.$$

We will now show that $\mathbb{S}_{(-)}$ is a surjection. Let $\mathbb{C} \leq \mathbb{A}$. Then

$$\mathbb{C} \leq \mathbb{S}_{\mathcal{SH}\Sigma(\mathbb{C})}.$$

Since $\mathbb{S}_{\mathcal{SH}\Sigma(\mathbb{C})} \leq \mathbb{A}$ and since $\mathbb{S}_{\mathcal{SH}\Sigma(\mathbb{C})} \in \mathcal{SH}\Sigma(\mathbb{C}) = \Sigma^C \mathcal{S}(\mathbb{C})$, it follows that

$$\mathbb{S}_{\mathcal{SH}\Sigma(\mathbb{C})} \in \Sigma^C(\{\mathbb{D}_j\}_{j \in J}),$$

where $\mathbb{D}_j \leq \mathbb{C}$. This means that for any $j \in J$, the coalgebra $\mathbb{S}_{\mathcal{SH}\Sigma(\mathbb{C})}$ contains a coalgebra $\tilde{\mathbb{D}}_j$ isomorphic to \mathbb{D}_j as its subcoalgebra. Hence

$$\tilde{\mathbb{D}}_j \leq \mathbb{S}_{\mathcal{SH}\Sigma(\mathbb{C})} \leq \mathbb{A}$$

for all $j \in J$, and $\mathbb{D}_j \leq \mathbb{C} \leq \mathbb{A}$. By Lemma 3.4, we have $\tilde{\mathbb{D}}_j = \mathbb{D}_j$. Therefore,

$$\mathbb{S}_{\mathcal{SH}\Sigma(\mathbb{C})} = \bigcup_{j \in J} \tilde{\mathbb{D}}_j = \bigcup_{j \in J} \mathbb{D}_j \leq \mathbb{C}$$

and $\mathbb{S}_{\mathcal{SH}\Sigma(\mathbb{C})} = \mathbb{C}$. Consequently the mapping $\mathbb{S}_{(-)}$ is a bijection. Since it is clear that $\mathbb{S}_{(-)}$ is order preserving we immediately get that $\mathbb{S}_{(-)}$ is the isomorphism from the lattice $L_{CV}(\mathcal{SH}\Sigma(\mathbb{A}))$ onto $(\mathcal{S}(\mathbb{A}), \cup, \cap)$. ■

For an F -coalgebra $\mathbb{A} = (A, \alpha)$ and a set B such that $A \subseteq B$, we define the following $B \times F$ -coalgebra:

$$\mathbb{A}_B := (A, (\subseteq_A^B, \alpha)).$$

The structure map of \mathbb{A}_B is the following:

$$(\subseteq_A^B, \alpha) : A \rightarrow B \times F(A); a \mapsto (a, \alpha(a)).$$

This easy trick allows us to force the $B \times F$ -coalgebra \mathbb{A}_B to be strongly simple and at the same time to leave the subcoalgebras of \mathbb{A} untouched. This property is formally described by the following lemmata.

Lemma 3.7. *Let $\mathbb{A} = (A, \alpha)$ be an F -coalgebra and let B be a set such that $A \subseteq B$. Then the $B \times F$ -coalgebra \mathbb{A}_B is strongly simple.*

Lemma 3.8. *Let $\mathbb{A} = (A, \alpha)$ be an F -coalgebra and let B be a set such that $A \subseteq B$. Then $(S(\mathbb{A}), \cup, \cap) = (S(\mathbb{A}_B), \cup, \cap)$.*

Corollary 3.9. *Let (X, τ) be a topological space. There exists a bounded functor $F : \mathbf{Set} \rightarrow \mathbf{Set}$ and a covariety \mathbf{K} of F -coalgebras such that $L_{\mathcal{CV}}(\mathbf{K})$ is isomorphic to the lattice (τ, \cup, \cap) of open sets in τ .*

Proof. It follows by Example 2.4, Lemma 3.8, Lemma 3.7 and Theorem 3.6. ■

4. COVARIETY LATTICES FOR FUNCTORS PRESERVING ARBITRARY INTERSECTIONS

Throughout this section we will assume that F is a bounded functor. Therefore, the collection of all covarieties of F -coalgebras is a set. It is worth noting that almost all of the results presented here naturally generalize to the case when classes of covarieties are allowed.

Given a strongly simple F -coalgebra \mathbb{A} , Theorem 3.6 describes the lattice of subcovarieties of the covariety $\mathcal{SH}\Sigma(\mathbb{A})$ in terms of the lattice of subcoalgebras of \mathbb{A} . The following question arises: can we describe the covariety lattice of any covariety \mathbf{K} of F -coalgebras in a similar way in terms of subcoalgebras of an F -coalgebra? In general the answer is “no”, which is seen in the Example 4.4. But first, we will characterize the lattice $L_{\mathcal{CV}}(\mathbf{Set}_F)$ in the case the functor F preserves arbitrary intersections.

An F -coalgebra \mathbb{A} is called *rooted* (or *one-generated*) if there exists an element $a \in A$, called a *root*, such that the coalgebra \mathbb{A} is the smallest subcoalgebra of \mathbb{A} containing the element a . If $a \in A$ is a root of a rooted coalgebra \mathbb{A} , then we say that \mathbb{A} is generated by a .

If $F : \mathbf{Set} \rightarrow \mathbf{Set}$ preserves arbitrary intersections, then all rooted F -coalgebras are of the following form

$$\langle a \rangle := \bigcap \{ \mathbb{S} \mid a \in S \text{ and } \mathbb{S} \leq \mathbb{A} \},$$

for some F -coalgebra \mathbb{A} and $a \in A$. For any F -coalgebra \mathbb{A} , we have $\mathbb{A} = \bigcup_{a \in A} \langle a \rangle$. It follows that $\mathbb{A} \in \Sigma^C(\{\langle a \rangle\}_{a \in A})$.

Let \mathbf{K} be a class of F -coalgebras. Let $\mathfrak{R}_{\mathbf{K}}$ denote the collection of rooted F -coalgebras consisting of exactly one representative from each class of isomorphic rooted F -coalgebras from the class \mathbf{K} . If $\mathbb{A}, \mathbb{B} \in \mathfrak{R}_{\mathbf{K}}$ and are isomorphic, then $\mathbb{A} = \mathbb{B}$. By the assumption of boundedness of F we know that $\mathfrak{R}_{\mathbf{K}}$ is a proper set. Let $\mathcal{D}(\mathfrak{R}_{\mathbf{K}})$ denote the set of subsets of $\mathfrak{R}_{\mathbf{K}}$ closed under taking subcoalgebras of homomorphic images, i.e.:

$$\mathcal{D}(\mathfrak{R}_{\mathbf{K}}) := \{ U \subseteq \mathfrak{R}_{\mathbf{K}} \mid \mathfrak{R}_{\mathbf{K}} \cap \mathcal{SH}(U) = U \}.$$

Theorem 4.1. *If $F : \mathbf{Set} \rightarrow \mathbf{Set}$ preserves arbitrary intersections then the lattice $L_{CV}(\mathbf{Set}_F)$ of subcovarieties of \mathbf{Set}_F is isomorphic to the lattice $(\mathcal{D}(\mathfrak{R}_{\mathbf{Set}_F}), \cup, \cap)$.*

Proof. Let \mathbf{K} be a covariety of F -coalgebras. Let $\mathbb{A} \in \mathfrak{R}_{\mathbf{K}}$. Then \mathbb{A} is a rooted coalgebra in the covariety \mathbf{K} . The rooted subcoalgebras of homomorphic images of \mathbb{A} are elements of the set $\mathfrak{R}_{\mathbf{K}}$. This means that $\mathfrak{R}_{\mathbf{K}} \in \mathcal{D}(\mathfrak{R}_{\mathbf{Set}_F})$. We define the following mapping.

$$r : L_{CV}(\mathbf{Set}_F) \rightarrow \mathcal{D}(\mathfrak{R}_{\mathbf{Set}_F}); \mathbf{K} \mapsto \mathfrak{R}_{\mathbf{K}}.$$

We will show that r is an isomorphism. Let \mathbf{K}_1 and \mathbf{K}_2 be two covarieties such that $r(\mathbf{K}_1) = r(\mathbf{K}_2)$. Let $\mathbb{A} \in \mathbf{K}_1$. For any $a \in A$ the rooted coalgebra $\langle a \rangle$ is a subcoalgebra of \mathbb{A} . Hence $\langle a \rangle \in \mathbf{K}_1$ and $\langle a \rangle \in \mathbf{K}_2$. Since $\mathbb{A} = \bigcup_{a \in A} \langle a \rangle$, the coalgebra \mathbb{A} belongs to \mathbf{K}_2 . Therefore, $\mathbf{K}_1 = \mathbf{K}_2$ and the mapping r is injective.

Now let $U \in \mathcal{D}(\mathfrak{R}_{\mathbf{Set}_F})$. The smallest covariety containing U is given by the class $\mathcal{SH}\Sigma(U)$. It is clear that $U \subseteq r(\mathcal{SH}\Sigma(U))$. Now let $\mathbb{A} \in r(\mathcal{SH}\Sigma(U))$. This means that \mathbb{A} is a rooted coalgebra, say $\mathbb{A} = \langle a \rangle$, and

is a subcoalgebra of \mathbb{B} , where $\mathbb{B} = h(\Sigma_{i \in I} \mathbb{C}_i)$ is a homomorphic image of the disjoint sum of a family $\{\mathbb{C}_i\}_{i \in I}$ of rooted coalgebras in U . Let $e_i : \mathbb{C}_i \rightarrow \Sigma_{i \in I} \mathbb{C}_i$ denote the canonical embeddings. It is easy to see that $\mathbb{B} = h(\Sigma_{i \in I} \mathbb{C}_i) = \bigcup_{i \in I} h(e_i(\mathbb{C}_i))$. Since $\langle a \rangle \leq \mathbb{B}$, it follows that $a \in h(e_j(\mathbb{C}_j))$ for some $j \in I$. Hence $\langle a \rangle \leq h(e_j(\mathbb{C}_j))$. Since U is closed under taking rooted subcoalgebras of homomorphic images, it follows that $\mathbb{A} = \langle a \rangle \in U$. Therefore $U = r(\mathcal{SH}\Sigma(U))$ and the mapping r is surjective. Consequently r is bijective. It is clear that the mapping r is an order embedding. Hence r is a lattice isomorphism. ■

Remark 4.2. It is worth noting that the mapping r in the proof of Theorem 4.1 is in fact a complete lattice isomorphism.

Corollary 4.3. *Let $F : \mathbf{Set} \rightarrow \mathbf{Set}$ preserve arbitrary intersections and let \mathbf{K} be a covariety of F -coalgebras. Then $L_{\mathcal{CV}}(\mathbf{K}) \cong (\mathcal{D}(\mathfrak{R}_{\mathbf{K}}), \cup, \cap)$.*

Example 4.4. We will describe the covariety lattice $L_{\mathcal{CV}}(\mathbf{Set}_{\mathcal{Id}})$. By Theorem 4.1, the first step is to find all rooted \mathcal{Id} -coalgebras. Note that \mathcal{Id} -coalgebras are exactly mono-unary algebras. Therefore, we can speak of an *index* and *period* of a rooted \mathcal{Id} -coalgebra. Let $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. It is easy to see that every rooted \mathcal{Id} -coalgebra can be represented by a pair $(i, p) \in \mathbf{N}_0 \times \mathbf{N} \cup \{(\infty, 0)\}$, where i denotes an index and p a period of a given coalgebra. E.g. $(0, 2)$ denotes the coalgebra given by the diagram $\bullet \rightleftharpoons \bullet$ and $(1, 2)$ by the diagram $\bullet \rightarrow \bullet \rightleftharpoons \bullet$. Given a finite rooted \mathcal{Id} -coalgebra (i, p) , it is not hard to notice that any rooted subcoalgebra of (i, p) is of the form (i', p) , where $i' \leq i$. Any subcoalgebra of $(\infty, 0)$ is of the form $(\infty, 0)$. Moreover, any rooted homomorphic image of a coalgebra (i, p) is of the form (i', p') , where $i' \leq i$ and $p' \mid p$. Therefore,

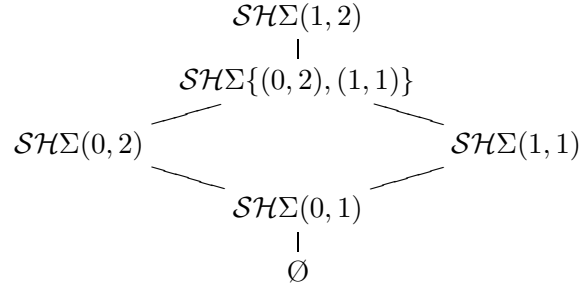
$$\mathcal{SH}((i, p)) = \{(i', p') \in \mathbf{N}_0 \times \mathbf{N} \cup \{(\infty, 0)\} \mid i' \leq i \text{ and } p' \mid p\}.$$

We can introduce a partial order on $\mathbf{N}_0 \times \mathbf{N} \cup \{(\infty, 0)\}$ as follows: $(i', p') \preceq (i, p) : \iff i' \leq i \text{ and } p' \mid p$. Then

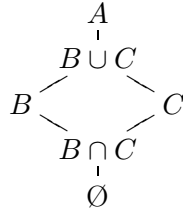
$$\mathcal{SH}((i, p)) = \downarrow (i, p) := \{(i', p') \mid (i', p') \preceq (i, p)\}.$$

By Theorem 4.1, the lattice $L_{\mathcal{CV}}(\mathbf{Set}_{\mathcal{Id}})$ of subcovarieties of $\mathbf{Set}_{\mathcal{Id}}$ is isomorphic to the lattice of downsets $(\mathcal{O}(\mathbf{N}_0 \times \mathbf{N} \cup \{(\infty, 0)\}), \cup, \cap)$ of the poset $\mathbf{N}_0 \times \mathbf{N} \cup \{(\infty, 0)\}$.

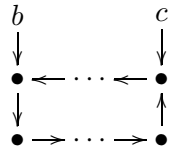
Now, consider the \mathcal{Id} -coalgebra $(1, 2)$. The covariety lattice of $\mathcal{SH}\Sigma(1, 2)$ looks as follows:



At the beginning of this section we stated a question whether it was possible to describe a covariety lattice $L_{CV}(\mathbf{K})$ of any covariety \mathbf{K} of F -coalgebras in terms of subcoalgebras of an F -coalgebra. We will show that it is impossible to construct an \mathcal{Id} -coalgebra \mathbb{A} , whose subcoalgebra lattice is isomorphic to the covariety lattice $\mathcal{SH}\Sigma(1, 2)$. By contradiction, assume that there exists \mathcal{Id} -coalgebra \mathbb{A} whose subcoalgebra lattice is the following:



Join irreducible elements, i.e. \mathbb{B}, \mathbb{C} and $\mathbb{B} \cap \mathbb{C}$, must be rooted \mathcal{Id} -coalgebras. The rooted coalgebra $\mathbb{B} \cap \mathbb{C}$ does not contain any proper subcoalgebras. This means that $\mathbb{B} \cap \mathbb{C}$ is a cycle. The coalgebras $\mathbb{B} = \langle b \rangle$ and $\mathbb{C} = \langle c \rangle$ cover the coalgebra $\mathbb{B} \cap \mathbb{C}$. Hence the coalgebra $\mathbb{B} \cup \mathbb{C}$ has the following form.



Since \mathbb{A} itself is join irreducible, it follows that it is rooted, i.e. $\mathbb{A} = \langle a \rangle$. On one hand the element a has to be connected directly with the element b and on the other with the element c , which is a contradiction. \square

Theorem 4.5. *Let $F : \mathbf{Set} \rightarrow \mathbf{Set}$ be a functor preserving arbitrary intersections. Then the lattice $L_{\mathcal{CV}}(\mathbf{K})$ of subcovarieties of a covariety \mathbf{K} of F -coalgebras is isomorphic to the lattice of subcoalgebras of some \mathcal{P}_κ -coalgebra.*

Conversely, for any \mathcal{P}_κ -coalgebra \mathbb{A} , there exists a functor $F : \mathbf{Set} \rightarrow \mathbf{Set}$ preserving arbitrary intersections and a covariety \mathbf{K} of F -coalgebras such that the lattice $L_{\mathcal{CV}}(\mathbf{K})$ is isomorphic to the lattice of subcoalgebras of \mathbb{A} .

Proof. If F preserves arbitrary intersection, then by Theorem 4.1, the lattice $L_{\mathcal{CV}}(\mathbf{Set}_F)$ of subcovarieties of \mathbf{Set}_F is isomorphic to the lattice $(\mathcal{D}(\mathfrak{R}_{\mathbf{Set}_F}), \cup, \cap)$. Take $\kappa := |\mathfrak{R}_{\mathbf{Set}_F}|$. Define a \mathcal{P}_κ -coalgebra $(\mathfrak{R}_{\mathbf{Set}_F}, \eta)$ as follows. For $\langle a \rangle \in \mathfrak{R}_{\mathbf{Set}_F}$ define

$$\eta(\langle a \rangle) := \mathcal{SH}(\langle a \rangle) \cap \mathfrak{R}_{\mathbf{Set}_F}.$$

Then clearly

$$\mathcal{S}((\mathfrak{R}_{\mathbf{Set}_F}, \eta)) \cong \mathcal{D}(\mathfrak{R}_{\mathbf{Set}_F}) \cong L_{\mathcal{CV}}(\mathbf{Set}_F).$$

Conversely let $\mathbb{A} = (A, \alpha)$ be a \mathcal{P}_κ -coalgebra. Then by Theorem 3.6, the lattice $L_{\mathcal{CV}}(\mathcal{SH}\Sigma(\mathbb{A}_A))$ of subcovarieties of the covariety $\mathcal{SH}\Sigma(\mathbb{A}_A)$ of $A \times \mathcal{P}_\kappa$ -coalgebras is isomorphic to $\mathcal{S}(\mathbb{A})$ and the functor $A \times \mathcal{P}_\kappa$ is bounded and preserves arbitrary intersections. ■

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