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THE ORDER OF NORMAL FORM HYPERSUBSTITUTIONS OF TYPE (2)

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Abstract

In [2] it was proved that all hypersubstitutions of type $\tau = (2)$ which are not idempotent and are different from the hypersubstitution which maps the binary operation symbol f to the binary term $f(y, x)$ have infinite order. In this paper we consider the order of hypersubstitutions within given varieties of semigroups. For the theory of hypersubstitution see [3].

Keywords: hypersubstitutions, terms, idempotent elements, elements of infinite order.

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1 Preliminaries

In [1] *hypersubstitutions* were defined to make the concept of a *hyperidentity* more precise. In this paper we consider the type $\tau = (2)$ and the binary operation symbol f. Type (2) hypersubstitutions seem to be simple enough to be accessible, yet rich enough to provide an interesting structure.

An identity $s \approx t$ of type $\tau = (2)$ is called a hyperidentity of a variety V of this type if for every substitution of terms built up by at most two variables (binary terms) for f in $s \approx t$, the resulting identity holds in V. This shows that we are interested in mappings

$$\sigma : \{f\} \to W(X_2),$$

where $W(X_2)$ is the set of all terms constructed by f and the variables from the two-element alphabet $X_2 = \{x, y\}$. Any such mapping is called a hypersubstitution of type $\tau = (2)$. By σ_t we denote the hypersubstitution $\sigma : \{f\} \to \{t\}$.

A hypersubstitutions σ can be uniquely extended to a mapping $\hat{\sigma}$ on $W(X)$ (the set of all terms built up by f and variables from the countably infinite alphabet $X = \{x, y, z, \cdots\}$) inductively defined by

(i) if $t = x$ for some variable x, then $\hat{\sigma}[t] = x$,

(ii) if $t = f(t_1, t_2)$ for some terms t_1, t_2, then $\hat{\sigma}[t] = \sigma(f)(\hat{\sigma}[t_1], \hat{\sigma}[t_2])$.

By *Hyp* we denote the set of all hypersubstitutions of type $\tau = (2)$. For any two hypersubstitutions σ_1, σ_2 we define a product

$$\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$$

and obtain together with $\sigma_{id} = \sigma_{xy}$, i.e., $\sigma_{id}(f) = xy$, a monoid $\underline{Hyp} = (Hyp; \circ_h, \sigma_{id})$. We will refer to this monoid as to $\underline{Hyp}$. In [2] Denecke and Wismath described all idempotent elements of *Hyp*.

We use the following denotation: Let W_x denote the set of all words using only the letter x, and dually for W_y. We set

$$E_x = \{\sigma_{xu} |\ u \in W_x\},\ \ E_y = \{\sigma_{vy} |\ v \in W_y\},\ \ E = E_x \cup E_y,$$

where xu abbreviates $f(x, u)$.

Clearly, for any element xu with $u \in W_x$ we have

$$\sigma_{xu} \circ_h \sigma_{xu} = \sigma_{xu}.$$

and for any element vy with $v \in W_y$ we have

$$\sigma_{vy} \circ_h \sigma_{vy} = \sigma_{vy}.$$

This shows that all elements of E are idempotent. The hypersubstitutions σ_x, σ_y mapping the binary operation symbol f to x and to y, respectively, and the identity hypersubstitution are also idempotent.

The hypersubstitution σ_{yx} satisfies the equation

$$\sigma_{yx} \circ_h \sigma_{yx} = \sigma_{xy}.$$

Further we have:

Proposition 1.1 (see [2]). *If $\sigma_s \circ_h \sigma_t = \sigma_{id}$, then either $\sigma_s = \sigma_t = \sigma_{id}$ or $\sigma_s = \sigma_t = \sigma_{yx}$.* ■

In the following theorem we will use the concept of the length of a term as number of occurrences of variables in the term.

In [2] was proved

Theorem 1.2.

(i) *If $\sigma \in Hyp$ is an idempotent, then $\sigma \in E \cup \{\sigma_x, \sigma_y, \sigma_{xy}\}$.*

(ii) *If $\sigma \in Hyp \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx}\})$, then $\sigma^n \neq \sigma^{n+1}$ for all $n \in \mathbb{N}$ with $n \geq 1$ (i.e. σ has infinite order).*

(iii) *If $\sigma \in Hyp \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx}\})$, then the length of the word $(\sigma \circ_h \sigma)(f)$ is greater than the length of $\sigma(f)$.* ■

If we set $G := Hyp \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx}\})$, then G does not form a subsemigroup of *Hyp*. In fact, we consider the hypersubstitution σ_{wx} where w is a term different from x and from y. Then $\sigma_{wx} \in G$. Let $u \in W_x$ and let $\overline{xu} \in W_x$ be the term formed from xu by substitution of all occurrences of the letters x by y, then $\sigma_{\overline{xu}} \in G$. But then we see

$$\sigma_{\overline{xu}} \circ_h \sigma_{wx} = \sigma_{xu}$$

and the product of these elements from G is outside of G.

If we want to check whether an equation $s \approx t$ is satisfied as a hyperidentity in a given variety V of semigroups, it is not necessary to test all hypersubstitutions from *Hyp*. Depending on the identities satisfied in V we may restrict ourselves to a smaller subset of *Hyp*. By definition of a binary operation on this subset, we will define a new algebra which, in general is not a monoid and will determine the order of elements of those algebras.

2 Normal Form hypersubstitutions

In [4] J. Płonka defined a binary relation on the set of all hypersubstitutions of an arbitrary type with respect to a variety of this type.

Definition 2.1. Let V be a variety of semigroups, and let $\sigma_1, \sigma_2 \in Hyp$. Then

$$\sigma_1 \sim_V \sigma_2 :\Leftrightarrow \sigma_1(f) \approx \sigma_2(f) \in IdV.$$

Clearly, the relation $\sim_V$ is an equivalence relation on *Hyp* and has the following properties:

Proposition 2.2 ([3]). *Let V be a variety of semigroups and let $\sigma_1, \sigma_2 \in Hyp$.*

(i) *If $\sigma_1 \sim_V \sigma_2$, then for any term t of type $\tau = (2)$ the equation $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ is an identity of V.*

(ii) *If $s \approx t \in IdV, \hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV$ and $\sigma_1 \sim_V \sigma_2 \in IdV$, then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV$.* ■

In general, the relation $\sim_V$ is not a congruence relation on *Hyp*. A variety is called *solid* if every identity in V is satisfied as a hyperidentity. For a solid variety V the relation $\sim_V$ is a congruence relation on *Hyp* and the factor monoid $\underline{Hyp/_{\sim_V}}$ exists.

In the arbitrary case we form also $Hyp/_{\sim_V}$ and consider a choice function

$$\varphi : Hyp/_{\sim_V} \to Hyp, \text{ with } \varphi([\sigma_{id}]_{\sim_V}) = \sigma_{id},$$

which selects from each equivalence class exactly one element. Then we obtain the set $Hyp_{N_\varphi}(V) := \varphi(Hyp/_{\sim_V})$ of all *normal form hypersubstitutions* with respect to V and φ.

On the set $Hyp_{N_\varphi}(V)$ we define a binary operation

$$\circ_N : Hyp_{N_\varphi}(V) \times Hyp_{N_\varphi}(V) \to Hyp_{N_\varphi}(V)$$

by $\sigma_1 \circ_N \sigma_2 = \varphi(\sigma_1 \circ_h \sigma_2)$. This mapping is well-defined, but in general not associative. Therefore, $(Hyp_{N_\varphi}(V); \circ_N, \sigma_{id})$ is not a monoid. We call this structure *groupoid of normal form hypersubstitutions*. We ask, how to characterize the idempotent elements of $Hyp_{N_\varphi}(V)$ since for practical work normal form hypersubstitutions are more important than usual hypersubstitutions.

Proposition 2.3. *Let V be a variety of semigroups and let*

$$\varphi : Hyp/_{\sim_V} \to Hyp$$

be a choice function. Then

(i) $\sigma \in Hyp_{N_\varphi}(V)$ *is an idempotent element iff* $\sigma \circ_h \sigma \sim_V \sigma$.

(ii) $\sigma_{yx} \circ_N \sigma_{yx} = \sigma_{xy}$ *if* $\sigma_{yx} \in Hyp_{N_\varphi}(V)$.

Proof. (i) If σ is an idempotent of $Hyp_{N_\varphi}(V)$, then $\sigma \circ_N \sigma = \sigma \sim_V \sigma \circ_h \sigma$. If conversely $\sigma \sim_V \sigma \circ_h \sigma$, then $\sigma \circ_N \sigma \sim_V \sigma$. But then $\sigma \circ_N \sigma = \sigma$ because of $\sigma \in Hyp_{N_\varphi}(V)$.

(ii) $\sigma_{yx} \circ_N \sigma_{yx} \sim_V \sigma_{yx} \circ_h \sigma_{yx} = \sigma_{xy} \in Hyp_{N_\varphi}(V)$. Therefore, $\sigma_{yx} \circ_N \sigma_{yx} = \sigma_{xy}$. ■

As a consequence we have: if σ is an idempotent of *Hyp* and $\sigma \in Hyp_{N_\varphi}(V)$, then it is also an idempotent in $Hyp_{N_\varphi}(V)$ for any variety V of semigroups and any choice function φ. But in general $Hyp_{N_\varphi}(V)$ has idempotents which are not idempotents in *Hyp*.

3 Idempotents in $Hyp_{N_\varphi}(V)$

Now we want to consider the following variety of semigroups: $V = Mod\{(xy)z \approx x(yz), xyuv \approx xuyv, x^3 \approx x\}$, i.e., the variety of all medial semigroups satisfying $x^3 \approx x$.

Let f be our binary operation symbol. As usual instead of $f(x,y)$ we will also write xy. The elements of $W(X_2)/IdV$ where $X_2 = \{x,y\}$ is a two-element alphabet, have the following form: $[x^n y^m]_{IdV}, [y^n x^m]_{IdV}$, $[xy^m x^n]_{IdV}, [yx^m y^n]_{IdV}$ where $0 \le m,n \le 2$. So we get the set

$$W(X_2)/IdV =$$
$$= \{[x]_{IdV}, [x^2]_{IdV}, [xy]_{IdV}, [xy^2]_{IdV}, [x^2y]_{IdV}, [xyx]_{IdV}, [x^2y^2]_{IdV}, [xy^2x]_{IdV},$$
$$[xyx^2]_{IdV}, [xy^2x^2]_{IdV}, [y]_{IdV}, [y^2]_{IdV}, [yx]_{IdV}, [yx^2]_{IdV}, [y^2x]_{IdV}, [yxy]_{IdV},$$
$$[y^2x^2]_{IdV}, [yx^2y]_{IdV}, [yxy^2]_{IdV}, [yx^2y^2]_{IdV}.\}$$

From each class we exchange a normal form term using a certain choice function φ and obtain the following set of normal form hypersubstitutions: $Hyp_{N_\varphi}(V) = \{\sigma_x, \sigma_{x^2}, \sigma_{xy}, \sigma_{xy^2}, \sigma_{x^2y}, \sigma_{xyx}, \sigma_{x^2y^2}, \sigma_{xy^2x}, \sigma_{xyx^2}, \sigma_{xy^2x^2}, \sigma_y, \sigma_{y^2}, \sigma_{yx}, \sigma_{yx^2}, \sigma_{y^2x}, \sigma_{yxy}, \sigma_{y^2x^2}, \sigma_{yx^2y}, \sigma_{yxy^2}, \sigma_{yx^2y^2}\}$.

The multiplication in the groupoid $(Hyp_{N_\varphi}(V); \circ_N, \sigma_{id})$ is given by the following table.

$\circ_N$	σ_x	σ_{x^2}	σ_{xy}	σ_{xy^2}	σ_{x^2y}	σ_{xyx}	$\sigma_{x^2y^2}$	σ_{xy^2x}	σ_{xyx^2}	$\sigma_{xy^2x^2}$	σ_y	σ_{y^2}	σ_{yx}	σ_{yx^2}	σ_{y^2x}	σ_{yxy}	$\sigma_{y^2x^2}$	σ_{yx^2y}	σ_{yxy^2}	$\sigma_{yx^2y^2}$
σ_x	σ_x	σ_x	σ_x	σ_x	σ_x	σ_x	σ_x	σ_x	σ_x	σ_x	σ_y	σ_y	σ_y	σ_y	σ_y	σ_y	σ_y	σ_y	σ_y	σ_y
σ_{x^2}	σ_x	σ_{x^2}	σ_{x^2}	σ_{x^2}	σ_{x^2}	σ_{x^2}	σ_{x^2}	σ_{x^2}	σ_{x^2}	σ_{x^2}	σ_y	σ_{y^2}	σ_{y^2}	σ_{y^2}	σ_{y^2}	σ_{y^2}	σ_{y^2}	σ_{y^2}	σ_{y^2}	σ_{y^2}
σ_{xy}	σ_x	σ_{x^2}	σ_{xy}	σ_{xy^2}	σ_{x^2y}	σ_{xyx}	$\sigma_{x^2y^2}$	σ_{xy^2x}	σ_{xyx^2}	$\sigma_{xy^2x^2}$	σ_y	σ_{y^2}	σ_{yx}	σ_{yx^2}	σ_{y^2x}	σ_{yxy}	$\sigma_{y^2x^2}$	σ_{yx^2y}	σ_{yxy^2}	$\sigma_{yx^2y^2}$
σ_{xy^2}	σ_x	σ_x	σ_{xy^2}	σ_{xy^2}	σ_{xy^2}	$\sigma_{xy^2x^2}$	σ_{xy^2}	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	σ_y	σ_y	σ_{yx^2}	σ_{yx^2}	σ_{yx^2}	$\sigma_{yx^2y^2}$	σ_{yx^2}	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$
σ_{x^2y}	σ_x	σ_x	σ_{x^2y}	σ_{x^2y}	σ_{x^2y}	$\sigma_{xy^2x^2}$	σ_{x^2y}	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	σ_y	σ_y	σ_{y^2x}	σ_{y^2x}	σ_{y^2x}	$\sigma_{yx^2y^2}$	σ_{y^2x}	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$
σ_{xyx}	σ_x	σ_x	σ_{xyx}	σ_{xyx}	σ_{xyx}	$\sigma_{xy^2x^2}$	σ_{xyx}	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	σ_y	σ_y	σ_{yxy}	σ_{yxy}	σ_{yxy}	$\sigma_{yx^2y^2}$	σ_{yxy}	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$
$\sigma_{x^2y^2}$	σ_x	σ_{x^2}	$\sigma_{x^2y^2}$	$\sigma_{x^2y^2}$	$\sigma_{x^2y^2}$	σ_{xy^2x}	$\sigma_{x^2y^2}$	σ_{xy^2x}	σ_{xy^2x}	σ_{xy^2x}	σ_y	σ_{y^2}	$\sigma_{y^2x^2}$	$\sigma_{y^2x^2}$	$\sigma_{y^2x^2}$	σ_{yx^2y}	$\sigma_{y^2x^2}$	σ_{yx^2y}	σ_{yx^2y}	σ_{yx^2y}
σ_{xy^2x}	σ_x	σ_{x^2}	σ_{xy^2x}	σ_{xy^2x}	σ_{xy^2x}	σ_{xy^2x}	σ_{xy^2x}	σ_{xy^2x}	σ_{xy^2x}	σ_{xy^2x}	σ_y	σ_{y^2}	σ_{yx^2y}	σ_{yx^2y}	σ_{yx^2y}	σ_{yx^2y}	σ_{yx^2y}	σ_{yx^2y}	σ_{yx^2y}	σ_{yx^2y}
σ_{xyx^2}	σ_x	σ_{x^2}	σ_{xyx^2}	$\sigma_{xy^2x^2}$	σ_{xyx}	σ_{xyx}	σ_{xy^2x}	σ_{xy^2x}	σ_{xyx^2}	$\sigma_{xy^2x^2}$	σ_y	σ_{y^2}	σ_{yxy^2}	$\sigma_{yx^2y^2}$	σ_{yxy}	σ_{yxy}	σ_{yx^2y}	σ_{yx^2y}	σ_{yxy^2}	$\sigma_{yx^2y^2}$
$\sigma_{xy^2x^2}$	σ_x	σ_x	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	σ_y	σ_y	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$
σ_y	σ_x	σ_x	σ_y	σ_y	σ_y	σ_x	σ_y	σ_x	σ_x	σ_x	σ_y	σ_y	σ_x	σ_x	σ_x	σ_y	σ_x	σ_y	σ_y	σ_y
σ_{y^2}	σ_x	σ_{x^2}	σ_{y^2}	σ_{y^2}	σ_{y^2}	σ_{x^2}	σ_{y^2}	σ_{x^2}	σ_{x^2}	σ_{x^2}	σ_y	σ_{y^2}	σ_{x^2}	σ_{x^2}	σ_{x^2}	σ_{y^2}	σ_{x^2}	σ_{y^2}	σ_{y^2}	σ_{y^2}
σ_{yx}	σ_x	σ_{x^2}	σ_{yx}	σ_{y^2x}	σ_{yx^2}	σ_{xyx}	$\sigma_{y^2x^2}$	σ_{xy^2x}	σ_{xyx^2}	$\sigma_{xy^2x^2}$	σ_y	σ_{y^2}	σ_{xy}	σ_{x^2y}	σ_{xy^2}	σ_{yxy}	$\sigma_{x^2y^2}$	σ_{yx^2y}	σ_{yxy^2}	$\sigma_{yx^2y^2}$
σ_{yx^2}	σ_x	σ_x	σ_{yx^2}	σ_{yx^2}	σ_{yx^2}	$\sigma_{xy^2x^2}$	σ_{yx^2}	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	σ_y	σ_y	σ_{xy^2}	σ_{xy^2}	σ_{xy^2}	$\sigma_{yx^2y^2}$	σ_{xy^2}	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$
σ_{y^2x}	σ_x	σ_x	σ_{y^2x}	σ_{y^2x}	σ_{y^2x}	$\sigma_{xy^2x^2}$	σ_{y^2x}	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	σ_y	σ_y	σ_{x^2y}	σ_{x^2y}	σ_{x^2y}	$\sigma_{yx^2y^2}$	σ_{x^2y}	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$
σ_{yxy}	σ_x	σ_x	σ_{yxy}	σ_{yxy}	σ_{yxy}	$\sigma_{xy^2x^2}$	σ_{yxy}	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	σ_y	σ_y	σ_{xyx}	σ_{xyx}	σ_{xyx}	$\sigma_{yx^2y^2}$	σ_{xyx}	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$
$\sigma_{y^2x^2}$	σ_x	σ_{x^2}	$\sigma_{y^2x^2}$	$\sigma_{y^2x^2}$	$\sigma_{y^2x^2}$	σ_{xy^2x}	$\sigma_{y^2x^2}$	σ_{xy^2x}	σ_{xy^2x}	σ_{xy^2x}	σ_y	σ_{y^2}	$\sigma_{x^2y^2}$	$\sigma_{x^2y^2}$	$\sigma_{x^2y^2}$	σ_{yx^2y}	$\sigma_{x^2y^2}$	σ_{yx^2y}	σ_{yx^2y}	σ_{yx^2y}
σ_{yx^2y}	σ_x	σ_{x^2}	σ_{yx^2y}	σ_{yx^2y}	σ_{yx^2y}	σ_{xy^2x}	σ_{yx^2y}	σ_{xy^2x}	σ_{xy^2x}	σ_{xy^2x}	σ_y	σ_{y^2}	σ_{xy^2x}	σ_{xy^2x}	σ_{xy^2x}	σ_{yx^2y}	σ_{xy^2x}	σ_{yx^2y}	σ_{yx^2y}	σ_{yx^2y}
σ_{yxy^2}	σ_x	σ_{x^2}	σ_{yxy^2}	σ_{yxy}	$\sigma_{yx^2y^2}$	σ_{xyx}	σ_{yx^2y}	σ_{xy^2x}	σ_{xyx^2}	$\sigma_{xy^2x^2}$	σ_y	σ_{y^2}	σ_{xyx^2}	σ_{xyx}	$\sigma_{xy^2x^2}$	σ_{yxy}	σ_{xy^2x}	σ_{yx^2y}	σ_{yxy^2}	$\sigma_{yx^2y^2}$
$\sigma_{yx^2y^2}$	σ_x	σ_x	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$	$\sigma_{yx^2y^2}$	$\sigma_{xy^2x^2}$	$\sigma_{yx^2y^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	σ_y	σ_y	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$	$\sigma_{xy^2x^2}$

The table shows that there are many idempotents in $Hyp_{N_\varphi}(V)$ which are not idempotents in *Hyp*.

The following example shows that $(Hyp_N(V); \circ_N, \sigma_{id})$ is not a monoid:

$$(\sigma_{x^2} \circ_N \sigma_{xy^2}) \circ_N \sigma_{x^2} = \sigma_{x^2} \circ_N \sigma_{x^2} = \sigma_{x^2},$$

$$\sigma_{x^2} \circ_N (\sigma_{xy^2} \circ_N \sigma_{x^2}) = \sigma_{x^2} \circ_N \sigma_x = \sigma_x.$$

All idempotent elements of $Hyp_N(V)$ are

$$\{\sigma_{xy}, \sigma_x, \sigma_{x^2}, \sigma_{xy^2}, \sigma_{x^2y}, \sigma_{x^2y^2}, \sigma_{xy^2x}, \sigma_{xyx^2}, \sigma_{xy^2x^2}, \sigma_y, \sigma_{y^2}, \sigma_{yx^2y}, \sigma_{yxy^2}, \sigma_{yx^2y^2}\}.$$

Now we ask for which varieties at most the idempotents of *Hyp* are idempotents of $Hyp_{N_\varphi}(V)$.

Theorem 3.1. *For a variety V of semigroups the following are equivalent:*

(i) $Mod\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V$,

(ii) $\{\sigma | \sigma \in Hyp_{N_\varphi}(V)$ and $\sigma \circ_N \sigma = \sigma\} = \{\sigma | \sigma \in Hyp$ and $\sigma \circ_h \sigma = \sigma\} \cap Hyp_{N_\varphi}(V)$ *for each choice function* φ.

Proof. "(i)⇒(ii)" Let φ be an arbitrary choice function and let $\sigma \in Hyp_{N_\varphi}(V)$ be an idempotent element of $Hyp_{N_\varphi}(V)$. Then $\sigma = \sigma \circ_N \sigma \sim_V \sigma \circ_h \sigma$. Let u and v be the words corresponding to σ and to $\sigma \circ_h \sigma$, respectively. By $\ell(u)$ we denote the length of u. Assume that $\sigma \notin E \cup \{\sigma_{id}, \sigma_x, \sigma_y\}$. By Theorem 1.2 (iii) the length of v is greater than that of u since $\sigma \neq \sigma_{f(y,x)}$ by Theorem 2.3 (ii). But then $u \approx v \notin IdMod\{x(yz) \approx (xy)z, xy \approx yx\}$ since from the associative and the commutative identity one can derive only identities $u \approx v$ with $\ell(u) = \ell(v)$. But by assumption, $u \approx v \in IdV \subseteq IdMod\{(xy)z \approx x(yz), xy \approx yx\}$, a contradiction. This shows

$$\{\sigma | \sigma \in Hyp_{N_\varphi}(V) \text{ and } \sigma \circ_N \sigma = \sigma\} \subseteq (E \cup \{\sigma_x, \sigma_y, \sigma_{id}\}) \cap Hyp_{N_\varphi}(V).$$

If conversely σ is an idempotent of *Hyp*, i.e. $\sigma \circ_h \sigma = \sigma$, then $\sigma \circ_N \sigma \sim_V \sigma \circ_h \sigma = \sigma$ and thus $\sigma \circ_N \sigma = \sigma$, since $\sigma \in Hyp_{N_\varphi}(V)$ and σ is an idempotent of $Hyp_{N_\varphi}(V)$. Therefore we have equality.

"(ii)⇒(i)" Assume that $Mod\{(xy)z \approx x(yz), xy \approx yx\} \nsubseteq V$. Then there exists an identity $x^k \approx x^n \in IdV$ with $1 \leq k < n \in \mathbb{N}$. Now we construct an idempotent element of $Hyp_{N_\varphi}(V)$ which is not in $E \cup \{\sigma_x, \sigma_y, \sigma_{id}\}$. We set $m := n - k$ and $w := x^2u)$ for some word $u \in W_x$ with $\ell(u) = 3km - 2$.

Clearly, $\sigma_w \notin E \cup \{\sigma_x, \sigma_y, \sigma_{id}\}$. It is easy to see that the length of w is $3km$ and the length of the word v corresponding to $\sigma_w \circ_h \sigma_w$ is $(3km)^2$. In fact, from $x^k \approx x^n \in IdV$ it follows $x^a \approx x^{a+bm} \in IdV$ for all natural numbers $a \geq k$ and $b \geq 1$ and in particular we have $x^{3km} \approx x^{3km+(9k^2m-3k)m} = x^{(3km)^2}$. Thus

$$(\sigma_w \circ_h \sigma_w)(f) \approx x^{(3km)^2} \approx x^{3km} \approx f(f(x,x),u) = \sigma_w(f).$$

Therefore, $\sigma_w \circ_h \sigma_w \sim_V \sigma_w$ and $\sigma_w \circ_N \sigma_w \sim_V \sigma_w \circ_h \sigma_w \sim_V \sigma_w$. Let φ be a choice function with $\sigma_w \in Hyp_{N_\varphi}(V)$. Then from $\sigma_w \circ_N \sigma_w \sim_V \sigma_w$ it follows $\sigma_w \circ_N \sigma_w = \sigma_w$, a contradiction. ■

4 Elements of infinite order

We remember that the order of an element of a groupoid is the cardinality of the subgroupoid generated by this element if this cardinality is finite and the order is infinite otherwise. By $O(\sigma)$ we denote the order of the hypersubstitution $\sigma \in Hyp_{N_\varphi}(V)$. By Theorem 1.2 (ii), the hypersubstitution $\sigma_{f(x,f(y,x))}$ has infinite order in *Hyp*, but in $Hyp_{N_\varphi}(V) = \{\sigma_x, \sigma_{x^2}, \sigma_{xy}, \sigma_{xy^2}, \sigma_{x^2y}, \sigma_{xyx}, \sigma_{x^2y^2}, \sigma_{xy^2x}, \sigma_{xyx^2}, \sigma_{xy^2x^2}, \sigma_y, \sigma_{y^2}, \sigma_{yx}, \sigma_{yx^2}, \sigma_{y^2x}, \sigma_{yxy}, \sigma_{y^2x^2}, \sigma_{yx^2y}, \sigma_{yxy^2}, \sigma_{yx^2y^2}\}$, where $V = Mod\{(xy)z \approx x(yz), xyuv \approx xuyv, x^3 \approx x\}$ we have

$$\sigma_{xyx} \circ_N \sigma_{xyx} = \sigma_{xy^2x^2}$$

and

$$\sigma_{xyx} \circ_N \sigma_{xy^2x^2} = \sigma_{xy^2x^2} = \sigma_{xy^2x^2} \circ_N \sigma_{xyx},$$

thus

$$\sigma_{xyx}^3 = \sigma_{xyx}^2$$

and σ_{xyx} has finite order. Now we characterize elements of infinite order with respect to varieties of semigroups which contain the variety of commutative semigroups.

By $\langle\sigma\rangle_{\circ_N}$ we denote the subgroupoid of $Hyp_{N_\varphi}(V)$ generated by the hypersubstitution σ.

Theorem 4.1. *Let V be a variety of semigroups. Then the following are equivalent:*

(i) $Mod\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V$

(ii) $\{\sigma|\sigma \in Hyp_{N_\varphi}(V)$ *and the order of* σ *is infinite*$\} = Hyp_{N_\varphi}(V) \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx}\} \cup A_1 \cup A_2)$, *where* $A_1 = \{\sigma|\sigma \in Hyp_{N_\varphi}(V) \cap (\{\sigma_v|v \in W_x\} \setminus (E_x \cup \{\sigma_x\})$ *and* $\langle\sigma\rangle_{\circ_N} \cap \{\sigma_{xu}|u \in W(X_2)\} \neq \emptyset\}$ *and* $A_2 = \{\sigma|\sigma \in Hyp_{N_\varphi}(V) \cap (\{\sigma_v|v \in W_y\} \setminus (E_y \cup \{\sigma_y\})$ *and* $\langle\sigma\rangle_{\circ_N} \cap \{\sigma_{uy}|u \in W(X_2)\} \neq \emptyset\}$ *for each choice function* φ.

Proof. "(i)⇒(ii)": Let φ be a choice function. Let σ be an element of $Hyp_{N_\varphi}(V)$ with $O(\sigma) = \infty$. By Theorem 3.1 and Proposition 2.3, $\sigma \notin E \cup \{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx}\}$.

If we assume that σ belongs to A_1, then there exists a word $u \in W(X_2)$ such that $\sigma_{xu} \in \langle\sigma\rangle_{\circ_N}$. Clearly, there exists a natural number $n \geq 1$ such that $\ell(\sigma_{xy}) = n$. Moreover, we have

$$\sigma \circ_N \sigma_{xu} \sim_V \sigma \circ_h \sigma_{xu} = \sigma,$$

since the word corresponding to σ is in W_x. Because of $\sigma \in Hyp_{N_\varphi}(V)$ we get

$$\sigma \circ_N \sigma_{xu} = \sigma$$

and $\ell(\sigma) + \ell(\sigma_{xu}) = n + 1$. But this means, $O(\sigma) \leq n$. Thus $\sigma \notin A_1$. In a similar way we show $\sigma \notin A_2$. This shows $\{\sigma|\sigma \in Hyp_{N_\varphi}(V)$ *and the order of* σ *is infinite*$\} \subseteq Hyp_{N_\varphi}(V) \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx}\} \cup A_1 \cup A_2)$.

Suppose that $\sigma \in Hyp_{N_\varphi}(V) \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx}\} \cup A_1 \cup A_2)$. Let u be the word corresponding to σ.

If $u \in W_x$, then $\langle\sigma\rangle_{Hyp_{N_\varphi}(V)} \subseteq \{\sigma_v|v \in W_x\}$. Otherwise there exists an identity $a \approx b \in IdV$ such that $a \in W_x$ and b uses the letter y. Clearly, $a \approx b \notin IdMod\{(xy)z \approx x(yz), xy \approx yx\}$ which contradicts $a \approx b \in IdV \subseteq IdMod\{(xy)z \approx x(yz), xy \approx yx\}$. Moreover, $\langle\sigma\rangle_{\circ_N} \cap \{\sigma_{xu}|u \in W(X_2)\} = \emptyset$ and $\sigma_x \notin \langle\sigma\rangle_{\circ_N}$. Therefore, for $\sigma_1, \sigma_2 \in \langle\sigma\rangle_{Hyp_{N_\varphi}(V)}$ the length of the word corresponding to $\sigma_1 \circ_h \sigma_2$ is greater than the length of u. Hence for each $\sigma' \in \langle\sigma\rangle_{\circ_N}$ with $\ell(\sigma') \geq 2$ the length of the word corresponding to σ' is greater than the length of u. Otherwise there would exist an identity $c \approx d \in IdV$ such that the length of d is greater than that of c. Clearly, $c \approx d \notin IdMod\{(xy)z \approx x(yz), xy \approx yx\}$, what contradicts $c \approx d \in IdV \subseteq IdMod\{(xy)z \approx x(yz), xy \approx yx\}$. Therefore, for all $\sigma_a, \sigma_b \in \langle\sigma\rangle_{\circ_N}$ there holds $\sigma_a \circ_N \sigma_b \neq \sigma$, i.e. $O(\sigma) = \infty$. If $u \in W_y$, then we get $O(\sigma) = \infty$ in the dual way.

If u uses both letters x and y, then $\langle\sigma\rangle_{\circ_N} \subseteq \{\sigma_v | v \in W(X_2) \setminus (W_x \cup W_y)\}$. Otherwise there is an identity $a \approx b \in IdV$ such that $a \in W_x \cup W_y$ and b uses both letters x and y. Clearly, $a \approx b \notin IdMod\{(xy)z \approx x(yz), xy \approx yx\}$ which contradicts $a \approx b \in IdV \subseteq IdMod\{(xy)z \approx x(yz), xy \approx yx\}$. The same argumentation as above (using also $\sigma \notin \{\sigma_{xy}, \sigma_{yx}\}$) shows that for each $\sigma' \in \langle\sigma\rangle_{\circ_N}$ with $\ell(\sigma') \geq 2$ the length of the word corresponding to σ' is greater than the length of u. This means there don't exist hypersubstitutins $\sigma_a, \sigma_b \in \langle\sigma\rangle_{\circ_N}$ such that $\sigma_a \circ_N \sigma_b = \sigma$ and hence $O(\sigma) = \infty$. This shows $\{\sigma | \sigma \in Hyp_{N_\varphi}(V) \textit{ and the order of } \sigma \textit{ is infinite}\} \supseteq Hyp_{N_\varphi}(V) \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx}\} \cup A_1 \cup A_2)$.

"(ii) $\Rightarrow$ (i)": Assume that $Mod\{(xy)z \approx x(yz), xy \approx yx\} \not\subseteq V$. Then there exists an identity $x^k \approx x^n \in IdV$ with $1 \leq k < n \in \mathbb{N}$. We set $m := n - k$ and $w := f(f(\ldots f(x,y), \ldots, y), y)$, where w has the length $km + 1$. It is easy to check that $(\sigma_w \circ_h \sigma_w)(f) = v \approx xy^{(km)^2}$. In fact, from $x^k \approx x^n \in IdV$ and $m := n - k$, it follows $x^{km} \approx x^c \in IdV$ with $c = km + (k^2m - k)m = k^2m^2$. Therefore, $(\sigma_w \circ_h \sigma_w)(f) = v \approx xy^{k^2m^2} \approx xy^{km} \approx \sigma_w(f)$, i.e. $\sigma_w \circ_h \sigma_w \sim_V \sigma_w$ and thus $\sigma_w \circ_N \sigma_w \sim_V \sigma_w \circ_h \sigma_w \sim_V \sigma_w$. Let φ be a choice function such that $\sigma_w \in Hyp_{N_\varphi}(V)$. Obviously, $\sigma_w \in Hyp_{N_\varphi}(V) \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{id}, \sigma_{f(y,x)}\} \cup A_1 \cup A_2)$ and thus $O(\sigma) = \infty$. But $\sigma_w \in Hyp_{N_\varphi}(V)$ forces $\sigma_w \circ_N \sigma_w = \sigma_w$ and $O(\sigma) = 2$, what contradicts $O(\sigma) = \infty$. Therefore $Mod\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V$. ∎

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