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MAXIMAL CLONES AND MAXIMAL PERMUTATION GROUPS

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In memoriam Professor Kazimierz Głazek

Abstract

A fundamental result in universal algebra is the theorem of Rosenberg describing the maximal subclones in the clone of all operations over a finite set. In group theory, the maximal subgroups of the symmetric groups are classified by the O'Nan–Scott Theorem. We shall explore the similarities and differences between these two analogous major results. In addition, we show that a primitive permutation group of diagonal type can be maximal in the symmetric group only if its socle is the direct product of two isomorphic simple groups, because if the number of simple factors of the socle is greater than two, then the group is contained in the alternating group.

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1. Introduction

Let X be a finite set, $|X| \geq 2$. In universal algebra one of the fundamental objects is $\mathrm{Clo}(X)$, the clone of all operations over X. In group theory one studies the symmetric group $\mathrm{Sym}(X)$, the group of all permutations of X. In this paper we are going to compare two fundamental results, one from 1965 due to Ivo Rosenberg [11] describing the maximal subclones in $\mathrm{Clo}(X)$, the other from 1979 due to Leonard L. Scott [15] and Michael O'Nan classifying the maximal subgroups in $\mathrm{Sym}(X)$.

These two results have quite different character. Rosenberg's Theorem gives a full description. All clones on his list are maximal and pairwise distinct, apart from some trivial cases (reversing the partial order; taking a power of the permutation of prime order), see [12]. Hence it is straightforward to enumerate all maximal clones on a given set X.

In contrast, the O'Nan–Scott Theorem is only a classification. Not all groups on their list are maximal. It has to be investigated individually whether a group listed is indeed maximal. That job has been accomplished by Liebeck, Praeger, and Saxl [5]. The present paper provides a small contribution to this by showing that a primitive group of diagonal type is not maximal in the symmetric group if its socle is a direct product of three or more simple factors, see Theorem 3. (Although such groups can be maximal in the alternating group as it follows from [5].) Furthermore, in order to know all maximal permutation groups of a given degree n one needs a list of all almost simple groups that contain a maximal subgroup of index n. Thus, the O'Nan–Scott Theorem yields no straightforward method to enumerate the maximal permutation groups of a given degree.

In Section 2 we present Rosenberg's Theorem. In Section 3 we formulate the O'Nan–Scott Theorem and prove our observation about the parity of permutations in primitive groups of diagonal type. Finally, in Section 4, for each maximal clone M we determine the permutations contained in M and tabulate those which contain a given maximal permutation group. We find that for four of the six types of maximal permutation groups any group in these classes can be obtained as the permutation part of a maximal clone.

2. ROSENBERG'S THEOREM ON MAXIMAL CLONES

In order to formulate Rosenberg's Theorem we require a number of definitions.

Let $\rho \subset X^h$ be an h-ary relation. We say that an n-ary operation $f\colon X^n \to X$ *preserves* ρ, if

$$(x_{11}, \dots, x_{1h}), \dots, (x_{n1}, \dots, x_{nh}) \in \rho$$

implies

$$(f(x_{11}, \dots, x_{n1}), \dots, f(x_{1h}, \dots, x_{nh})) \in \rho.$$

The set of all operations preserving ρ is closed under substitutions and contains all projections, so it is a *clone*, which will be denoted by $\mathrm{Pol}(\rho)$ (see [9, pp. 47–48]).

It is well known what is meant by a partial order relation, by the smallest and largest element with respect to a partial order, by a nontrivial equivalence relation, and by the graph (as a binary relation) of a function. Three further types of relations need to be defined.

If X is an elementary abelian group (i.e., the direct sum of cyclic groups of the same prime order; in other words, the additive group of a vector space over a field of prime number of elements), then the *affine relation* over X is defined as

$$\{(x, y, u, v) \in X^4 \mid x + y = u + v\},$$

and it is easily seen to be preserved exactly by the quasi-linear operations of the form

$$\sum_{i=1}^{n} \epsilon_i(x_i) + c \quad (\epsilon_i \in \mathrm{End}(X), c \in X).$$

A relation $\rho \subsetneq X^h$ is called *central* if it is totally symmetric, totally reflexive, and has nonempty center

$$Z = \{x \in X \mid \forall x_2, \dots, x_h : (x, x_2, \dots, x_h) \in \rho\}.$$

By total symmetry of the relation we mean that for every permutation $g \in \mathrm{Sym}(h)$, if $(x_1, \dots, x_h) \in \rho$, then $(x_{g(1)}, \dots, x_{g(h)}) \in \rho$. Moreover, total reflexivity means that $(x_1, \dots, x_h) \in \rho$, whenever any two of the arguments $x_1, \dots, x_h$ are equal. Note that we must have $h + |Z| \leq |X|$.

Finally, let $\phi : X \twoheadrightarrow \{1,\dots,h\}^m$ be a surjective, but not necessarily injective coordinatization of the set X with $m \geq 1$ coordinates, each in the range $\{1,\dots,h\}$ with $h \geq 3$, and let $\pi_i : \{1,\dots,h\}^m \to \{1,\dots,h\}$ be the i-th projection ($i = 1,\dots,m$). Then the corresponding *h-regular relation* ρ is defined by $(x_1,\dots,x_h) \in \rho$ iff for every $i = 1,\dots,m$ not all elements $x_1,\dots,x_h$ have distinct i-th coordinates:

$$|\{\pi_i(\phi(x_1)),\dots,\pi_i(\phi(x_h))\}| < h.$$

Now we can formulate Rosenberg's fundamental theorem.

Theorem 1. *Over a finite set X ($|X| \geq 2$) all maximal clones have the form* $\mathrm{Pol}(\rho)$ *for some relation ρ of one of the following six types:*

(a) *a partial order with smallest and largest element;*

(b) *the graph of a fixed-point-free permutation of prime order;*

(c) *the affine relation determined by an elementary abelian group;*

(d) *a nontrivial equivalence relation;*

(e) *a central relation;*

(f) *an h-regular relation determined by a mapping $\phi : X \twoheadrightarrow \{1,\dots,h\}^m$.*

In its original formulation Rosenberg's Theorem was given as a primality criterion, stating that a collection of functions is complete in the sense that it generates the clone of all functions, if and only if there is no relation of any of the six types above that is preserved by every function in the given collection. The result was announced in 1965 [11] and the proof appeared in a 91-page paper in 1970 [13]. Unfortunately, due to political reasons, this issue of *Rozpravy* was not sent to many libraries and it is still missing from their collections. In the book of Pöschel and Kalužnin [9] it is proved that the clones determined by relations of types (a)–(f) are indeed maximal, but the proof that there are no other maximal clones is not given there. A proof of this part, shorter than the original one, can be found in a paper of Quackenbush [10]. For the number of maximal clones over a given finite set see [14].

3. THE O'NAN–SCOTT THEOREM ON MAXIMAL PERMUTATION GROUPS

Here again we need some definitions. If X and Y are disjoint nonempty sets then let $\mathrm{Sym}(X) \times \mathrm{Sym}(Y) \leq \mathrm{Sym}(X \cup Y)$ be the intransitive permutation group consisting of the permutations mapping both X and Y to itself. Furthermore, we define two actions of the *wreath product* of the two symmetric groups. Let $\mathrm{Sym}(X) \wr \mathrm{Sym}(Y) \leq \mathrm{Sym}(X \times Y)$ consist of the permutations of the form $(x, y) \mapsto (g_y(x), h(y))$, where $h \in \mathrm{Sym}(Y)$ and for each $y \in Y$, $g_y \in \mathrm{Sym}(X)$ are arbitrary permutations. The *power action* (also called the *product action*) of the wreath product is the permutation group $\mathrm{Sym}(X) \uparrow \mathrm{Sym}(Y) \leq \mathrm{Sym}(X^Y)$ consisting of permutations of the form $f \mapsto f'$, where $f : Y \to X$ and $f'(y) = g_y(f(h^{-1}(y)))$. If $|X| = n$ we may write $\mathrm{Sym}(n)$ for $\mathrm{Sym}(X)$.

If X is equipped with the structure of a d-dimensional vector space over the p-element field (p a prime), then the *affine group* $\mathrm{AGL}(d, p)$ consists of the permutations of the form $x \mapsto \alpha(x) + c$, where $\alpha \in \mathrm{Aut}(X)$ is an invertible linear transformation and $c \in X$.

Let S be a a nonabelian simple group, $k \geq 2$, and let $D = \{(s, \ldots, s) \mid s \in S\} \leq S^k$ be the diagonal subgroup. Consider the action of S^k on the left cosets of D. Let G be the normalizer of this permutation group in the symmetric group of degree $|S|^{k-1}$. Then G is a primitive permutation group with $G/S^k \cong \mathrm{Out}(S) \times \mathrm{Sym}(k)$, where $\mathrm{Out}(S)$ is the outer automorphism group of S defined as the quotient group of the full automorphism group by the group of inner automorphisms. We say that this G is of *diagonal type*.

Finally, we say that G is an *almost simple group*, if G has a unique minimal normal subgroup S, and S is a nonabelian simple group. Then $S \leq G \leq \mathrm{Aut}(S)$. If G acts on the cosets of a maximal subgroup of index n not containing S, then we obtain a primitive permutation representation of degree n of G.

We give the following formulation of the O'Nan–Scott Theorem as it can be found in [4, p. 268], or in [3, Theorem 4.8]. Note that the original version in [15] contained some inaccuracies that were corrected in a paper of Aschbacher and Scott [1, Appendix], see also [6].

Theorem 2. *All maximal subgroups of* $\mathrm{Sym}(n)$ $(n \geq 5)$ *belong to one of the following six classes:*

(a) $\mathrm{Sym}(n_1) \times \mathrm{Sym}(n_2)$, $n = n_1 + n_2$ *(intransitive);*

(b) $\mathrm{Sym}(m) \wr Sym(k)$, $n = mk$ *(imprimitive);*

(c) $\mathrm{Sym}(m) \uparrow Sym(k)$, $n = m^k$ *(product [power] action);*

(d) $\mathrm{AGL}(d, p)$, $n = p^d$ *(affine);*

(e) *a group of diagonal type;*

(f) *a primitive almost simple group.*

Note that if $2 \leq n \leq 4$, i.e., if $\mathrm{Sym}(n)$ is solvable, then we have to replace the almost simple groups of case (f) by the alternating group $\mathrm{Alt}(n)$ (which belongs to case (f) if $n \geq 5$).

The subgroups given in Theorem 2 are not necessarily maximal. A complete list of exceptions was given by Liebeck, Praeger, and Saxl [5]. They show that in most of the cases these groups are maximal either in the symmetric group $\mathrm{Sym}(n)$ or in the alternating group $\mathrm{Alt}(n)$. Of course, the alternating group is one of the maximal subgroups in the symmetric group. However, it has no analogue among the maximal clones. Therefore, we restrict ourselves to classifying the maximal subgroups only in the symmetric group, thus eliminating the proper subgroups of the alternating groups from the list. Exercise 4.10 in [3] asks for investigating which of the groups listed in the theorem (Theorem 4.8 in [3]) contain odd permutations. In most cases it is a routine task. However, for primitive groups of diagonal type it leads to the following observation that seems to be new. Note that this does not affect the O'Nan–Scott Theorem on the classification of primitive permutation groups, where–of course–groups of diagonal type with $k \geq 3$ do occur as well.

Theorem 3. *Let* G *be a primitive group of diagonal type with socle* S^k*, where* S *is a nonabelian simple group. If* $k \geq 3$*, then* G *is contained in the alternating group.*

Proof. By definition, G acts on the cosets of the diagonal subgroup

$$D = \{(s, \ldots, s) \mid s \in S\}$$

in S^k and G/S^k is isomorphic to a subgroup of $\mathrm{Out}(S) \times \mathrm{Sym}(k)$. The degree of the permutation group G is $|S^k : D| = |S|^{k-1}$.

Any permutation action of S^k is obviously contained in the alternating group, since S is a nonabelian simple group.

Next, we have to prove that for every automorphism α of S, the permutation of the cosets

$$(x_1, \ldots, x_k)D \mapsto (\alpha(x_1), \ldots, \alpha(x_k))D$$

is an even permutation. Every coset has a unique representative of the form $(x_1, \ldots, x_{k-1}, 1)$, so our task is to show that

$$(x_1, \ldots, x_{k-1}) \mapsto (\alpha(x_1), \ldots, \alpha(x_{k-1}))$$

is an even permutation of S^{k-1}. More generally, we show this for every permutation α of the set S. Clearly, it is enough to verify this statement for a set of generators of $\mathrm{Sym}(S)$, for example, for the transpositions. So let α be a transposition on the set S. Then the corresponding permutation of S^{k-1} has order 2, so it is the product of disjoint 2-cycles. The number of 2-cycles can be calculated by subtracting the number of fixed points from the number of all elements and then dividing this number of moved points by 2. We obtain that the number of 2-cycles in the permutation of S^{k-1} corresponding to a transposition α on S is

$$\frac{1}{2}\left(|S|^{k-1} - (|S| - 2)^{k-1}\right).$$

This number is even, since $|S|$ is even and $k - 1 \geq 2$.

Finally, we have to prove that any permutation of the components gives rise to an even permutation of the cosets of D. Clearly, it is enough to show it for a transposition of the components, and by symmetry, we can restrict ourselves to interchanging the first two components. So we treat the permutation

$$(x_1, x_2, \ldots, x_k)\, D \mapsto (x_2, x_1, \ldots, x_k)\, D,$$

and the parity of this permutation can be calculated using the same method as above. We obtain that it is the product of

$$\frac{1}{2}\left(|S|^{k-1} - |S|^{k-2}\right)$$

2-cycles. This number is even, as the order of every nonabelian finite simple group is divisible by 4, and $k - 2 \geq 1$.

Since we have shown that a generating set of a group containing G consists of even permutations only, we get that G is contained in the alternating group. ■

Hence in any maximal subgroup of the symmetric group among the primitive groups of diagonal type we must have that the minimal normal subgroup is a direct product of two isomorphic simple groups. Such groups were called *groups of biregular type* by F. Buekenhout [2]. In this case the group can be described as the following group of permutations of the elements of the simple group S:

$$\{x \mapsto \alpha(x)^\epsilon s \mid \alpha \in \mathrm{Aut}(S), \epsilon \in \{+1, -1\}, s \in S\} < \mathrm{Sym}(S).$$

This group may contain odd permutations, for example, it does for $\mathrm{PSL}(2, q)$ for all odd prime powers q, and also for $\mathrm{Alt}(7)$, M_{11}, etc. However, it is contained in $\mathrm{Alt}(S)$ for many simple groups S, for example, if $S = \mathrm{Alt}(n)$ for $n \geq 8$, $S = M_{12}$, etc. It would be interesting to have a complete list of those simple groups for which this "extended holomorph" contains odd permutations.

In the other five cases, referring to Liebeck, Praeger, and Saxl [5], we get the following.

(a) $\mathrm{Sym}(n_1) \times \mathrm{Sym}(n_2)$ $(n_1, n_2 \geq 1)$ is maximal in $\mathrm{Sym}(n_1 + n_2)$, except when $n_1 = n_2$ – in this case it is contained in $\mathrm{Sym}(n/2) \wr Sym(2)$.

(b) $\mathrm{Sym}(m) \wr \mathrm{Sym}(k)$ $(m, k \geq 2)$ is always maximal in $\mathrm{Sym}(mk)$.

(c) $\mathrm{Sym}(m) \uparrow \mathrm{Sym}(k)$ $(m \geq 5,\ k \geq 2)$ is maximal either in $\mathrm{Sym}(m^k)$ or in $\mathrm{Alt}(m^k)$. The latter occurs if either $k = 2$ and $4 \mid m$ or $k \geq 3$ and $2 \mid m$. If $m \leq 4$, then the power action is contained in an affine group, namely,

$$\mathrm{Sym}(2) \uparrow \mathrm{Sym}(k) < \mathrm{AGL}(k, 2),$$

$$\mathrm{Sym}(3) \uparrow \mathrm{Sym}(k) < \mathrm{AGL}(k, 3),$$

$$\mathrm{Sym}(4) \uparrow \mathrm{Sym}(k) < \mathrm{AGL}(2k, 2).$$

(d) $\mathrm{AGL}(d, p)$ $(d \geq 1,\ p$ prime, $p^d \geq 5)$ is maximal either in $\mathrm{Sym}(p^d)$ or in $\mathrm{Alt}(p^d)$. The latter occurs if $p = 2$ and $d \geq 3$. For $p^d \leq 4$ we have $\mathrm{AGL}(d, p) = \mathrm{Sym}(p^d)$.

(f) The numerous exceptional cases for almost simple groups are listed in [5].

4. PERMUTATIONS IN MAXIMAL CLONES

The symmetric group $\mathrm{Sym}(X)$ can be considered as a clone consisting of the operations that depend on just one variable, i.e., operations of the form

$$f(x_1, \ldots, x_n) = g(x_i),$$

where $g \in \mathrm{Sym}(X)$ and $1 \leq i \leq n$. We shall consider "traces" of maximal clones in the symmetric group, that is, the intersection $M \cap \mathrm{Sym}(X)$ for the maximal clones M. The following may happen for a maximal clone M:

(1) M contains $\mathrm{Sym}(X)$;

(2) $M \cap \mathrm{Sym}(X)$ is a maximal permutation group;

(3) $M \cap \mathrm{Sym}(X)$ is properly contained in a maximal permutation group.

Furthermore, we may have that

(4) a maximal permutation group is not of the form $M \cap \mathrm{Sym}(X)$ for any maximal clone M.

Since the lattice of clones does not satisfy any nontrivial lattice identity (in particular, it is not modular), we can expect that in most cases the possibility (3) will occur, and there will be many maximal permutation groups with property (4). It is indeed the case, however, the goal of the present section is to establish that case (2), quite surprisingly, does occur quite often as well.

In a somewhat different setting, the study of similar "traces" of maximal clones in the monoid of all unary operations was started by Maja Ponjavić and Dragan Mašulović [8, 7]. They proved that these traces form a very complex poset containing, for example, arbitrarily long chains.

Let M be a maximal clone. Then it has the form $M = \mathrm{Pol}(\rho)$. Now

$$M \cap \mathrm{Sym}(X) = \mathrm{Aut}(\rho),$$

the automorphism group of the relation ρ. We investigate $\mathrm{Aut}(\rho)$ for each of the six types of relations given in Rosenberg's Theorem 1. In our discussion, three of the six cases will be further subdivided into subcases.

(a) The automorphism group of a bounded partial order fixes the smallest and the largest element. If $|X| = 2$, then we get the trivial group, which is maximal in $\mathrm{Sym}(X)$ in this case. Otherwise, for $|X| > 2$, the automorphism group has at least three orbits and hence it is not maximal.

(b1) If $|X| = p$ is a prime, then the automorphism group of the graph of a cycle of length p is the centralizer of the cyclic permutation and has order p, while it is contained in the normalizer of the cyclic group generated by the given p-cycle, and this normalizer has order $(p-1)p$. So if $|X| = 2$, then $\mathrm{Aut}(\rho) = \mathrm{Sym}(X)$; if $|X| = 3$, then $\mathrm{Aut}(\rho) = \mathrm{Alt}(X)$ is maximal in $\mathrm{Sym}(X)$; and if $|X| > 3$, then $\mathrm{Aut}(\rho)$ is properly contained in a maximal permutation group.

(b2) If $|X| = pk$ with $k > 1$, then the automorphism group of the graph of a fixed-point-free permutation consisting of k cycles of length p is the wreath product $C_p \wr \mathrm{Sym}(k)$ of a cyclic group of order p with the symmetric group of degree k.

maximal permutation group $\mathrm{Sym}(2) \wr \mathrm{Sym}(k)$; otherwise, if $p > 2$, then this automorphism group is properly contained in the maximal permutation group $\mathrm{Sym}(p) \wr \mathrm{Sym}(k)$.

(c) The automorphism group of the affine relation is the affine group. As it was mentioned in Section 3, this group is the full symmetric group if $|X| \leq 4$; it is properly contained in the alternating group if $|X| \geq 8$ is a power of 2; and it is a maximal permutation group if $|X| \geq 5$ is a power of an odd prime number.

(d1) If the equivalence relation ρ has uniform class size m and there are k classes, then $\mathrm{Aut}(\rho) = \mathrm{Sym}(m) \wr \mathrm{Sym}(k)$ is a maximal permutation group.

(d2) If the classes of the equivalence relation are not uniform, then $\mathrm{Aut}(\rho)$ is not transitive. It is a maximal permutation group if either the equivalence relation ρ has just two classes or it has only one non-singleton class and the size of this class is not $|X|/2$; otherwise, it is properly contained in an intransitive maximal permutation group.

(e) Let Z be the center of a central relation ρ. Then $\mathrm{Aut}(\rho) \subseteq \mathrm{Sym}(Z) \times \mathrm{Sym}(X \setminus Z)$, with equality if and only if ρ is a "pure" central relation, i.e.,

$$\rho = \{(x_1, \dots, x_h) \mid \exists i\colon x_i \in Z \text{ or } \exists i \neq j\colon x_i = x_j\}.$$

In that case $\mathrm{Aut}(\rho)$ is maximal, except when $|Z| = |X|/2$. In this exceptional case, as well as in any other case the automorphism group of a central relation is not maximal.

(f1) If the h-regular relation ρ is determined by a one-to-one map $\phi : X \twoheadrightarrow \{1, \dots, h\}^m$, then $\mathrm{Aut}(\rho) = \mathrm{Sym}(h) \uparrow \mathrm{Sym}(m)$. If $m \geq 2$, then we get the power action. Its maximality was discussed in Section 3. If $m = 1$, then $\mathrm{Aut}(\rho)$ is the full symmetric group. In that case $\mathrm{Pol}(\rho)$ is the so-called Słupecki clone.

(f2) If ϕ is not injective, then let ρ' denote the kernel of ϕ. Clearly, $\mathrm{Aut}(\rho)$ preserves ρ', since $\rho' = \{(x_1, x_2)\colon \forall x_3, \dots, x_h : (x_1, x_2, \dots, x_h) \in \rho\}$. If $m = 1$, then we have $\mathrm{Aut}(\rho) = \mathrm{Aut}(\rho')$, so we have reduced the maximality question to cases (d1)–(d2). Since the number of classes of ρ' is equal to $h \geq 3$, the arity of the relation ρ, here the equivalence relation ρ' cannot have just two classes.

Now let $m > 1$. If ρ' has at least two classes of the same size, then $\mathrm{Aut}(\rho)$ is properly contained in $\mathrm{Aut}(\rho')$, hence it is not maximal. Otherwise, $\mathrm{Aut}(\rho')$ has $h^m \geq 9$ orbits, hence in this case $\mathrm{Aut}(\rho) = \mathrm{Aut}(\rho')$ is not maximal either.

We have obtained that the first four types (corresponding to sums, products, powers, and vector spaces) from the O'Nan–Scott Theorem do arise as traces of maximal clones. The remaining two types, namely those, where the construction involves nonabelian simple groups, do not arise in this way, they fall into category (4).

We summarize our observations in the following theorems.

Theorem 4. *Let* $\mathrm{Pol}(\rho)$ *be a maximal clone over a finite set* X*, where* ρ *is one of the relations described in Theorem* 1*. Then* $\mathrm{Pol}(\rho)$ *contains the full symmetric group* $\mathrm{Sym}(X)$ *in exactly the following cases:*

(i) $|X| \geq 3$*,* ρ *is the* h*-regular relation with* $h = |X|$*, where* $(x_1, \dots, x_h) \in \rho$ *iff there are equal elements* $x_i = x_j$ $(i \neq j)$ *among the coordinates of the* h*-tuple;*

(ii) $2 \le |X| \le 4$ *and* ρ *is the (unique) affine relation on* X*;*

(iii) $|X| = 2$ *and* ρ *is the graph of the transposition interchanging the two elements of* X *(i.e.,* ρ *is the nonequality relation on the* 2*-element set).*

Theorem 5. *Let* $\mathrm{Pol}(\rho)$ *be a maximal clone over a finite set* X*, where* ρ *is one of the relations described in Theorem* 1*. Then* $\mathrm{Pol}(\rho) \cap \mathrm{Sym}(X) = \mathrm{Aut}(\rho)$ *is a maximal permutation group in exactly the following cases:*

(a) $\mathrm{Aut}(\rho) = \mathrm{Sym}(X_1) \times \mathrm{Sym}(X_2)$*, where* $X = X_1 \cup X_2$ *is a disjoint union with* $|X_1| \ne |X_2|$ *or* $|X| = 2$ *:*

(a1) $|X| \ge 3$ *and* ρ *is the equivalence relation with classes* X_1 *and* X_2*;*

(a2) ρ *is an* h*-ary "pure" central relation*

$$\rho = \{(x_1, \dots, x_h) \mid \exists i \colon x_i \in Z \text{ or } \exists i \ne j \colon x_i = x_j\},$$

with center $Z = X_1$ *or* $Z = X_2$*, where* $1 \le h \le |X| - |Z|$*;*

(a3) $|X_1|, |X_2| \ge 2$ *and* ρ *is the equivalence relation with one non-singleton class* X_1 *or* X_2*;*

(a4) $|X_1|, |X_2| \ge 2$ *and* ρ *is the* h*-regular relation corresponding to* $\phi : X \twoheadrightarrow \{1, \dots, h\}$ *where the kernel of* ϕ *has one non-singleton class* X_1 *or* X_2 *(then* $h = |X_2| + 1$*, or* $h = |X_1| + 1$*, respectively);*

(a5) $|X| = 2$ *and* ρ *is the order relation of the* 2*-element chain.*

(b) $\mathrm{Aut}(\rho) = \mathrm{Sym}(X_1) \wr \mathrm{Sym}(X_2)$*, where* $X = X_1 \times X_2$ *with* $|X_1|, |X_2| \ge 2$ *:*

(b1) ρ *is the equivalence relation* $(x_1, x_2)\rho(x_1', x_2')$ *iff* $x_2 = x_2'$*;*

(b2) $|X_2| \ge 3$ *and* ρ *is the* h*-regular relation* $(h = |X_2|, m = 1)$ *corresponding to the second projection mapping* $\phi : X_1 \times X_2 \twoheadrightarrow X_2$*;*

(b3) $|X_1| = 2$ *and* ρ *is the binary relation* $(x_1, x_2)\rho(x_1', x_2')$ *iff* $x_1 \ne x_1'$ *and* $x_2 = x_2'$ *(the graph of the fixed-point-free permutation of order two interchanging the elements with the same second coordinate).*

(c) $\mathrm{Aut}(\rho) = \mathrm{Sym}(X_1) \uparrow \mathrm{Sym}(X_2)$*, where* $X = X_1^{X_2}$*,* $|X_1| \ge 5$*,* $|X_2| \ge 2$*, moreover* 4 *does not divide* $|X_1|$ *if* $|X_2| = 2$*, and* 2 *does not divide* $|X_1|$ *if* $|X_2| \ge 3$ *:*

(c1) *ρ is the h-regular relation corresponding to the one-to-one mapping $\phi : X \twoheadrightarrow X_1^{X_2}$, where $h = |X_1|$.*

(d) $\mathrm{Aut}(\rho)$ *is the affine group* $\mathrm{AGL}(d,p)$*, where $p^d \geq 5$ is odd:*

(d1) *ρ is the affine relation (corresponding to the vector space structure determining the affine group).*

(e) $\mathrm{Aut}(\rho) = \mathrm{Alt}(X)$:

(e1) *$|X| = 3$ and ρ is the graph of a 3-cycle.*

Counting the various possibilities we obtain the number of maximal clones containing a given maximal permutation group.

Corollary 6. *Let G be either the full symmetric group* $\mathrm{Sym}(X)$ *or a maximal permutation group of any of the types* (a)–(d) *in the O'Nan–Scott Theorem. Then the number N of maximal clones* $\mathrm{Pol}(\rho)$ *on the set X such that* $G = \mathrm{Pol}(\rho) \cap \mathrm{Sym}(X)$ *is as follows:*

(o) *For* $G = \mathrm{Sym}(X)$ *we have*

$$N = \begin{cases} 2, & \text{if } |X| = 2,3,4; \\ \\ 1, & \text{if } |X| \geq 5. \end{cases}$$

(a) *For* $G = \mathrm{Sym}(X_1) \times \mathrm{Sym}(X_2)$*, where $X = X_1 \cup X_2$ is a disjoint union with $|X_1| \neq |X_2|$ or $|X| = 2$, we have*

$$N = \begin{cases} |X| + 1, & \text{if } \min(|X_1|, |X_2|) = 1; \\ \\ |X| + 5, & \text{if } \min(|X_1|, |X_2|) \geq 2. \end{cases}$$

(b) *For* $G = \mathrm{Sym}(X_1) \wr \mathrm{Sym}(X_2)$*, where $X = X_1 \times X_2$, and $|X_1|, |X_2| \geq 2$ we have*

$$N = \begin{cases} 1, & \text{if } |X_1| \geq 3 \text{ and } |X_2| = 2; \\ \\ 2, & \text{if } |X_1| = 2 \text{ and } |X_2| = 2; \\ \\ 2, & \text{if } |X_1| \geq 3 \text{ and } |X_2| \geq 3; \\ \\ 3, & \text{if } |X_1| = 2 \text{ and } |X_2| \geq 3. \end{cases}$$

(c) *For* $G = \mathrm{Sym}(X_1) \uparrow \mathrm{Sym}(X_2)$, *where* $X = X_1^{X_2}$ *and* $|X_1| \geq 5$, $|X_2| \geq 2$, *moreover* 4 *does not divide* $|X_1|$ *if* $|X_2| = 2$, *and* 2 *does not divide* $|X_1|$ *if* $|X_2| \geq 3$, *we have* $N = 1$.

(d) *For* $G = \mathrm{AGL}(d, p)$, *where* $p^d \geq 5$ *is odd, we have* $N = 1$.

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