

## SUPERIOR SUBALGEBRAS AND IDEALS OF BCK/BCI-ALGEBRAS

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### Abstract

The notions of superior subalgebras and (commutative) superior ideals are introduced, and their relations and related properties are investigated. Conditions for a superior ideal to be commutative are provided.

**Keywords:** superior mapping, superior subalgebra, (commutative) superior ideal.

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### 1. INTRODUCTION

Algebras have played an important role in pure and applied mathematics and have its comprehensive applications in many aspects including dynamical systems and genetic code of biology (see [1, 2, 6], and [11]). Starting from the four DNA bases order in the Boolean lattice, Sánchez *et al.* [10] proposed a novel Lie Algebra of the genetic code which shows strong connections among algebraic

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relationship, codon assignments and physicochemical properties of amino acids. A BCK/BCI-algebra (see [3, 4, 9]) is an important class of logical algebras introduced by Iséki and was extensively investigated by several researchers. Jun and Song [5] introduced the notion of BCK-valued functions and investigated several properties. They established block-codes by using the notion of BCK-valued functions, and shown that every finite BCK-algebra determines a block-code.

In this paper, we first introduce the notion of superior mapping by using partially ordered sets. Using the superior mapping, we introduce the concept of superior subalgebras and (commutative) superior ideals in BCK/BCI-algebras, and investigate related properties. We discuss relations among a superior subalgebra, a superior ideal and a commutative superior ideal.

## 2. PRELIMINARIES

We display basic definitions and properties of BCK/BCI-algebras that will be used in this paper. For more details of BCK/BCI-algebras, we refer the reader to [3, 7, 8] and [9].

An algebra  $\mathcal{L} := (L; *, 0)$  of type  $(2, 0)$  is called a BCI-algebra if it satisfies the following conditions:

- (I)  $(\forall x, y, z \in L) (((x * y) * (x * z)) * (z * y) = 0),$
- (II)  $(\forall x, y \in L) ((x * (x * y)) * y = 0),$
- (III)  $(\forall x \in L) (x * x = 0),$
- (IV)  $(\forall x, y \in L) (x * y = 0, y * x = 0 \Rightarrow x = y).$

If a BCI-algebra  $\mathcal{L}$  satisfies the following identity:

- (V)  $(\forall x \in L) (0 * x = 0),$

then  $\mathcal{L}$  is called a BCK-algebra.

Any BCK/BCI-algebra  $\mathcal{L}$  satisfies the following conditions:

- (2.1)  $(\forall x \in L) (x * 0 = x),$
- (2.2)  $(\forall x, y, z \in L) (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x),$
- (2.3)  $(\forall x, y, z \in L) ((x * y) * z = (x * z) * y),$
- (2.4)  $(\forall x, y, z \in L) ((x * z) * (y * z) \leq x * y)$

where  $x \leq y$  if and only if  $x * y = 0$ .

A BCK-algebra  $\mathcal{L}$  is said to be commutative if  $x \wedge y = y \wedge x$  for all  $x, y \in L$  where  $x \wedge y = y * (y * x)$ .

A nonempty subset  $S$  of a BCK/BCI-algebra  $\mathcal{L}$  is called a subalgebra of  $\mathcal{L}$  if  $x * y \in S$  for all  $x, y \in S$ . A subset  $A$  of a BCK/BCI-algebra  $\mathcal{L}$  is called an ideal of  $\mathcal{L}$  if it satisfies:

$$(2.5) \quad 0 \in A,$$

$$(2.6) \quad (\forall x, y \in L) (x * y \in A, y \in A \Rightarrow x \in A).$$

A subset  $A$  of a BCK-algebra  $\mathcal{L}$  is called a commutative ideal of  $\mathcal{L}$  if it satisfies (2.5) and

$$(2.7) \quad (\forall x, y, z \in L) ((x * y) * z \in A, z \in A \Rightarrow x * (y * (y * x)) \in A).$$

### 3. SUPERIOR MAPPINGS

Let  $L$  be a set of parameters and let  $U$  be a partially ordered set with the partial ordering  $\preceq$  and the first element  $e$ . For a mapping  $\tilde{f} : L \rightarrow \mathcal{P}(U)$ , we consider the mapping

$$(3.1) \quad ||\tilde{f}|| : L \rightarrow U, x \mapsto \begin{cases} \sup \tilde{f}(x) & \text{if } \exists \sup \tilde{f}(x), \\ e & \text{otherwise,} \end{cases}$$

which is called the superiormapping of  $L$  with respect to  $(\tilde{f}, L)$ . In this case, we say that  $(\tilde{f}, L)$  is a pair on  $(U, \preceq)$ .

**Example 3.1.** Let  $U = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$  be ordered by the relation “ $x$  divides  $y$ ”. The Hasse diagram of  $U$  appears in Figure 1.

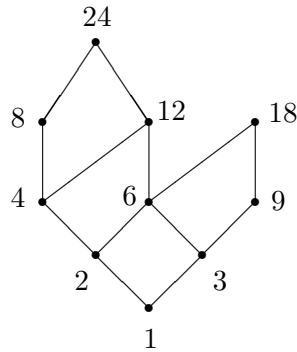


Figure 1

For a set  $L = \{a, b, c, d\}$  of parameters, let  $(\tilde{f}, L)$  be a pair on  $(U, \preceq)$  where  $\tilde{f}$  is

given as follows:

$$\tilde{f} : L \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \{2, 4, 6\} & \text{if } x = a, \\ \{8, 12, 18\} & \text{if } x = b, \\ \{1, 3, 6, 9\} & \text{if } x = c, \\ \{4, 6, 8, 12\} & \text{if } x = d. \end{cases}$$

Then the superiormapping of  $L$  with respect to  $(\tilde{f}, L)$  is described as follows:  $\|\tilde{f}\|(a) = 12$ ,  $\|\tilde{f}\|(c) = 18$  and  $\|\tilde{f}\|(d) = 24$ , but  $\|\tilde{f}\|(b) = 1$  because there does not exist the supremum of  $\tilde{f}(b)$ .

**Example 3.2.** For any positive integer  $m$ , we will let  $\mathbf{D}_m$  denote the set of divisors of  $m$  ordered by divisibility. The Hasse diagram of

$$\mathbf{D}_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$$

appears in Figure 2.

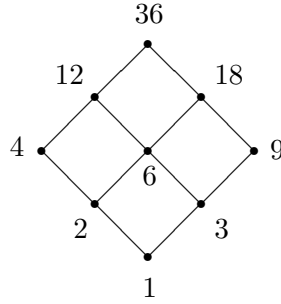


Figure 2

For a set  $L = \{a_1, a_2, a_3, a_4, a_5, a_6\}$  of parameters, let  $(\tilde{f}, L)$  be a pair on  $(U, \preceq)$  with  $U = \mathbf{D}_{36}$  in which  $\tilde{f}$  is defined as follows:

$$\tilde{f} : L \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \{1, 2, 3\} & \text{if } x = a_1, \\ \{2, 3, 6\} & \text{if } x = a_2, \\ \{2, 3, 4, 6\} & \text{if } x = a_3, \\ \{12, 36\} & \text{if } x = a_4, \\ \{4, 6, 9\} & \text{if } x = a_5, \\ \{3, 4, 6, 9\} & \text{if } x = a_6. \end{cases}$$

Then the superiormapping of  $L$  with respect to  $(\tilde{f}, L)$  is described as follows:  $\|\tilde{f}\|(a_1) = \|\tilde{f}\|(a_2) = 6$ ,  $\|\tilde{f}\|(a_3) = 12$ , and  $\|\tilde{f}\|(a_4) = \|\tilde{f}\|(a_5) = \|\tilde{f}\|(a_6) = 36$ .

## 4. SUPERIOR SUBALGEBRAS AND IDEALS

**Definition 4.1.** Let  $\mathcal{L} := (L, *, 0)$  be a BCK/BCI-algebra and let  $(\tilde{f}, L)$  be a pair on  $(U, \preceq)$ . By a superior subalgebra on  $(\mathcal{L}, \tilde{f})$ , we mean the superiormapping  $\|\tilde{f}\|$  of  $\mathcal{L}$  with respect to  $(\tilde{f}, L)$  which satisfies:

$$(4.1) \quad (\forall x, y \in L) \left( \|\tilde{f}\|(x * y) \preceq \sup\{\|\tilde{f}\|(x), \|\tilde{f}\|(y)\} \right)$$

whenever there exists  $\sup\{\|\tilde{f}\|(x), \|\tilde{f}\|(y)\}$  for any  $x, y \in L$ .

**Example 4.2.** Let  $L = \{0, a, b, c\}$  be a set with a binary operation  $*$  shown in Table 1.

Table 1. Cayley table for the binary operation  $*$ .

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Then  $\mathcal{L} := (L, *, 0)$  is a BCK-algebra (see [9]). Consider the poset  $(U, \preceq)$  which is given in Example 3.1.

(1) Let  $(\tilde{f}, L)$  be a pair on  $(U, \preceq)$  where  $\tilde{f}$  is given by

$$\tilde{f} : L \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \{1, 2\} & \text{if } x = 0, \\ \{4, 6, 8\} & \text{if } x = a, \\ \{2, 3, 4, 6\} & \text{if } x = b, \\ \{1, 2, 3, 6\} & \text{if } x = c. \end{cases}$$

Then the superiormapping of  $\mathcal{L}$  with respect to  $(\tilde{f}, L)$  is described as follows:  $\|\tilde{f}\|(0) = 2$ ,  $\|\tilde{f}\|(a) = 24$ ,  $\|\tilde{f}\|(b) = 12$  and  $\|\tilde{f}\|(c) = 6$ , and it is a superior subalgebra on  $(\mathcal{L}, \tilde{f})$ .

(2) Let  $(\tilde{g}, L)$  be a pair on  $(U, \preceq)$  in which  $\tilde{g}$  is provided as follows:

$$\tilde{g} : L \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \{2, 4, 6\} & \text{if } x \in \{0, a\}, \\ \{1, 3, 6, 9\} & \text{if } x = b, \\ \{4, 6, 8, 12\} & \text{if } x = c. \end{cases}$$

Then the superiormapping of  $\mathcal{L}$  with respect to  $(\tilde{g}, L)$  is described as follows:  $\|\tilde{g}\|(0) = \|\tilde{g}\|(a) = 12$ ,  $\|\tilde{g}\|(b) = 18$  and  $\|\tilde{g}\|(c) = 24$ , and it is not a superior subalgebra on  $(\mathcal{L}, \tilde{g})$  since  $\|\tilde{g}\|(b * b) = \|\tilde{g}\|(0) = 12$  and  $\sup\{\|\tilde{g}\|(b), \|\tilde{g}\|(b)\} = 18$  are noncomparable.

(3) Let  $(\tilde{h}, L)$  be a pair on  $(U, \preceq)$  in which  $\tilde{h}$  is given as follows:

$$\tilde{h} : L \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \{2, 4, 6\} & \text{if } x = 0, \\ \{8, 12, 18\} & \text{if } x = a, \\ \{1, 3, 6, 9\} & \text{if } x = b, \\ \{2, 3, 9\} & \text{if } x = c. \end{cases}$$

Then the superiormapping of  $\mathcal{L}$  with respect to  $(\tilde{h}, L)$  is described as follows:  $\|\tilde{g}\|(0) = 12$ ,  $\|\tilde{g}\|(a) = 1$ , and  $\|\tilde{g}\|(b) = \|\tilde{g}\|(c) = 18$ . Since

$$\|\tilde{g}\|(a * a) = \|\tilde{g}\|(0) = 12 \not\leq 1 = \sup\{\|\tilde{g}\|(a), \|\tilde{g}\|(a)\},$$

$\|\tilde{f}\|$  is not a superiorsubalgebra on  $(\mathcal{L}, \tilde{f})$ .

**Example 4.3.** Let  $L = \{0, 1, 2, a, b\}$  be a set with a binary operation  $*$  shown in Table 2.

Table 2. Cayley table for the binary operation  $*$ .

$*$	0	1	2	$a$	$b$
0	0	0	0	$a$	$a$
1	1	0	1	$b$	$a$
2	2	2	0	$a$	$a$
$a$	$a$	$a$	$a$	0	0
$b$	$b$	$a$	$b$	1	0

Then  $\mathcal{L} := (L, *, 0)$  is a BCI-algebra (see [9]). Consider the poset  $(U, \preceq)$  which is given in Example 3.2. Let  $(\tilde{f}, L)$  be a pair on  $(U, \preceq)$  where  $\tilde{f}$  is defined by

$$\tilde{f} : L \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \{1\} & \text{if } x = 0, \\ \{4, 6, 9, 12\} & \text{if } x \in \{1, b\}, \\ \{2, 3\} & \text{if } x = 2, \\ \{3, 6, 9\} & \text{if } x = a. \end{cases}$$

Then the superiormapping of  $\mathcal{L}$  with respect to  $(\tilde{f}, L)$  is described as follows:  $\|\tilde{f}\|(0) = 1$ ,  $\|\tilde{f}\|(a) = 18$ ,  $\|\tilde{f}\|(b) = \|\tilde{f}\|(1) = 36$  and  $\|\tilde{f}\|(2) = 6$ , and it is a superiorsubalgebra on  $(\mathcal{L}, \tilde{f})$ .

**Definition 4.4.** Let  $\mathcal{L} := (L, *, 0)$  be a BCK/BCI-algebra and let  $(\tilde{f}, L)$  be a pair on  $(U, \preceq)$ . By a superiorideal on  $(\mathcal{L}, \tilde{f})$ , we mean the superiormapping  $\|\tilde{f}\|$  of  $\mathcal{L}$  with respect to  $(\tilde{f}, L)$  which satisfies the following conditions:

$$(4.2) \quad (\forall x \in L) \left( \|\tilde{f}\|(0) \preceq \|\tilde{f}\|(x) \right),$$

$$(4.3) \quad (\forall x, y \in L) \left( \|\tilde{f}\|(x) \preceq \sup\{\|\tilde{f}\|(x * y), \|\tilde{f}\|(y)\} \right),$$

whenever there exists  $\sup\{\|\tilde{f}\|(x), \|\tilde{f}\|(y)\}$  for any  $x, y \in L$ .

**Example 4.5.**

- (1) In Example 4.2(1), the superiormapping  $\|\tilde{f}\|$  of  $\mathcal{L}$  with respect to  $(\tilde{f}, L)$  is a superiorideal on  $(\mathcal{L}, \tilde{f})$ .
- (2) In Example 4.2(2), the superiormapping  $\|\tilde{g}\|$  of  $\mathcal{L}$  with respect to  $(\tilde{g}, L)$  is not a superiorideal on  $(\mathcal{L}, \tilde{g})$ .
- (3) In Example 4.2(3), the superiormapping  $\|\tilde{h}\|$  of  $\mathcal{L}$  with respect to  $(\tilde{h}, L)$  is not a superiorideal on  $(\mathcal{L}, \tilde{h})$ .

**Proposition 4.6.** *Let  $\mathcal{L} := (L, *, 0)$  be a BCK/BCI-algebra. Then every superior-subalgebra  $\|\tilde{f}\|$  on  $(\mathcal{L}, \tilde{f})$  satisfies the condition (4.2).*

**Proof.** Since  $x * x = 0$  for all  $x \in L$ , it is clear. ■

**Theorem 4.7.** *Let  $\mathcal{L} := (L, *, 0)$  be a BCK/BCI-algebra. If  $\|\tilde{f}\|$  is a superior-subalgebra (ideal) on  $(\mathcal{L}, \tilde{f})$ , then the nonempty set*

$$\|\tilde{f}\|_\alpha := \{x \in L \mid \|\tilde{f}\|(x) \preceq \alpha\}$$

*is a subalgebra (ideal) of  $\mathcal{L}$  for all  $\alpha \in U$ .*

**Proof.** Assume that  $\|\tilde{f}\|$  is a superior-subalgebra on  $(\mathcal{L}, \tilde{f})$ . Let  $\alpha \in U$  and suppose that  $\|\tilde{f}\|_\alpha \neq \emptyset$ . If  $x, y \in \|\tilde{f}\|_\alpha$ , then  $\|\tilde{f}\|(x) \preceq \alpha$  and  $\|\tilde{f}\|(y) \preceq \alpha$ . It follows from (4.1) that

$$\|\tilde{f}\|(x * y) \preceq \sup\{\|\tilde{f}\|(x), \|\tilde{f}\|(y)\} \preceq \alpha$$

and that  $x * y \in \|\tilde{f}\|_\alpha$ . Therefore  $\|\tilde{f}\|_\alpha$  is a subalgebra of  $\mathcal{L}$ . Now, suppose that  $\|\tilde{f}\|$  is a superiorideal on  $(\mathcal{L}, \tilde{f})$ . Let  $\alpha \in U$  be such that  $\|\tilde{f}\|_\alpha \neq \emptyset$ . Then there exists  $x \in L$  such that  $\|\tilde{f}\|(x) \preceq \alpha$ , and so  $\|\tilde{f}\|(0) \preceq \|\tilde{f}\|(x) \preceq \alpha$ . Thus  $0 \in \|\tilde{f}\|_\alpha$ . Let  $x, y \in L$  be such that  $x * y \in \|\tilde{f}\|_\alpha$  and  $y \in \|\tilde{f}\|_\alpha$ . Then  $\|\tilde{f}\|(x * y) \preceq \alpha$  and  $\|\tilde{f}\|(y) \preceq \alpha$ . It follows from (4.3) that

$$\|\tilde{f}\|(x) \preceq \sup\{\|\tilde{f}\|(x * y), \|\tilde{f}\|(y)\} \preceq \alpha.$$

Thus  $x \in \|\tilde{f}\|_\alpha$ , and therefore  $\|\tilde{f}\|_\alpha$  is an ideal of  $\mathcal{L}$ . ■

The following example illustrates Theorem 4.7.

**Example 4.8.** (1) Consider the BCK-algebra  $\mathcal{L}$  and the poset  $(U, \preceq)$  which are given in Examples 4.2 and 3.1, respectively. Let  $(\tilde{f}, L)$  be a pair on  $(U, \preceq)$  where  $\tilde{f}$  is given as follows:

$$\tilde{f} : L \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \{1, 3\} & \text{if } x = 0, \\ \{2, 3, 6\} & \text{if } x = a, \\ \{1, 3, 9\} & \text{if } x = b, \\ \{2, 3, 6, 9\} & \text{if } x = c. \end{cases}$$

Then the superiormapping of  $\mathcal{L}$  with respect to  $(\tilde{f}, L)$  is described as follows:  $\|\tilde{f}\|(0) = 3$ ,  $\|\tilde{f}\|(a) = 6$ ,  $\|\tilde{f}\|(b) = 9$  and  $\|\tilde{f}\|(c) = 18$ , and it is a superiorsubalgebra on  $(\mathcal{L}, \tilde{f})$ . It is routine to verify that  $\|\tilde{f}\|_\alpha$  is a subalgebra of  $\mathcal{L}$  for all  $\alpha \in U$ .

(2) Consider the BCI-algebra  $\mathcal{L}$  and the poset  $(U, \preceq)$  which are given in Examples 4.3 and 3.1, respectively. Let  $(\tilde{f}, L)$  be a pair on  $(U, \preceq)$  where  $\tilde{f}$  is defined by

$$\tilde{f} : L \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \{1\} & \text{if } x = 0, \\ \{4, 6\} & \text{if } x \in \{1, b\}, \\ \{1, 3\} & \text{if } x = 2, \\ \{2, 3, 6\} & \text{if } x = a. \end{cases}$$

Then the superiormapping  $\|\tilde{f}\|$  of  $\mathcal{L}$  with respect to  $(\tilde{f}, L)$  is described as follows:  $\|\tilde{f}\|(0) = 1$ ,  $\|\tilde{f}\|(1) = 12$ ,  $\|\tilde{f}\|(2) = 3$ ,  $\|\tilde{f}\|(a) = 6$ , and  $\|\tilde{f}\|(b) = 12$ . It is routine to verify that  $\|\tilde{f}\|$  is a superiorideal on  $(\mathcal{L}, \tilde{f})$ . Thus  $\|\tilde{f}\|_8 = \|\tilde{f}\|_4 = \|\tilde{f}\|_2 = \emptyset$ , and  $\|\tilde{f}\|_{24} = \|\tilde{f}\|_{12} = L$ ,  $\|\tilde{f}\|_{18} = \|\tilde{f}\|_6 = \{0, 2, a\}$ ,  $\|\tilde{f}\|_9 = \|\tilde{f}\|_3 = \{0, 2\}$ ,  $\|\tilde{f}\|_1 = \{0\}$  which are ideals of  $\mathcal{L}$ .

**Proposition 4.9.** *If  $\|\tilde{f}\|$  is a superiorideal on  $(\mathcal{L}, \tilde{f})$ , then  $\|\tilde{f}\|(x) \preceq \|\tilde{f}\|(y)$  for all  $x, y \in L$  with  $x \leq y$ .*

**Proof.** Let  $x, y \in L$  be such that  $x \leq y$ . Using (4.3) and (4.2), we have

$$\|\tilde{f}\|(x) \preceq \sup\{\|\tilde{f}\|(x * y), \|\tilde{f}\|(y)\} = \sup\{\|\tilde{f}\|(0), \|\tilde{f}\|(y)\} = \|\tilde{f}\|(y),$$

proving the result. ■

**Theorem 4.10.** *Let  $\mathcal{L}$  be a BCK-algebra. Every superiorideal on  $(\mathcal{L}, \tilde{f})$  is a superiorsubalgebra on  $(\mathcal{L}, \tilde{f})$ .*

**Proof.** Let  $\|\tilde{f}\|$  be a superiorideal on  $(\mathcal{L}, \tilde{f})$ . Since  $x * y \leq x$  for all  $x, y \in L$ , it follows from Proposition 4.9 that

$$\|\tilde{f}\|(x * y) \preceq \|\tilde{f}\|(x) \preceq \sup\{\|\tilde{f}\|(x * y), \|\tilde{f}\|(y)\} \preceq \sup\{\|\tilde{f}\|(x), \|\tilde{f}\|(y)\}$$

and that  $\|\tilde{f}\|$  is a superiorsubalgebra on  $(\mathcal{L}, \tilde{f})$ . ■



The converse of Theorem 4.10 may not be true as seen in the following example.

**Example 4.11.** Let  $L = \{0, 1, 2, 3\}$  be a set with a binary operation ‘ $*$ ’ shown in Table 3.

Table 3. Cayley table for the binary operation ‘ $*$ ’.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Then  $\mathcal{L} := (L, *, 0)$  is a BCK-algebra (see [9]). Let  $U = \{a, b, c, d, e, f\}$  be ordered as pictured in Figure 3.

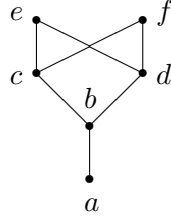


Figure 3

Let  $(\tilde{f}, L)$  be a pair on  $(U, \preceq)$  where  $\tilde{f}$  is given as follows:

$$\tilde{f} : L \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \{a, b\} & \text{if } x = 0, \\ \{a, b, c\} & \text{if } x = 1, \\ \{b, c, d, f\} & \text{if } x \in \{2, 3\}. \end{cases}$$

Then the superiormapping  $\|\tilde{f}\|$  of  $\mathcal{L}$  with respect to  $(\tilde{f}, L)$  is described as follows:  $\|\tilde{f}\|(0) = b$ ,  $\|\tilde{f}\|(1) = c$  and  $\|\tilde{f}\|(2) = \|\tilde{f}\|(3) = f$ . By routine calculations, we know that  $\|\tilde{f}\|$  is a superiorsubalgebra on  $(\mathcal{L}, \tilde{f})$ , but it is not a superiorideal on  $(\mathcal{L}, \tilde{f})$  because

$$\|\tilde{f}\|(2) = f \not\preceq c = \sup\{\|\tilde{f}\|(2 * 1), \|\tilde{f}\|(1)\}.$$

**Proposition 4.12.** *Every superiorideal  $\|\tilde{f}\|$  on  $(\mathcal{L}, \tilde{f})$  satisfies the following assertion.*

$$(4.4) \quad (\forall x, y, z \in L) \left( x * y \leq z \Rightarrow \|\tilde{f}\|(x) \preceq \sup\{\|\tilde{f}\|(y), \|\tilde{f}\|(z)\} \right).$$

**Proof.** Let  $x, y, z \in L$  be such that  $x * y \leq z$ . Then  $(x * y) * z = 0$ , and so

$$\|\tilde{f}\|(x * y) \preceq \sup\{\|\tilde{f}\|((x * y) * z), \|\tilde{f}\|(z)\} = \sup\{\|\tilde{f}\|(0), \|\tilde{f}\|(z)\} = \|\tilde{f}\|(z)$$

by (4.3) and (4.2). It follows that

$$\|\tilde{f}\|(x) \preceq \sup\{\|\tilde{f}\|(x * y), \|\tilde{f}\|(y)\} \preceq \sup\{\|\tilde{f}\|(z), \|\tilde{f}\|(y)\}.$$

This completes the proof. ■

**Theorem 4.13.** Let  $\|\tilde{f}\|$  be the superiormapping of  $\mathcal{L}$  with respect to  $(\tilde{f}, L)$ . If  $\|\tilde{f}\|$  satisfies two conditions (4.2) and (4.4), then  $\|\tilde{f}\|$  is a superiorideal on  $(\mathcal{L}, \tilde{f})$ .

**Proof.** Since  $x * (x * y) \leq y$  for all  $x, y \in L$ , it follows from (4.4) that

$$\|\tilde{f}\|(x) \preceq \sup\{\|\tilde{f}\|(x * y), \|\tilde{f}\|(y)\}$$

for all  $x, y \in L$ . Therefore  $\|\tilde{f}\|$  is a superiorideal on  $(\mathcal{L}, \tilde{f})$ . ■

## 5. COMMUTATIVE SUPERIORIDEALS

**Definition 5.1.** Let  $\mathcal{L} := (L, *, 0)$  be a BCK-algebra and let  $(\tilde{f}, L)$  be a pair on  $(U, \preceq)$ . By a commutative superiorideal on  $(\mathcal{L}, \tilde{f})$ , we mean the superiormapping  $\|\tilde{f}\|$  of  $\mathcal{L}$  with respect to  $(\tilde{f}, L)$  which satisfies the condition (4.2) and

$$(5.1) \quad (\forall x, y, z \in L) \left( \|\tilde{f}\|(x * (y * (y * x))) \preceq \sup\{\|\tilde{f}\|((x * y) * z), \|\tilde{f}\|(z)\} \right)$$

whenever there exists  $\sup\{\|\tilde{f}\|(x), \|\tilde{f}\|(y)\}$  for any  $x, y \in L$ .

**Example 5.2.** Let  $U = \{1, 2, 3, \dots, 8\}$  be ordered as pictured in Figure 4.

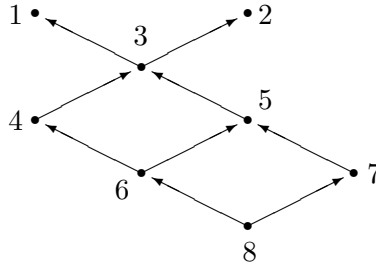


Figure 4

Table 4. Cayley table for the binary operation ‘\*’.

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	b	0
c	c	a	c	0	c
d	d	d	d	d	0

Let  $L = \{0, a, b, c, d\}$  be a set with a binary operation ‘\*’ shown in Table 4.

Then  $\mathcal{L} := (L, *, 0)$  is a BCK-algebra (see [9]).

Let  $(\tilde{f}, L)$  be a pair on  $(U, \preceq)$  where  $\tilde{f}$  is given as follows:

$$\tilde{f} : L \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \{6, 8\} & \text{if } x \in \{0, b\}, \\ \{4, 6, 7\} & \text{if } x = d, \\ \{2, 3, 5, 6, 7\} & \text{if } x \in \{a, c\}. \end{cases}$$

Then the superiormapping  $\|\tilde{f}\|$  of  $\mathcal{L}$  with respect to  $(\tilde{f}, L)$  is described as follows:  $\|\tilde{f}\|(0) = \|\tilde{f}\|(b) = 6$ ,  $\|\tilde{f}\|(d) = 3$ , and  $\|\tilde{f}\|(a) = \|\tilde{f}\|(c) = 2$ . It is routine to check that  $\|\tilde{f}\|$  is a commutative superiorideal on  $(\mathcal{L}, \tilde{f})$ .

**Theorem 5.3.** *If  $\mathcal{L}$  is a BCK-algebra, then every commutative superiorideal on  $(\mathcal{L}, \tilde{f})$  is a superiorideal on  $(\mathcal{L}, \tilde{f})$ .*

**Proof.** Let  $\|\tilde{f}\|$  be a commutative superiorideal on  $(\mathcal{L}, \tilde{f})$  where  $\mathcal{L}$  is a BCK-algebra. Using (V), (2.1) and (5.1), we have

$$\begin{aligned} \|\tilde{f}\|(x) &= \|\tilde{f}\|(x * (0 * (0 * x))) \\ &\preceq \sup\{\|\tilde{f}\|((x * 0) * z), \|\tilde{f}\|(z)\} \\ &= \sup\{\|\tilde{f}\|(x * z), \|\tilde{f}\|(z)\} \end{aligned}$$

for all  $x, z \in L$ . Hence  $\|\tilde{f}\|$  is a superiorideal on  $(\mathcal{L}, \tilde{f})$ . ■

The following example shows that the converse of Theorem 5.3 is not true in general.

**Example 5.4.** Let  $L = \{0, a, b, c, d\}$  be a set with a binary operation ‘\*’ shown in Table 5.

Table 5. Cayley table for the binary operation ‘\*’.

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0

Then  $\mathcal{L} := (L, *, 0)$  is a BCK-algebra (see [9]). Consider the poset  $(U, \preceq)$  which is given in Example 5.2. Let  $(\tilde{f}, L)$  be a pair on  $(U, \preceq)$  where  $\tilde{f}$  is given as follows:

$$\tilde{f} : L \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \{8\} & \text{if } x = 0, \\ \{5, 6, 7\} & \text{if } x = a, \\ \{3, 4, 5, 7\} & \text{if } x \in \{b, c, d\}. \end{cases}$$

Then the superiormapping  $\|\tilde{f}\|$  on  $(\mathcal{L}, \tilde{f})$  is described as follows:  $\|\tilde{f}\|(0) = 8$ ,  $\|\tilde{f}\|(a) = 5$  and  $\|\tilde{f}\|(b) = \|\tilde{f}\|(c) = \|\tilde{f}\|(d) = 3$ . Routine calculations show that  $\|\tilde{f}\|$  is a superiorideal on  $(\mathcal{L}, \tilde{f})$ . But it is not a commutative superiorideal on  $(\mathcal{L}, \tilde{f})$  since

$$\|\tilde{f}\|(b * (c * (c * b))) \not\preceq \sup\{\|\tilde{f}\|((b * c) * 0), \|\tilde{f}\|(0)\}.$$

**Proposition 5.5.** *Let  $\|\tilde{f}\|$  be a commutative superiorideal on  $(\mathcal{L}, \tilde{f})$  where  $\mathcal{L}$  is a BCK-algebra. Then the following assertion is valid.*

$$(5.2) \quad (\forall x, y \in L) \left( \|\tilde{f}\|(x * (y * (y * x))) \preceq \|\tilde{f}\|(x * y) \right).$$

**Proof.** Taking  $z = 0$  in (5.1) and using (4.2) and (2.1), we have the desired result. ■

We provide conditions for a superiorideal to be commutative.

**Theorem 5.6.** *Let  $\|\tilde{f}\|$  be a superiorideal on  $(\mathcal{L}, \tilde{f})$  where  $\mathcal{L}$  is a BCK-algebra. If the condition (5.2) is valid, then  $\|\tilde{f}\|$  is commutative.*

**Proof.** Assume that  $\|\tilde{f}\|$  satisfies the condition (5.2). The condition (4.3) induces

$$(5.3) \quad \|\tilde{f}\|(x * y) \preceq \sup\{\|\tilde{f}\|((x * y) * z), \|\tilde{f}\|(z)\}$$

for all  $x, y, z \in L$ . Combining (5.3) and (5.2), we know that

$$\|\tilde{f}\|(x * (y * (y * x))) \preceq \sup\{\|\tilde{f}\|((x * y) * z), \|\tilde{f}\|(z)\}$$

for all  $x, y, z \in L$ . Therefore  $\|\tilde{f}\|$  is a commutative superiorideal on  $(\mathcal{L}, \tilde{f})$ . ■

Combining Theorems 4.13 and 5.6, we have the following corollary.

**Corollary 5.7.** *Let  $\|\tilde{f}\|$  be the superiormapping of a BCK-algebra  $\mathcal{L}$  with respect to  $(\tilde{f}, L)$ . If  $\|\tilde{f}\|$  satisfies (4.2), (4.4) and (5.2), then  $\|\tilde{f}\|$  is a commutative superiorideal on  $(\mathcal{L}, \tilde{f})$ .*

**Theorem 5.8.** *In a commutative BCK-algebra, every superiorideal is a commutative superiorideal.*

**Proof.** Let  $\|\tilde{f}\|$  be a superiorideal on  $(\mathcal{L}, \tilde{f})$  where  $\mathcal{L}$  is a commutative BCK-algebra. Note that

$$\begin{aligned} & ((x * (y * (y * x))) * ((x * y) * z)) * z \\ &= ((x * (y * (y * x))) * z) * ((x * y) * z) \\ &\leq (x * (y * (y * x))) * (x * y) \\ &= (x * (x * y)) * (y * (y * x)) = 0, \end{aligned}$$

that is,  $(x * (y * (y * x))) * ((x * y) * z) \leq z$  for all  $x, y, z \in L$ . It follows from Proposition 4.12 that

$$\|\tilde{f}\|(x * (y * (y * x))) \preceq \sup\{\|\tilde{f}\|((x * y) * z), \|\tilde{f}\|(z)\}$$

for all  $x, y, z \in L$ . Therefore  $\|\tilde{f}\|$  is a commutative superiorideal on  $(\mathcal{L}, \tilde{f})$ . ■

**Corollary 5.9.** *If a BCK-algebra  $\mathcal{L}$  satisfies the following condition:*

$$(5.4) \quad (\forall x, y \in L) (x * (x * y) \leq y * (y * x)),$$

*then every superiorideal is a commutative superiorideal.*

**Lemma 5.10** [9]. *Let  $A$  be an ideal of a BCK-algebra  $\mathcal{L}$ . Then  $A$  is commutative if and only if the following assertion holds.*

$$(5.5) \quad (\forall x, y \in A) (x * y \in A \Rightarrow x * (y * (y * x)) \in A).$$

**Theorem 5.11.** *If  $\|\tilde{f}\|$  is a commutative superiorideal on  $(\mathcal{L}, \tilde{f})$  where  $\mathcal{L}$  is a BCK-algebra, then the nonempty set*

$$\|\tilde{f}\|_\alpha := \{x \in L \mid \|\tilde{f}\|(x) \preceq \alpha\}$$

*is a commutative ideal of  $\mathcal{L}$  for all  $\alpha \in U$ .*

**Proof.** Assume that  $\|\tilde{f}\|$  is a commutative superiorideal on  $(\mathcal{L}, \tilde{f})$  where  $\mathcal{L}$  is a BCK-algebra. Then  $\|\tilde{f}\|$  is a superiorideal on  $(\mathcal{L}, \tilde{f})$  by Theorem 5.3. Hence if  $\|\tilde{f}\|_\alpha \neq \emptyset$ , then  $\|\tilde{f}\|_\alpha$  is an ideal of  $\mathcal{L}$  for all  $\alpha \in U$ . Let  $x, y \in L$  be such that  $x * y \in \|\tilde{f}\|_\alpha$ . Using (5.2), we have  $\|\tilde{f}\|((x * (y * (y * x))) \preceq \|\tilde{f}\|(x * y) \preceq \alpha$  and so  $x * (y * (y * x)) \in \|\tilde{f}\|_\alpha$ . Hence, by Lemma 5.10,  $\|\tilde{f}\|_\alpha$  is a commutative ideal of  $\mathcal{L}$  for all  $\alpha \in U$ . ■

**Theorem 5.12.** Let  $\|\tilde{f}\|$  and  $\|\tilde{g}\|$  be superiorideals on  $(\mathcal{L}, \tilde{f})$  and  $(\mathcal{L}, \tilde{g})$ , respectively, where  $\mathcal{L}$  is a BCK-algebra such that  $\|\tilde{f}\|(0) = \|\tilde{g}\|(0)$  and  $\|\tilde{g}\|(x) \preceq \|\tilde{f}\|(x)$  for all  $x (\neq 0) \in L$ . If  $\|\tilde{f}\|$  is a commutative superiorideal on  $(\mathcal{L}, \tilde{f})$ , then  $\|\tilde{g}\|$  is a commutative superiorideal on  $(\mathcal{L}, \tilde{g})$ .

**Proof.** For any  $x, y \in L$ , let  $u = x * y$ . Using hypothesis, (5.2), (2.3) and (III), we have

$$\begin{aligned} \|\tilde{g}\|((x * u) * (y * (y * (x * u)))) &\preceq \|\tilde{f}\|((x * u) * (y * (y * (x * u)))) \\ &\preceq \|\tilde{f}\|((x * u) * y) = \|\tilde{f}\|((x * y) * u) \\ &= \|\tilde{f}\|(0) = \|\tilde{g}\|(0), \end{aligned}$$

and so  $\|\tilde{g}\|((x * u) * (y * (y * (x * u)))) = \|\tilde{g}\|(0)$ . Note that

$$\begin{aligned} &(x * (y * (y * x))) * (x * (y * (y * (x * u)))) \\ &\leq (y * (y * (x * u))) * (y * (y * x)) \\ &\leq (y * x) * (y * (x * u)) \\ &\leq (x * u) * x = 0 * u = 0, \end{aligned}$$

and thus  $(x * (y * (y * x))) * (x * (y * (y * (x * u)))) = 0$ . It follows from (4.3), (4.2) and (2.3) that

$$\begin{aligned} \|\tilde{g}\|(x * (y * (y * x))) &\preceq \sup\{\|\tilde{g}\|((x * (y * (y * x))) * (x * (y * (y * (x * u))))), \\ &\quad \|\tilde{g}\|(x * (y * (y * (x * u))))\} \\ &= \sup\{\|\tilde{g}\|(0), \|\tilde{g}\|(x * (y * (y * (x * u))))\} \\ &= \|\tilde{g}\|(x * (y * (y * (x * u)))) \\ &\preceq \sup\{\|\tilde{g}\|((x * (y * (y * (x * u)))) * u), \|\tilde{g}\|(u)\} \\ &= \sup\{\|\tilde{g}\|((x * u) * (y * (y * (x * u)))) , \|\tilde{g}\|(u)\} \\ &= \sup\{\|\tilde{g}\|(0), \|\tilde{g}\|(u)\} \\ &= \|\tilde{g}\|(u) = \|\tilde{g}\|(x * y). \end{aligned}$$

Therefore  $\|\tilde{g}\|$  is a commutative superiorideal on  $(\mathcal{L}, \tilde{g})$  by Theorem 5.6. ■

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