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DIRECT DECOMPOSITIONS OF DUALLY RESIDUATED LATTICE ORDERED MONOIDS

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Abstract

The class of dually residuated lattice ordered monoids ($DR\ell$-monoids) contains, in an appropriate signature, all ℓ-groups, Brouwerian algebras, MV- and GMV-algebras, BL- and pseudo BL-algebras, etc. In the paper we study direct products and decompositions of $DR\ell$-monoids in general and we characterize ideals of $DR\ell$-monoids which are direct factors. The results are then applicable to all above mentioned special classes of $DR\ell$-monoids.

Keywords: $DR\ell$-monoid, lattice-ordered monoid, ideal, normal ideal, polar, direct factor.

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1. Introduction

Commutative dually residuated lattice-ordered monoids (in short: $DR\ell$-monoids) were introduced and studied by K.L.N. Swamy in [20], [21], [22] as a common generalization of commutative lattice-ordered groups (ℓ-groups) and Brouwerian algebras. The papers [23], [24], [9]–[15], [4] and the part of the thesis [5] engaged the further research of structure properties of commutative $DR\ell$-monoids. It was shown that MV-algebras (see [13]) and BL-algebras (see [14]) which are an algebraic counterpart of the Łukasiewicz infinite valued logic and Hájek basic fuzzy logic, respectively, can be understood as special cases of commutative $DR\ell$-monoids. General $DR\ell$-monoids (i.e., not necessarily commutative), the special case of which are also all ℓ-groups, were introduced by Kovář in [5]. GMV-algebras were defined as a non-commutative generalization of MV-algebras in [16] and it was shown there that they are special cases of $DR\ell$-monoids. This fact was then used when studying GMV-algebras in [17] and [18]. Similarly, it was proved in [6] that pseudo BL-algebras (defined in [2] as a non-commutative generalization of BL-algebras) are also a special case of $DR\ell$-monoids. $DR\ell$-monoids were further studied in [8], [7] and [19].

In the paper we shall study direct products and direct decompositions of $DR\ell$-monoids. The general results are then applicable for all mentioned special cases of $DR\ell$-monoids.

2. Basic notions and notation

Definition. An algebra $M=(\mathrm{M};+,0,\vee,\wedge,\rightharpoonup,\leftharpoondown)$ of signature $\langle 2,0,2,2,2,2\rangle$ is called a *dually residuated (non-commutative) lattice-ordered monoid (* a *$DR\ell$-monoid)* if

(M1) $(M;+,0,\vee,\wedge)$ is a lattice-ordered monoid (ℓ-monoid), that is, $(M;+,0)$ is a (non-commutative) monoid, $(M,\vee,\wedge)$ is a lattice, and for any $x,y,u,v\in M$, the following identities are satisfied:

$$u+(x\vee y)+v=(u+x+v)\vee(u+y+v),$$

$$u+(x\wedge y)+v=(u+x+v)\wedge(u+y+v);$$

(M2) if $\leq$ denotes the order on M induced by the lattice $(M;\vee,\wedge)$, then, for any x, $y\in M$, we have

$x \rightharpoonup y$ is the least element $s \in M$ such that $s + y \geq x$,

$x \leftharpoondown y$ is the least element $t \in M$ such that $y + t \geq x$;

(M3) M fulfils the identities

$$((x \rightharpoonup y) \vee 0) + y \leq x \vee y, \;\; y + ((x \leftharpoondown y) \vee 0) \leq x \vee y,$$

$$x \rightharpoonup x \geq 0, \;\; x \leftharpoondown x \geq 0.$$

Commutative $DR\ell$-monoids (called $DR\ell$-semigroups) were introduced by K.L.N. Swamy in [20] as a common generalization of commutative ℓ-groups and Brouwerian algebras. The present definition of a non-commutative extension of $DR\ell$-monoids is due to [5]. Also, for basic properties of non-commutative $DR\ell$-monoids see [5].

Let us denote by $M^+ = \{x \in \mathrm{M} : 0 \leq x\}$ the set of all positive elements in M.

Examples.

a) Let $G = (G; +, 0, -(\cdot), \vee, \wedge)$ be an ℓ-group. Set $x \rightharpoonup y = x - y$ and $x \leftharpoondown y = -y + x$ for any $x, y \in G$. Then $(G; +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ is a $DR\ell$-monoid.

b) Let G be an ℓ-group and G^+ be its positive cone, i.e.: $G^+ = \{x \in \mathrm{G} : 0 \leq x\}$. Set $x \rightharpoonup y = (x - y) \vee 0$ and $x \leftharpoondown y = (-y + x) \vee 0$ for any elements $x, y \in G^+$. Then $(G^+; +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ is a $DR\ell$-monoid.

c) Let $B = (B; \vee, \wedge)$ be a Brouwerian algebra, i.e. a dually relative pseudo-complemented lattice with the largest element (that means, for any $a, b \in B$, there exists the smallest element $x \in B$ such that $b \vee x \geq a$). Let us denote by $a - b$ this relative pseudocomplement x of the element b with respect to the element a. The lattice $(B; \vee, \wedge)$ has the smallest element 0 and if we set $a + b = a \vee b$ and $a \rightharpoonup b = a \leftharpoondown b = a - b$ for every $a, b \in B$, then $(B; +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ is a commutative $DR\ell$-monoid.

d) Let $A = (A; \oplus, \neg, \sim, 0, 1)$ be a GMV-algebra (see, e.g., [16]), i.e. a non-commutative generalization of an MV-algebra. For any $x, y \in A$, put $x \odot y = \sim(\neg x \oplus \neg y)$, $x \rightharpoonup y = \neg y \odot x$ and $x \leftharpoondown y = x \odot \sim y$. If we denote $x \vee y = x \oplus (y \odot \sim x)$ and $x \wedge y = x \odot (y \oplus \sim x)$, then $(A; \vee, \wedge)$ is a bounded distributive lattice and the algebra $(A; \oplus, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ is a (bounded) $DR\ell$-monoid. If the addition $\oplus$ is commutative, then the negations $\neg$ and $\sim$ coincide, A is an MV-algebra, and the induced $DR\ell$-monoid is commutative.

Let M be a $DR\ell$-monoid and $x \in M$. Then the *absolute value of an element* x is $|x| = x \vee (0 \rightharpoonup x)$.

Definitions.

a) If M is a $DR\ell$-monoid and $\emptyset \neq I \subseteq M$, then I is called an *ideal of* M if the following conditions are satisfied:

(1) $x, y \in I \implies x + y \in I$;

(2) $x \in I,\ y \in M,\ |y| \leq |x| \implies y \in I$.

b) An ideal I is said to be *normal* if for each $x, y \in M$ the equivalence:

$$x \rightharpoonup y \in I \Longleftrightarrow x \leftharpoondown y \in I$$

is satisfied.

Remark. By [8], normal ideals are just kernels of $DR\ell$-homomorphisms.

It is proved in [8] that the set $\mathcal{C}(M)$ of all ideals of an arbitrary $DR\ell$-monoid M, ordered by set inclusion, is an algebraic Brouwerian lattice in which infima coincide with set intersections. Further, by Lemma 21 of [8], if I and J are normal ideals of a $DR\ell$-monoid M, then their join $I \vee J$ in $\mathcal{C}(M)$ is the following set:

$$I \vee J = \{x \in \mathrm{M} : |x| \leq a + b, \text{ for some } a \in I^+,\ b \in J^+\}.$$

Definitions.

a) Let M be a $DR\ell$-monoid and $X \subseteq M$. Then the set

$$X^\perp = \{y \in \mathrm{M} : |x| \wedge |y| = 0, \text{ for each } x \in X\}$$

is called the *polar of* X *in* M.

b) A subset $X \subseteq M$ is a *polar in* M if there exists $Y \subseteq M$ such that $X = Y^\perp$.

By [7], every polar in M belongs to $\mathcal{C}(M)$ and it is a polar of some ideal of M. The polar of any ideal $I \in \mathcal{C}(M)$ is its pseudocomplement in the lattice $\mathcal{C}(M)$ and therefore the set $\mathcal{P}(M)$ of all polars in M is a complete Boolean algebra with respect to set inclusion.

3. Direct products and decompositions

In this section we will study properties of direct products of $DR\ell$-monoids, in particular with respect to possibilities of introduction of inner direct products.

Lemma 1. *Let M be a $DR\ell$-monoid. Then for any $v, w \in M$ we have $v \rightharpoonup w = 0$ if and only if $v \leftharpoondown w = 0$.*

Proof. If $v \rightharpoonup w = 0$ and $x \in M$, then $x + v \geq w$ if and only if $x \geq 0$. Hence $w = 0 + w \geq v$. Then also $w + 0 \geq v$, thus $0 \geq v \leftharpoondown w$. At the same time $w \geq v$ implies $v \leftharpoondown w \geq 0$; therefore, $v \leftharpoondown w = 0$. ∎

Let B and C be $DR\ell$-monoids and let $M = B \times C$ be their direct product. Denote $\widetilde{B}$, $\widetilde{C} \subseteq M$ such that

$$\widetilde{B} = \{(x_1, 0) \ : \ x_1 \in B\},$$

$$\widetilde{C} = \{(0, x_2) \ : \ x_2 \in C\}.$$

The following proposition seems to be well-known as a folklore:

Proposition 2. *If B and C are $DR\ell$-monoids and $M = B \times C$ then $\widetilde{B}$ and $\widetilde{C}$ are normal ideals of $DR\ell$-monoid M and it holds:*

a) $\widetilde{B} + \widetilde{C} = M, \quad \widetilde{B} \cap \widetilde{C} = \{0\}$;

b) $x + y = x' + y'$ *implies* $x = x'$, $y = y'$ *for each* $x, x' \in \widetilde{B}$ *and* $y, y' \in \widetilde{C}$.

∎

Proposition 3. *If $M = B \times C$, then*

$$\widetilde{B} = \widetilde{C}^{\perp} \quad \text{and} \quad \widetilde{C} = \widetilde{B}^{\perp}.$$

Proof. For any elements $x_1 \in B$ and $y_2 \in C$ it is satisfied

$$|(x_1, 0)| \wedge |(0, y_2)| \in \widetilde{B} \cap \widetilde{C} \ = \ \{(0, 0)\}.$$

Therefore, $\widetilde{B} \subseteq \widetilde{C}^{\perp}$ and $\widetilde{C} \subseteq \widetilde{B}^{\perp}$.

Conversely, let $(z_1, z_2) \in \left(\widetilde{B}^{\perp}\right)^{+}$. Then

$$(z_1, z_2) = (z_1, 0) + (0, z_2) \text{ and } (z_1, 0) = (z_1, 0) \wedge (z_1, z_2) = (0, 0).$$

Thus $\left(\widetilde{B}^{\perp}\right)^{+} \subseteq \widetilde{C}$, therefore also $\widetilde{B}^{\perp} \subseteq \widetilde{C}$, it means $\widetilde{B}^{\perp} = \widetilde{C}$.

Analogously, $\widetilde{C}^{\perp} \subseteq \widetilde{B}$. ■

Now we will deal with possibility of introduction of an inner direct decomposition of $DR\ell$-monoids.

At first, we will prove the following lemma.

Lemma 4. *Let M be a $DR\ell$-monoid and let I, $J \in \mathcal{C}(M)$ be such that $I + J = M$ and $I \cap J = \{0\}$. If $a \in M$ and $a_1 \in I$, $a_2 \in J$ are such that $a = a_1 + a_2$, then $a \geq 0$ if and only if $a_1 \geq 0$ and $a_2 \geq 0$.*

Proof. Suppose $0 \leq a = a_1 + a_2$. Then $0 \rightharpoonup a_2 \leq (a_1 + a_2) \rightharpoonup a_2$. Since, by Lemma 1.1.19 of [5], it holds $(p+q) \rightharpoonup r \leq p + (q \rightharpoonup r)$ for any p, q, $r \in M$, in our case we obtain $(a_1 + a_2) \rightharpoonup a_2 \leq a_1 + (a_2 \rightharpoonup a_2) = a_1$. So $0 \rightharpoonup a_2 \leq a_1$. Therefore, $0 \leq (0 \rightharpoonup a_2) \vee 0 \leq a_1 \vee 0 \in I$. Hence $(0 \rightharpoonup a_2) \vee 0 \in I \cap J$, that means $(0 \rightharpoonup a_2) \vee 0 = 0$. Thus $0 \rightharpoonup a_2 \leq 0$. By Lemma 1.1.16 of [5], $p \geq q$ if and only if $q \rightharpoonup p \leq 0$, for any p, $q \in M$. Thus we have $a_2 \geq 0$. Similarly, $a_1 \geq 0$.

The converse implication is obvious. ■

Definitions.

a) An element y of a $DR\ell$-monoid M is called *singular* if $0 \rightharpoonup y = 0$ (or equivalently, by Lemma 1, $0 \leftharpoondown y = 0$).

b) An element $x \in M$ is called *invertible* if there exists an inverse element for it in the monoid $(M; +, 0)$.

Denote by $\mathrm{Sing}(M)$ the set of all singular elements in M and by $\mathrm{Inv}(M)$ the set of all invertible elements in M.

Remarks. Kovář proved in [5] (see Theorem 1.2.16 and Lemma 1.2.11) that $\mathrm{Sing}(M) \in \mathcal{C}(M)$, $\mathrm{Sing}(M) \subseteq M^{+}$ and 0 is the least element in $\mathrm{Sing}(M)$. Further, by Theorems 1.2.1 and 1.2.4 of [5], $\mathrm{Inv}(M)$ is also an ideal of M which is, moreover, an ℓ-group. The ideals $\mathrm{Sing}(M)$ and $\mathrm{Inv}(M)$ play an important role in the study of structure properties of $DR\ell$-monoids because, by Theorem 1.3.6 of [5], each $DR\ell$-monoid M is isomorphic to the direct product of the $DR\ell$-monoids $\mathrm{Sing}(M)$ and $\mathrm{Inv}(M)$.

At the same time, extreme case can arise, because if M is an ℓ-group, then $\mathrm{Sing}(M) = \{0\}$ and $\mathrm{Inv}(M) = M$. If M is a Brouwerian algebra, then, conversely, $\mathrm{Sing}(M) = M$ and $\mathrm{Inv}(M) = \{0\}$. Consequently, the cardinality of $\mathrm{Sing}(M)$ determines the degree of dissimilarity of properties of a given $DR\ell$-monoid from properties of an ℓ-group.

Proposition 5. *If M is a $DR\ell$-monoid and a, $b \in M$ are orthogonal* (*i.e.* $|a| \wedge |b| = 0$)*, then $a + b = b + a$.*

Proof. a) Assume $a, b \in M^+$ and $a \wedge b = 0$. By Lemmas 1.1.5 and 1.1.9 of [5], for any $x, y, z \in M$ it holds $x \rightharpoonup x = 0$ and $x \rightharpoonup (y \wedge z) = (x \rightharpoonup y) \vee (x \rightharpoonup z)$, hence in our case we have

$$(a \rightharpoonup (a \wedge b)) + b = ((a \rightharpoonup a) \vee (a \rightharpoonup b)) + b = (0 \vee (a \rightharpoonup b)) + b = a \vee b,$$

therefore $a + b = a \vee b = b + a$, in our case.

b) Now, let a, b be arbitrary elements in M such that $|a| \wedge |b| = 0$. As mentioned in the previous remark, by Theorem 1.3.6 of [5], M is the direct product of its ideals $\mathrm{Sing}(M)$ and $\mathrm{Inv}(M)$. Hence there are a', $b' \in \mathrm{Sing}(M)$ and x_a, $x_b \in \mathrm{Inv}(M)$ such that $a = a' + x_a$, $b = b' + x_b$. By [5], $|a| = a' + |x_a|$, $|b| = b' + |x_b|$. Therefore, the assumption $|a| \wedge |b| = 0$ implies $a' \wedge b' = 0$ and $|x_a| \wedge |x_b| = 0$.

By the part a), we obtain $a' + b' = b' + a'$. As $\mathrm{Inv}(M)$ is an ℓ-group, it holds that $|x_a| \wedge |x_b| = 0$ entails $x_a + x_b = x_b + x_a$. Moreover, since M is isomorphic to the direct product of $\mathrm{Sing}(M)$ and $\mathrm{Inv}(M)$, elements in $\mathrm{Sing}(M)$ commute with those in $\mathrm{Inv}(M)$. Thus

$$a + b = (a' + x_a) + (b' + x_b) = a' + b' + x_a + x_b =$$

$$= b' + a' + x_b + x_a = (b' + x_b) + (a' + x_a) = b + a.$$

■

Theorem 6. *Let M be a $DR\ell$-monoid and I, $J \in \mathcal{C}(M)$. Let the following conditions be satisfied:*

1. $I + J = M, \quad I \cap J = \{0\};$
2. $\forall x, x' \in I,\ y, y' \in J;\ \ x + y = x' + y' \implies x = x',\ y = y'.$

If $\overline{M} = I \times J$ *is the direct product of the* $DR\ell$*-monoids* I *and* J, *then* M *and* $\overline{M}$ *are isomorphic.*

Proof. The conditions 1 and 2 obviously yield that for every element $a \in M$ there exist unique elements $a_1 \in I$ and $a_2 \in J$ such that $a = a_1 + a_2$. Hence the mapping $f:\ a \longmapsto (a_1, a_2)$ is a bijection of M onto $\overline{M}$.

Let us suppose $x \in I$ and $y \in J$. Then $|x| \in I$, $|y| \in J$ and $|x| \wedge |y| \in I \cap J = \{0\}$. It follows that x and y are orthogonal. Therefore, $x + y = y + x$ by Proposition 5. For this reason it holds for any elements $a, b \in M$

$$a + b = (a_1 + a_2) + (b_1 + b_2) = (a_1 + b_1) + (a_2 + b_2),$$

therefore

$$f(a + b) = (a_1 + b_1, a_2 + b_2) = f(a) + f(b).$$

Assume again $a = a_1 + a_2$, $b = b_1 + b_2 \in M$, $a_1, b_1 \in I$, $a_2, b_2 \in J$ and let $a \leq b$. By Lemma 1.1.14 of [5], there exists $x \in M^+$ such that $a + x = b$. Let $x = x_1 + x_2$, where $x_1 \in I$, $x_2 \in J$. By Lemma 4, it holds $x_1 \in I^+$ and $x_2 \in J^+$. From this we have $(a_1 + x_1) + (a_2 + x_2) = b_1 + b_2$, i.e. $a_1 + x_1 = b_1$, $a_2 + x_2 = b_2$, where $0 \leq x_1$, $0 \leq x_2$. As $0 \leq x_1$, it holds $a_1 \leq a_1 + x_1 = b_1$. Similarly, $a_2 \leq b_2$.

Hence, for any $a, b \in M$, $a \leq b$ if and only if $f(a) \leq f(b)$.

We have proved that f is an isomorphism of lattice-ordered monoids $(M; +, 0, \vee, \wedge)$ and $(\overline{M}; +, 0, \vee, \wedge)$. Since the values of the operations $\rightharpoonup$ and $\leftharpoondown$ are uniquely determined in both the $DR\ell$-monoids $M = (M; +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ and $(\overline{M}; +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ in the same manner by means of the operation $+$ and the order relation $\leq$, $DR\ell$-monoids $M = I + J$ and $\overline{M} = I \times J$ are also isomorphic. ∎

Remarks.

a) Let $\widetilde{I} = \{(x, 0);\ x \in I\}$ and $\widetilde{J} = \{(0, y);\ y \in J\}$. Since $I \cong \widetilde{I}$ and $J \cong \widetilde{J}$, the ideals I and J are (by Proposition 2 and Theorem 6) normal in M.

b) By Theorem 6 and Proposition 3, the set of all direct factors of a $DR\ell$-monoid M is a subset of the set of all polars in M. In particular, $\mathrm{Sing}(M)$ and $\mathrm{Inv}(M)$ are polars in M. It holds

$$(\mathrm{Sing}(M))^{\perp} = \mathrm{Inv}(M) \quad \text{and} \quad (\mathrm{Inv}(M))^{\perp} = \mathrm{Sing}(M).$$

If M is a $DR\ell$-monoid and $I \in \mathcal{C}(M)$, let us denote by $D(I)$ the join of ideals I and $I^{\perp}$ in the lattice $\mathcal{C}(M)$.

Proposition 7. *If M is a $DR\ell$-monoid, $I \in \mathcal{C}(M)$ and I is a direct factor of $DR\ell$-monoid $D(I)$, then $I + I^{\perp} \in \mathcal{C}(D(I))$ and $I + I^{\perp} = D(I) = I \vee I^{\perp}$ (in the sense of $\mathcal{C}(M)$).*

Proof. Since I and $I^{\perp}$ are normal ideals of $D(I)$, by Lemma 21 of [8], it holds that

$$I \vee I^{\perp} = \{x \in D(I);\ \ |x| \leq a + b, \ \text{ where } \ a \in I,\ b \in I^{\perp}\}.$$

in $\mathcal{C}(D(I))$ (consequently, also in $\mathcal{C}(M)$).

By Proposition 5, $a+b = b+a = a \vee b$. By Theorem 1.1.23 of [5], the underlying lattice $(M; \vee, \wedge)$ is distributive, therefore the lattice $(D(I)^{+}; \vee, \wedge)$ is also distributive. For this reason, from the inequality $|x| \leq a + b$, where $a \in I^{+}$ and $b \in \left(I^{\perp}\right)^{+}$, it follows the existence of elements $0 \leq a_1 \leq a$, $0 \leq b_1 \leq b$ in $D(I)$ such that $|x| = a_1 \vee b_1 = a_1 + b_1$. At the same time $a_1 \in I$, $b_1 \in I^{\perp}$ and hence $I \vee I^{\perp} = I + I^{\perp}$. ■

Corollary 8. *If M is a $DR\ell$-monoid and $I \in \mathcal{C}(M)$, then $DR\ell$-monoids $I + I^{\perp}$ and $I \times I^{\perp}$ are isomorphic if and only if $x + y = x' + y'$ implies $x = x'$ and $y = y'$, for any x, $x' \in I$ and y, $y' \in I^{\perp}$.* ■

By Proposition 15 of [8], for any ideal I of a $DR\ell$-monoid M (and hence also for each polar in M) it holds that its polar $I^{\perp}$ is the pseudocomplement of I in $\mathcal{C}(M)$. We can specify this result for the direct factors of M.

Proposition 9. *If an ideal I of a $DR\ell$-monoid M is a direct factor in M, then the polar $I^{\perp}$ is the complement of I in the lattice $\mathcal{C}(M)$.* ■

Now we can prove the following proposition:

Proposition 10. *If M is an arbitrary $DR\ell$-monoid, then ideals I of M, for which there exists an ideal $J \in \mathcal{C}(M)$ such that I and J satisfy condition* 1 *from Theorem* 6*, form a Boolean lattice. This lattice is a sublattice of $\mathcal{C}(M)$.*

Proof. Let I and J satisfy the given assumptions. Then from distributivity of the lattice $\mathcal{C}(M)$ we obtain

$$(I \vee J) \cap \left(I^{\perp} \cap J^{\perp}\right) = \left(I \cap I^{\perp} \cap J^{\perp}\right) \vee \left(J \cap I^{\perp} \cap J^{\perp}\right) = \{0\},$$

$$(I \vee J) \vee \left(I^{\perp} \cap J^{\perp}\right) = \left(I \vee J \vee I^{\perp}\right) \cap \left(I \vee J \vee J^{\perp}\right) = M,$$

hence $I \vee J$ and $I^{\perp} \cap J^{\perp}$ satisfy condition 1.

The remaining part of the assertion follows from Proposition 9. ∎

Let us consider the following condition of uniqueness of decomposition for $DR\ell$-monoids M:

(UD)
$$\begin{aligned}&\text{If } I,\ J \in \mathcal{C}(M),\ I \cap J = \{0\},\ x,\ x' \in I,\ y,\ y' \in J\\ &\text{and } \ x + y = x' + y',\ \text{ then } \ x = x' \ \text{ and } \ y = y'.\end{aligned}$$

Theorem 11. *If $DR\ell$-monoid M satisfies condition* (UD)*, then the direct factors in M form a Boolean sublattice of the lattice $\mathcal{C}(M)$.*

Proof. If condition (UD) holds in M, then $I \in \mathcal{C}(M)$ is a direct factor if and only if I and $J = I^{\perp}$ satisfy condition 1. Therefore, the theorem follows from Proposition 10. ∎

Remark. If G is an ℓ-group, then $DR\ell$-monoids G and G^{+} satisfy condition (UD). Hence their direct factors form a Boolean lattice.

References

[1] R.L.O. Cignoli, I.M.L. D'Ottaviano and D. Mundici, *Foundations of Many-valued Reasoning*, Kluwer Acad. Publ., Dordrecht 2000.

[2] A. Di Nola, G. Georgescu and A. Iorgulescu, *Pseudo BL-algebras: Part I,* Multiple-Valued Logic **8** (2002), 673–714.

[3] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Acad. Publ., Dordrecht 1998.

[4] M.E. Hansen, *Minimal prime ideals in autometrized algebras*, Czechoslovak Math. J. **44** (**119**) (1994), 81–90.

[5] T. Kovář, *A general theory of dually residuated lattice-ordered monoids*, Ph.D. Thesis, Palacký Univ., Olomouc 1996.

[6] J. Kühr, *Pseudo BL-algebras and DRℓ-monoids*, Math. Bohemica **128** (2003), 199–208.

[7] J. Kühr, *Prime ideals and polars in DRℓ-monoids and pseudo BL-algebras*, Math. Slovaca **53** (2003), 233–246.

[8] J. Kühr, *Ideals of noncommutative DRℓ-monoids*, Czechoslovak Math. J. (to appear).

[9] J. Rachůnek, *Prime ideals in autometrized algebras*, Czechoslovak Math. J. **37** (**112**) (1987), 65–69.

[10] J. Rachůnek, *Polars in autometrized algebras*, Czechoslovak Math. J. **39** (**114**) (1989), 681–685.

[11] J. Rachůnek, *Regular ideals in autometrized algebras*, Math. Slovaca **40** (1990), 117–122.

[12] J. Rachůnek, *DRℓ-semigroups and MV-algebras*, Czechoslovak Math. J. **48** (**123**) (1998), 365–372.

[13] J. Rachůnek, *MV-algebras are categorically equivalent to a class of DRℓ-semigroups*, Math. Bohemica **123** (1998), 437–441.

[14] J. Rachůnek, *A duality between algebras of basic logic and bounded representable DRℓ-monoids*, Math. Bohemica **126** (2001), 561–569.

[15] J. Rachůnek, *Polars and annihilators in representable DRℓ-monoids and MV-algebras*, Math. Slovaca **51** (2001), 1–12.

[16] J. Rachůnek, *A non-commutative generalization of MV-algebras*, Czechoslovak Math. J. **52** (**127**) (2002), 255–273.

[17] J. Rachůnek, *Prime ideals and polars in generalized MV-algebras*, Multiple-Valued Logic **8** (2002), 127–137.

[18] J. Rachůnek, *Prime spectra of non-commutative generalizations of MV-algebras*, Algebra Univers. **48** (2002), 151–169.

[19] D. Šalounová, *Lex-ideals of DRℓ-monoids and GMV-algebras*, Math. Slovaca **53** (2003), 321–330.

[20] K.L.N. Swamy, *Dually residuated lattice-ordered semigroups I*, Math. Ann. **159** (1965), 105–114.

[21] K.L.N. Swamy, *Dually residuated lattice-ordered semigroups II*, Math. Ann. **160** (1965), 64–71.

[22] K.L.N. Swamy, *Dually residuated lattice-ordered semigroups III*, Math. Ann. **167** (1966), 71–74.

[23] K.L.N. Swamy and N.P. Rao, *Ideals in autometrized algebras*, J. Austral. Math. Soc. Ser. A **24** (1977), 362–374.

[24] K.L.N. Swamy, and B.V. Subba Rao, *Isometries in dually residuated lattice-ordered semigroups*, Math. Sem. Notes Kobe Univ. **8** (1980), 369–379.

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