

## ON LATTICE-ORDERED MONOIDS

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### Abstract

In the paper lattice-ordered monoids and specially normal lattice-ordered monoids which are a generalization of dually residuated lattice-ordered semigroups are investigated. Normal lattice-ordered monoids are metricless normal lattice-ordered autometrized algebras. It is proved that in any lattice-ordered monoid  $A$ ,  $a \in A$  and  $na \geq 0$  for some positive integer  $n$  imply  $a \geq 0$ . A necessary and sufficient condition is found for a lattice-ordered monoid  $A$ , such that the set  $I$  of all invertible elements of  $A$  is a convex subset of  $A$  and  $A^- \subseteq I$ , to be the direct product of the lattice-ordered group  $I$  and a lattice-ordered semigroup  $P$  with the least element 0.

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Normal lattice-ordered autometrized algebras were investigated in [4], [10], [11], [12], [13], [19]. Swamy ([16], [17], [18]) introduced and studied dually residuated lattice-ordered semigroups (notation *DRL*-semigroups) as a common abstraction of Boolean rings and abelian lattice ordered groups (notations *l*-groups). Swamy and Subba Rao ([20]) investigated isometries in *DRL*-semigroups. They proved that any isometry fixing zero in a representable *DRL*-semigroup is an involutory semigroup automorphism. In [5], it was shown that to each weak isometry  $f$  fixing zero in a *DRL*-semigroup  $G$  there exists a direct decomposition  $G = A \times B$ , where  $A$  is a *DRL*-semigroup and  $B$  is an *l*-group, such that  $f(x) = x_A + (0 - x_B)$  for each  $x \in G$ .

Kovář in [6] proved that any *DRL*-semigroup  $A$  is the direct product of the  $l$ -group of all invertible elements of  $A$  and a *DRL*-semigroup with the least element and showed in [8] that conditions (1), (2) and (3) imply the condition (4) in the definition of a *DRL*-semigroup. In [9], he studied the group of zero fixing isometries of a *DRL*-semigroup. Prime ideals in *DRL*-semigroups were investigated by Hansen in [4]. Rachůnek ([14], [15]) proved that *MV*-algebras are in a one-to-one correspondence with special kinds of bounded *DRL*-semigroups. In [11], [12], he studied ideals and polars in *DRL*-semigroups.

Let us review some notions and notations used in the paper.

A system  $A = (A; +, \leq)$  is called a *partially ordered semigroup* (*po-semigroup*) if and only if

- (1)  $(A; +)$  is a semigroup,
- (2)  $(A; \leq)$  is a partially ordered set,
- (3)  $a \leq b$  implies  $a + x \leq b + x$  and  $x + a \leq x + b$  for all  $a, b, x \in A$ .

A *po*-semigroup  $(A; +, \leq)$  is called a *lattice-ordered semigroup* (*l-semigroup*) if and only if

- (1)  $(A; \leq)$  is a lattice with lattice operations  $\vee$  and  $\wedge$ ,
- (2)  $a + (b \vee c) = (a + b) \vee (a + c)$ ,  $(b \vee c) + a = (b + a) \vee (c + a)$ ,  
 $a + (b \wedge c) = (a + b) \wedge (a + c)$ ,  $(b \wedge c) + a = (b + a) \wedge (c + a)$

for each  $a, b, c \in A$ .

An  $l$ -semigroup with zero element  $0$  is called a *lattice-ordered monoid* (*l-monoid*).

A system  $A = (A; +, \leq, -)$  is called a *dually residuated lattice-ordered semigroup* (*DRL-semigroup*) if and only if

- (1)  $(A; +, \leq)$  is a commutative  $l$ -monoid,
- (2) for given  $a, b$  in  $A$  there exists a least  $x \in A$  such that  $b + x \geq a$ , and this  $x$  is denoted by  $a - b$ ,
- (3)  $(a - b) \vee 0 + b \leq a \vee b$  for all  $a, b \in A$ ,
- (4)  $(a - a) \geq 0$  for each  $a \in A$ .

Partially ordered semigroup  $A$  with a zero element is said to be the *direct product* of its partially ordered subsemigroups  $P$  and  $Q$  (notation  $A = P \times Q$ ) if the following conditions are fulfilled:

- (1) if  $a \in P$  and  $b \in Q$ , then  $a + b = b + a$ ,
- (2) each element  $c \in A$  can be uniquely represented in the form  $c = c_1 + c_2$ , where  $c_1 \in P$ ,  $c_2 \in Q$ ,
- (3) if  $a, b \in A$ ,  $a = a_1 + a_2$ ,  $b = b_1 + b_2$ , where  $a_1, b_1 \in P$ ,  $a_2, b_2 \in Q$ , then  $a \geq b$  if and only if  $a_1 \geq b_1$  and  $a_2 \geq b_2$ .

If  $A = P \times Q$ , then for  $x \in A$  we denote by  $x_P$  and  $x_Q$  the components of  $x$  in the direct factors  $P$  and  $Q$ , respectively.

An element  $x$  of an  $l$ -monoid  $A$  is called *positive* (*negative*) if  $x \geq 0$  ( $x \leq 0$ , resp.). The set of all positive (negative) elements of an  $l$ -monoid  $A$  will be denoted by  $A^+$  ( $A^-$ , resp.). For each element  $x$  of a lattice-ordered group  $G$ ,  $|x| = x \vee (-x)$ . (Throughout this paper  $0$  will denote a zero element. We use  $\mathbb{N}$  for the set of all positive integers).

Kovář showed in [10] (Theorem 1) that the set  $I$  of all invertible elements of a normal lattice-ordered autometrized algebra is an  $l$ -group. Analogous assertion is valid for  $l$ -monoids.

**Theorem 1.** *The set  $I$  of all invertible elements of an  $l$ -monoid  $A$  is an  $l$ -group and a sublattice of  $A$ .*

**Proof.** Clearly,  $I$  is a group. Let  $a, b \in I$ . Then we have  $a \vee b + (-a) \wedge (-b) = [a \vee b + (-a)] \wedge [a \vee b + (-b)] = [0 \vee (b-a)] \wedge [(a-b) \vee 0] \geq 0$ ,  $a \vee b + (-a) \wedge (-b) = [a + (-a) \wedge (-b)] \vee [b + (-a) \wedge (-b)] = [0 \wedge (a-b)] \vee [(b-a) \wedge 0] \leq 0$ . Thus  $a \vee b + (-a) \wedge (-b) = 0$ . Analogously, we obtain  $(-a) \wedge (-b) + a \vee b = 0$ . Hence  $-(a \vee b) = (-a) \wedge (-b)$ . Similarly, we can prove that  $-(a \wedge b) = (-a) \vee (-b)$ . Therefore,  $a \vee b, a \wedge b \in I$ . Hence  $I$  is a sublattice of  $A$ . ■

As a consequence of Theorem 1, we obtain:

**Corollary 1.** *The set  $I$  of all invertible elements of a lattice-ordered autometrized algebra  $A$  is an  $l$ -group.* ■

**Lemma 1.** *Let  $A$  be an  $l$ -monoid,  $a, b, c \in A$ .*

*If  $a \wedge b = 0$  and  $a \wedge c = 0$ , then  $a \wedge (b + c) = 0$ .*

*If  $a \vee b = 0$  and  $a \vee c = 0$ , then  $a \vee (b + c) = 0$ .*

The proof is the same as in the case of  $l$ -groups. See [1], p. 294. ■

**Remark 1.** Choudhury showed in [2] (p. 72) that  $a + b = a \vee b + a \wedge b$  for each elements  $a, b$  of a commutative  $l$ -semigroup.

**Theorem 2.** *Let  $A$  be an  $l$ -monoid. Let  $I$  be the set of all invertible elements of  $A$ , and let  $P = \{y \in A : y \wedge |z| = 0 \text{ for each } z \in I\}$ , where  $|z|$  is the absolute value of  $z$  in  $I$ . Then*

- (i)  $P \subseteq A^+$ ,  $0$  is the least element of  $P$ ,  $I \cap P = \{0\}$ ,
- (ii)  $P$  is a convex subset of  $A$ ,
- (iii)  $P$  is a sublattice of  $A$  and an  $l$ -semigroup.

**Proof.** (i): Clearly,  $0 \in P \subseteq A^+$ . Hence  $0$  is the least element of  $P$ . Let  $a \in I \cap P$ . Then  $a = a \wedge |a| = 0$ .

(ii): Let  $a, b \in P$ ,  $x \in A$ , and  $a \geq x \geq b$ . Then  $0 = a \wedge |z| \geq x \wedge |z| \geq b \wedge |z| = 0$  for each  $z \in I$ . Thus  $x \wedge |z| = 0$  for each  $z \in I$  and hence  $x \in P$ .

(iii): Let  $a, b \in P$ . By Lemma 1,  $a + b \in P$ . Hence  $P$  is a subsemigroup of  $A$ . Since  $a \geq 0$ ,  $b \geq 0$ , we have  $a + b \geq b$ ,  $a + b \geq a$ . Thus  $a + b \geq a \vee b \geq a \wedge b \geq 0$ . From the convexity of  $P$ , it follows that  $a \vee b, a \wedge b \in P$ . ■

**Theorem 3.** *Let  $A$  be a commutative  $l$ -monoid. Let each negative element of  $A$  be invertible and the set  $I$  of all invertible elements of  $A$  be a convex subset of  $A$ . Let  $P$  be as in Theorem 2. Then  $A$  is the direct product of the  $l$ -group  $I$  and the  $l$ -semigroup  $P$  with the least element  $0$  if and only if  $A$  satisfies the following condition:*

(C) *For each  $a \in A^+ \setminus I$  the set  $M_a = \{a \wedge x : x \in I^+\}$  has the greatest element.*

**Proof.** Let the set  $I$  of all invertible elements in  $A$  be a convex subset of  $A$  and let  $A^- \subseteq I$ . By Theorem 1,  $I$  is an  $l$ -group. By Theorem 2,  $P$  is an  $l$ -monoid with the least element  $0$ .

Suppose that  $A$  satisfies the condition (C). Assume that  $a \in A^+ \setminus I$ . Since  $0 \leq a \wedge x \leq x$  for each  $x \in I^+$ , from the convexity of  $I$  it follows that  $a \wedge x \in I^+$  and hence  $M_a \subseteq I^+$ . Let  $a_1$  be the greatest element of  $M_a$ ,  $a_2 = a + (-a_1)$ . Then  $0 \leq a_1 \leq a$ ,  $0 \leq a_2 \leq a$ . Hence  $a = a_1 + a_2$ , where  $a_1 \in I^+$ . Now we prove that  $a_2 \in P$ . Let  $b \in I$ ,  $d = |b| \vee a_1$ . Then  $d + a_1 \in I^+$ . Then (C) yields  $a \wedge (d + a_1) \leq a_1$ . This implies  $a_2 \wedge d = [a + (-a_1)] \wedge [d + a_1 + (-a_1)] = [a \wedge (d + a_1)] + (-a_1) \leq 0$ . Clearly,  $0 \leq a_2 \wedge d$ . Thus  $a_2 \wedge d = 0$ . Then  $0 \leq |b| \leq d$  yields  $0 = 0 \wedge a_2 \leq |b| \wedge a_2 \leq d \wedge a_2 = 0$ . Hence  $a_2 \wedge |b| = 0$ . Therefore,  $a_2 \in P$ .

If  $a \in I^+$ , then we can write  $a = a + 0$  and hence each  $a \in A^+$  can be written in the form  $a = a_1 + a_2$ , where  $a_1 \in I^+$ ,  $a_2 \in P$ .

Let  $g \in A$ . In view of Remark 1, we have  $g = g \wedge 0 + g \vee 0$ . Then  $g \vee 0 = (g \vee 0)_1 + (g \vee 0)_2$ , where  $(g \vee 0)_1 \in I$ ,  $(g \vee 0)_2 \in P$ . Let  $g_1 = g \wedge 0 + (g \vee 0)_1$ ,  $g_2 = (g \vee 0)_2$ . Since  $g \wedge 0 \in I$ , we have  $g_1 \in I$ . Thus  $g = g_1 + g_2$ , where  $g_1 \in I$ ,  $g_2 \in P$ .

Let  $g = g_1 + g_2 = h_1 + h_2$ , where  $h_1 \in I$ ,  $h_2 \in P$ . Thus  $g_2 = h_1 + h_2 + (-g_1)$ . Then  $g_2 \vee h_2 = [h_1 + h_2 + (-g_1)] \vee h_2 = [h_1 + (-g_1)] \vee 0 + h_2 \in P$ . Since  $h_2 = g_1 + g_2 + (-h_1)$ , we have  $g_2 \vee h_2 = g_2 \vee [g_1 + g_2 + (-h_1)] = 0 \vee [g_1 + (-h_1)] + g_2 \in P$ . Hence  $2(g_2 \vee h_2) = [h_1 + (-g_1)] \vee 0 + h_2 + 0 \vee [g_1 + (-h_1)] + g_2 = g_2 + h_2 + (h_1 - g_1) \vee 0 \vee (g_1 - h_1) = g_2 + h_2 + |h_1 + (-g_1)| \in P$ . Since  $h_1 + (-g_1) \in I$  and  $2(g_2 \vee h_2) \in P$ , we get  $0 = 2(g_2 \vee h_2) \wedge |h_1 + (-g_1)| = [g_2 + h_2 + |h_1 + (-g_1)|] \wedge |h_1 + (-g_1)| = |h_1 + (-g_1)| + [(g_2 + h_2) \wedge 0] = |h_1 + (-g_1)|$ . This implies  $h_1 = g_1$ . Then clearly  $g_2 = h_2$ . Therefore, each  $g \in A$  is uniquely represented in the form  $g = g_1 + g_2$ , where  $g_1 \in I$ ,  $g_2 \in P$ . Clearly, if  $g \in A^+$ , then  $g$  is uniquely represented in the form  $g = g_1 + g_2$ , where  $g_1 \in I^+$ ,  $g_2 \in P$ .

Let  $f, h \in A$ ,  $f \geq h$ ,  $f = f_1 + f_2$ ,  $h = h_1 + h_2$ , where  $f_1, h_1 \in I$ ,  $f_2, h_2 \in P$ . From  $f_1 + f_2 \geq h_1 + h_2$ , we get  $f_1 - h_1 + f_2 \geq h_2 \geq 0$ . This and  $f_1 - h_1 \in I$ ,  $f_2 \in P$  yield  $f_1 - h_1 \geq 0$ . Hence  $f_1 \geq h_1$ . Since  $|f_1 - h_1| \wedge h_2 = 0$  and  $h_2 \leq f_1 - h_1 + f_2 \leq |f_1 - h_1| + f_2$ , we get  $h_2 = h_2 \wedge (f_2 + h_2) \leq (|f_1 - h_1| + f_2) \wedge (f_2 + h_2) = (|f_1 - h_1| \wedge h_2) + f_2 = f_2$ . Therefore,  $A = I \times P$ .

Let  $A = I \times P$ ,  $a \in A^+$ . Then  $a = a_I + a_P$ , where  $a_I \in I^+$ ,  $a_P \in P$ . Since  $a \geq a_I$ , we have  $a \wedge a_I = a_I \in M_a$ . Let  $x \in I^+$ . From the convexity of  $I$  and  $x \geq a \wedge x \geq 0$ , it follows that  $a \wedge x \in I$ . Then  $a \geq a \wedge x$  implies  $a_I \geq (a \wedge x)_I = a \wedge x$ . Therefore,  $a_I$  is the greatest element of  $M_a$ . ■

A commutative  $l$ -monoid  $A$  is called a *normal  $l$ -monoid* if for each  $a, b \in A$  such that  $a \leq b$  there exists  $x \in A^+$  such that  $a + x = b$ .

**Remark 2.** In the definition of the normal  $l$ -monoid it suffices to require for each  $a, b \in A$  such that  $a \leq b$  the existence  $x \in A$  such that  $a + x = b$ , since then there exists also a positive element  $y \in A$  such that  $a + y = b$ . In fact, if we put  $y = x \vee 0$ , then we get  $a + y = a + x \vee 0 = (a + x) \vee a = b \vee a = b$ .

**Theorem 4.** Let  $A$  be a normal  $l$ -monoid,  $a, b \in A$ ,  $a \leq b$ ,  $S_{ab} = \{x \in A; a + x = b\}$ . Then  $S_{ab}$  is a sublattice of  $A$ .

**Proof.** Let  $a, b, x, y \in A$ ,  $a \leq b$ ,  $a + x = b$ ,  $a + y = b$ . Then  $a + x \vee y = (a + x) \vee (a + y) = b$ ,  $a + x \wedge y = (a + x) \wedge (a + y) = b$ . ■

As a consequence of Theorem 2, we obtain:

**Corollary 2.** *If  $A$  is a normal  $l$ -monoid and  $I, P$  are as in Theorem 2, then  $P$  is a normal  $l$ -monoid.*

**Proof.** Let  $a, b \in P, a \leq b$ . Then there exists  $x \in A^+$  such that  $a + x = b$ . Since  $0 \leq a$ , we get  $0 \leq x \leq a + x = b$ . By Theorem 2 (ii),  $x \in P$ . The rest also follows by Theorem 2. ■

**Lemma 2.** *Each negative element of a normal  $l$ -monoid  $A$  is invertible.*

**Proof.** If  $a \in A, a \leq 0$ , then there exists  $x \in A^+$  such that  $a + x = 0$ . ■

**Theorem 5.** *The set  $I$  of all invertible elements of a normal  $l$ -monoid  $A$  is a convex subset of  $A$ .*

The proof is similar to the proof of Theorem 2 of [10]. ■

By Theorem 3, Corollary 2, Lemma 2, and Theorem 5, we get the following theorem:

**Theorem 6.** *Let  $A$  be a normal  $l$ -monoid. Let  $I$  and  $P$  be as in Theorem 2. Then  $A$  is the direct product of the  $l$ -group  $I$  and the normal  $l$ -emigroup  $P$  with the least element 0 if and only if  $A$  satisfies condition (C). ■*

The following example shows that there exists a commutative  $l$ -monoid  $H$  such that each negative element of  $H$  is invertible, the set  $I$  of all invertible elements of  $H$  is a convex subset of  $H$  and  $H$  satisfies the condition (C), but  $H$  is not a normal  $l$ -monoid.

**Example.** Let  $(B, \leq_1)$  be the interval  $\langle 0, 1 \rangle$  of real line with the natural order. Let  $x \oplus y = y \oplus x = 1$  for each  $x, y \in (0, 1)$  and let  $0 \oplus z = z \oplus 0 = z$  for each  $z \in \langle 0, 1 \rangle$ . Then  $(B, \oplus, \leq_1)$  is a commutative  $l$ -monoid with the least element 0, but not a normal  $l$ -monoid. Let  $(\mathbb{Z}, +, \leq)$  be the additive group of all integers with the natural order. Then the direct product  $H$  of  $(\mathbb{Z}, +, \leq)$  and  $(B, \oplus, \leq_1)$  has the above mentioned properties.

**Theorem 7.** *Each DRL-semigroup  $A$  is a normal  $l$ -monoid and satisfies the condition (C) from Theorem 3. Moreover,  $0 - (0 - a)$  is the greatest element of  $M_a = \{a \wedge x : x \in I^+\}$  for each  $a \in A^+ \setminus I$ , where  $I$  is the set of all invertible elements of  $A$ .*

**Proof.** Let  $A$  be a *DRL*-semigroup. Let  $x, y \in A, x \leq y$ . From Lemma 8 of [16] and Corollary of [16], p. 107, it follows that  $x + (y - x) = y, y - x \geq 0$ . Hence  $A$  is a normal  $l$ -monoid. Let  $a \in A^+ \setminus I$ . Since  $a \geq 0$ , from Lemmas 1 and 3 of [16], we get  $0 - a \leq 0$  and  $0 \leq 0 - (0 - a)$ . By Lemma 1.2 of [5],  $0 - (0 - a)$  is the inverse of  $0 - a$  and hence  $0 - a, 0 - (0 - a) \in I$ . In view of Lemma 13 of [16],  $0 - (0 - a) \leq a$ . Then  $a \wedge (0 - (0 - a)) = 0 - (0 - a)$ . Hence  $0 - (0 - a) \in M_a$ . Let  $x \in I^+$ . From the convexity of  $I$  and  $x \geq a \wedge x \geq 0$  we obtain  $a \wedge x \in I^+$ . By Lemma 1.1 (i) of [5],  $0 - (0 - (a \wedge x)) = a \wedge x$ . Since  $a \geq a \wedge x$ , in view of Lemma 3 of [16], we have  $0 - (a \wedge x) \geq 0 - a$  and  $0 - (0 - a) \geq 0 - (0 - (a \wedge x)) = a \wedge x$ . Hence  $0 - (0 - a)$  is the greatest element of  $M_a$ . ■

The following example shows that there exists a normal  $l$ -monoid satisfying (C) which is not a *DRL*-semigroup.

**Example.** Let  $(\mathbb{R}, +, \leq)$  be the additive group of all real number with the natural order. Let  $(G, \leq)$  be the interval  $\langle 0, 1 \rangle$  of real line with the natural order. Let  $G^\infty = G \cup \{\infty\}$ . We put  $x \leq_1 y$  if  $x \leq y, x, y \in \langle 0, 1 \rangle$  and  $x \leq_1 \infty$  for each  $x \in G^\infty$ . Further, for each  $x, y \in \langle 0, 1 \rangle$  we put  $x \oplus y = y \oplus x = x + y$  if  $x + y \leq 1$  and  $x \oplus y = y \oplus x = \infty$  if  $x + y > 1$ . Let  $x \oplus \infty = \infty \oplus x = \infty$  for each  $x \in G^\infty$ . Then  $(G^\infty, \oplus, \leq_1)$  is a normal  $l$ -monoid with the least element 0, but not a *DRL*-semigroup because  $\infty - 1$  there does not exist in  $G^\infty$ . Then the direct product of  $(\mathbb{R}, +, \leq)$  and  $(G^\infty, \oplus, \leq_1)$  is a normal  $l$ -monoid satisfying (C), but not a *DRL*-semigroup.

Hence Theorems 3 and 6 generalize The Representation Theorem 12 of Kovař in [6].

**Theorem 8.** *Let  $A$  be a finite normal  $l$ -monoid with the least element 0. Then  $A$  is a *DRL*-semigroup and  $a + (b - a) = a \vee b$  for each  $a, b \in A$ .*

**Proof.** Let  $a, b \in A$ . Since  $a \leq a \vee b$ , there exists  $a_1 \in A$  such that  $a + a_1 = a \vee b \geq b$ . Let  $a_1, \dots, a_n$  be all elements of  $A$  such that  $a + a_i \geq b, i = 1, \dots, n$ . Then  $a + a_1 \wedge \dots \wedge a_n = (a + a_1) \wedge \dots \wedge (a + a_n) \geq b$ . Hence  $a_1 \wedge \dots \wedge a_n = b - a$ . Since  $a_1 \geq a_1 \wedge \dots \wedge a_n = b - a$ , we have  $a + (b - a) \leq a + a_1 = a \vee b$ . From  $b - a \geq 0$ , we obtain  $a + (b - a) \geq a$ . Then  $a + (b - a) \geq a \vee b$ . Therefore,  $a \vee b = a + (b - a) = a + (b - a) \vee 0$ . ■

**Theorem 9.** *In any finite normal  $l$ -monoid, 0 is the least element.*

**Proof.** If  $x$  is an element of a finite normal  $l$ -monoid  $A$ , then from Theorem 1 and Lemma 1, it follows that  $x \wedge 0$  is an element of the  $l$ -group  $I$  of all

invertible elements of  $A$ . If  $x \wedge 0 < 0$ , then  $I$  is infinite, a contradiction. Hence  $x \wedge 0 = 0$ . Therefore,  $0$  is the least element of  $A$ . ■

From Theorems 8 and 9, we immediately obtain:

**Theorem 10.** *Any finite normal  $l$ -monoid is a  $DRI$ -semigroup.* ■

Let  $x$  be an element of an  $l$ -group  $G$ . Then  $x^+ = x \vee 0$  is called the positive part of  $x$  and  $x^- = x \wedge 0$  is called the negative part of  $x$ . (See Birkhof [1], p. 293, Fuchs [3], p. 75.) We use the same definition  $x^+$  and  $x^-$  also for an element  $x$  of an  $l$ -monoid  $A$ .

**Remark 3.** For the negative part  $x^-$  of an element  $x$  in an  $l$ -group, formula  $x^- = (-x) \vee 0$  was used in [6]. Hansen defined in [4] the negative part  $x^-$  of an element  $x$  of a  $DRI$ -semigroup by the formula  $x^- = (0 - x) \vee 0$ . The negative part  $x^-$  of  $x$  is a positive element in these cases.

The elements  $x^+$  and  $x^-$  in an  $l$ -monoid  $A$  have analogous properties as in an  $l$ -group.

**Theorem 11.** *Let  $A$  be an  $l$ -monoid,  $x, y \in A$ . Then*

- (i)  $x = x^+$  if and only if  $x \geq 0$ ,
- (ii)  $x = x^-$  if and only if  $x \leq 0$ ,
- (iii)  $(x + y)^+ \leq x^+ + y^+$ ,  $(x + y)^- \geq x^- + y^-$ .

The proof is obvious. ■

**Lemma 3.** *Let  $A$  be a normal  $l$ -monoid. Let  $x \in A$ , and let  $b, c \in A^+$  such that  $x \wedge 0 + b = x$ ,  $x \wedge 0 + c = 0$ . Then  $b = x \vee 0 = x + c$ ,  $b + c = b \vee c$ ,  $b \wedge c = 0$ .*

**Proof.** Let  $x \in A$ ,  $b, c \in A^+$ ,  $x \wedge 0 + b = x$ ,  $x \wedge 0 + c = 0$ . In view of Remark 1, we have  $b = b + 0 = b + x \wedge 0 + c = x + c = x \vee 0 + x \wedge 0 + c = x \vee 0$ . Further we get  $b + c = x \vee 0 + c = b \vee c$ ,  $b \wedge c = (x + c) \wedge c = (x \wedge 0) + c = 0$ . ■

**Theorem 12.** *Let  $A$  be a normal  $l$ -monoid,  $x \in A$ ,  $n \in \mathbb{N}$ . Then  $n(x^-) = (nx)^-$ ,  $n(x^+) = (nx)^+$ .*

**Proof.** Let  $x \in A$ ,  $n \in \mathbb{N}$ . Let  $b, c \in A^+$  such that  $x \wedge 0 + b = x$ ,  $x \wedge 0 + c = 0$ . By Lemma 3,  $b \wedge c = 0$ . From Lemma 1 and Remark 1, it follows that  $nb \wedge nc = 0$ ,  $nb + nc = nb \vee nc$ . Then  $(nx)^- = (nx) \wedge (n0) = [n(x \wedge 0) + nb] \wedge [n(x \wedge 0) + nc] = n(x \wedge 0) + nb \wedge nc = n(x \wedge 0) = n(x^-)$ . In view of Lemma 3, we have  $(nx)^+ = (nx) \vee (n0) = [n(x \wedge 0) + nb] \vee [n(x \wedge 0) + nc] = n(x \wedge 0) + nc \vee nb = n(x \wedge 0) + nc + nb = n(x \wedge 0 + c) + nb = nb = n(x \vee 0) = n(x^+)$ . ■

**Remark 4.** Kovař showed in [7] (p. 16) that for any element  $x$  of an  $l$ -monoid  $A$  the following assertions are valid:

- (i)  $a = a \wedge 0 + a \vee 0 = a \vee 0 + a \wedge 0$ ,
- (ii)  $n(a \wedge 0) = na \wedge (n-1)a \wedge \cdots \wedge a \wedge 0$ , where  $n \in \mathbb{N}$ .

Then, clearly,  $a + a \wedge 0 = a \wedge 0 + a$ ,  $a + a \vee 0 = a \vee 0 + a$ ,  $n(a \vee 0) = na \vee (n-1)a \vee \cdots \vee a \vee 0$  for any  $a \in A$ ,  $n \in \mathbb{N}$ .

**Lemma 4.** Let  $A$  be an  $l$ -monoid,  $a \in A$ , and  $n \in \mathbb{N}$ . Then:

- (i) If  $2a \geq 0$ , then  $a \geq 0$ ;
- (ii) If  $na \geq 0$ , then  $(n+1)a \geq 0$ .

**Proof.** (i): Let  $a \in A$ , and  $2a \geq 0$ . Then  $2(a \wedge 0) = 2a \wedge a \wedge 0 = a \wedge 0$ . Hence  $a = a \wedge 0 + a \vee 0 = 2(a \wedge 0) + a \vee 0 = a \wedge 0 + a$ . Further, we have  $2(a \vee 0) = 2a \vee a \vee 0 = 2a \vee a = a + a \vee 0$ . Then  $a = a \wedge 0 + a \vee 0 = (a + a \vee 0) \wedge (a \vee 0) = 2(a \vee 0) \wedge (a \vee 0) = a \vee 0$ . Therefore,  $a \geq 0$ .

(ii): Let  $a \in A$ ,  $n \in \mathbb{N}$ , and  $na \geq 0$ . Then  $n(a \wedge 0) = na \wedge (n-1)a \wedge \cdots \wedge a \wedge 0 = (n-1)a \wedge \cdots \wedge a \wedge 0 = (n-1)(a \wedge 0)$ . Hence  $(n+1)a = (n+1)(a \vee 0) + a \wedge 0 + n(a \wedge 0) = (n+1)(a \vee 0) + a \wedge 0 + (n-1)(a \wedge 0) = a \vee 0 + n(a \vee 0) + n(a \wedge 0) = a \vee 0 + na \geq 0$ . ■

The following theorem generalizes Lemmas 16 and 17 of paper [16] by Swamy.

**Theorem 13.** Let  $A$  be an  $l$ -monoid,  $a \in A$ , and  $n \in \mathbb{N}$ . Then:

- (i) If  $na \geq 0$ , then  $a \geq 0$ ;
- (ii) If  $na \leq 0$ , then  $a \leq 0$ ;
- (iii) If  $na = 0$ , then  $a = 0$ .

**Proof.** (i): We prove this statement by induction on  $n$ . The statement is valid for  $n = 1$ . Suppose that the statement is valid for all  $k \in \mathbb{N}$ , such that  $k \leq n$ .

Let  $(n + 1)a \geq 0$ . If  $n + 1$  is an even number, then  $n + 1 = 2m$ , where  $m \in \mathbb{N}$ , and  $m \leq n$ . In view of Lemma 4 (i), from  $0 \leq (n + 1)a = 2(ma)$ , we obtain  $0 \leq ma$ . Hence  $0 \leq a$ . If  $n + 1$  is an odd number, then  $n + 2$  is an even number and  $n + 2 = 2s$ , where  $s \in \mathbb{N}$ , and  $s \leq n$ . By Lemma 4 (ii),  $0 \leq (n + 2)a = 2(sa)$ . Then, from Lemma 4 (i), it follows that  $0 \leq sa$ . Therefore,  $0 \leq a$ .

Assertion (ii) can be proved dually.

(iii): It follows from (i) and (ii). ■

**Theorem 14.** *Let  $S$  be an  $l$ -monoid, and  $S = A \times B$ . Then:*

- (i)  $A, B$  are convex sublattices of  $S$ ,
- (ii)  $(x \wedge y)_A = x_A \wedge y_A, (x \wedge y)_B = x_B \wedge y_B$  for each  $x, y \in S$ ,
- (iii)  $(x \vee y)_A = x_A \vee y_A, (x \vee y)_B = x_B \vee y_B$  for each  $x, y \in S$ ,
- (iv) if  $S$  is normal, then  $A$  and  $B$  are normal  $l$ -monoids.

**Proof.** (i): Let  $u, v \in A, z \in S$ , and  $u \leq z \leq v$ . Then  $0 = u_B \leq z_B \leq v_B = 0$ . Thus  $z_B = 0$  and hence  $z = z_A \in A$ . Let  $x, y \in A$ . Since  $x \wedge y \leq x, y$ , we have  $(x \wedge y)_A \leq x_A = x, (x \wedge y)_A \leq y_A = y$ . Thus  $(x \wedge y)_A \leq x \wedge y \leq x$ . From the convexity of  $A$ , we get  $x \wedge y \in A$ . Similarly,  $x \vee y \in A$ . Analogously, we can show that  $B$  is a convex sublattice of  $S$ .

(ii) and (iii) are obvious.

(iv): Let  $x, y \in A, x \leq y$ . Then there exists  $z \in S^+$ , such that  $x + z = y$ . Hence  $x_A + z_A = y_A, x_B + z_B = y_B$ . Since  $x_B = y_B = 0$ , we have  $z_B = 0$ . Therefore,  $z = z_A \in A^+$ . By (i),  $A$  is a lattice. The rest is obvious. Similarly,  $B$  is a normal  $l$ -monoid. ■

If  $A$  is a commutative  $l$ -monoid and  $A = I \times P$ , where  $I$  and  $P$  are as in Theorem 2, then  $A$  is called a decomposable  $l$ -monoid.

**Theorem 15.** *Let  $A$  be a decomposable  $l$ -monoid,  $x \in A$ . Then:*

- (i)  $x^+ = (x_I)^+ + x_P$ ,
- (ii)  $x^- = (x_I)^-$ ,
- (iii)  $x^+ \wedge (-x^-) = 0$ .

**Proof.** (i): In view of Theorem 14, we have  $x^+ = x \vee 0 = x_I \vee 0 + x_P \vee 0 = (x_I)^+ + x_P$ .

The proof of (ii) is analogous.

(iii): Since  $a^+ \wedge (-a)^+ = 0$  for any element  $a$  of an  $l$ -group (see [1], p. 295), in view of (i), (ii), Theorem 1 and 14, we have  $x^+ \wedge (-x^-) = [(x_I)^+ + x_P] \wedge [-(x_I)^-] = (x_I)^+ \wedge [-(x_I)^-] + x_P \wedge 0 = (x_I)^+ \wedge [-(x_I \wedge 0)] = (x_I)^+ \wedge (-x_I)^+ = 0$  for each  $x \in A$ . ■

The absolute value  $|x|$  of an element  $x$  in an  $l$ -group  $G$  is defined by the formula  $|x| = (-x) \vee x$ . We cannot use this definition for decomposable  $l$ -monoids. But we can define the absolute value  $|x|$  of an element  $x$  in a decomposable  $l$ -monoid analogously as in [6]:  $|x| = |x_I| + x_P$ , where  $|x_I|$  is the absolute value of  $x_I$  in the  $l$ -group  $I$ . A such defined absolute value has analogous properties as the absolute value in an  $l$ -group.

**Theorem 16.** *Let  $A$  be a decomposable  $l$ -monoid,  $x, y \in A, n \in N$ . Then*

- (i)  $|x| = 0$  if and only if  $x = 0$ ,
- (ii) if  $x \geq 0$ , then  $|x| = x$ ,
- (iii) if  $x \leq 0$ , then  $|x| = -x$ ,
- (iv)  $|x| = x^+ - x^- = x^+ \vee (-x^-)$ ,
- (v)  $n|x| = |nx|$ ,
- (vi)  $|x| + |y| \geq |x + y|$ ,
- (vii)  $|x| + |y| \geq |x| \vee |y| \geq |x \vee y|, |x| \vee |y| \geq |x \wedge y|$ .

**Proof.** (i): Let  $|x| = 0$ . Then  $|x_I| + x_P = 0$  implies  $x_I = 0, x_P = 0$ . Hence  $x = 0$ . If  $x = 0$ , then  $x_I = 0, x_P = 0$ . Hence  $|x| = 0$ .

(ii): Let  $x \geq 0$ . Then  $x_I \geq 0$ . Thus  $|x| = |x_I| + x_P = x_I + x_P = x$ .

(iii): Let  $x \leq 0$ . Then  $x_I \leq 0, x_P = 0$ . Hence  $|x| = |x_I| + x_P = -x_I = -x$ .

(iv): In view of Theorem 15 and Theorem 7 of [1] (p. 295), we have  $x^+ - (x^-) = (x_I)^+ + x_P - [(x_I)^-] = |x_I| + x_P = |x|$ . By Remark 1 and Theorem 15 (iii),  $x^+ - (x^-) = x^+ \vee (-x^-) + x^+ \wedge (-x^-) = x^+ \vee (-x^-)$ .

(v): By Theorem 8 of [1] (p. 296), we get  $n|x| = n(|x_I| + x_P) = n|x_I| + nx_P = |nx_I| + (nx)_P = |(nx)_I| + (nx)_P = |nx|$ .

(vi): If  $a, b$  are elements of a commutative  $l$ -group, then  $|a| + |b| \geq |a + b|$  (see [3], p. 76). Let  $x, y \in A$ . Then, we have  $|x| + |y| = |x_I| + x_P + |y_I| + y_P = |x_I| + |y_I| + (x + y)_P \geq |x_I + y_I| + (x + y)_P = |(x + y)_I| + (x + y)_P = |x + y|$ .

(vii): Let  $x, y \in A$ . Since  $x_P, y_P \geq 0$ , we get  $x_P + y_P \geq x_P \vee y_P$ . By assertion M) of [3] (p. 76),  $|a| + |b| \geq |a| \vee |b| \geq |a \vee b|$  for any elements  $a, b$  of an  $l$ -group. In view of Theorem 14, we obtain  $|x| + |y| = |x_I| + x_P + |y_I| + y_P \geq |x_I| \vee |y_I| + x_P \vee y_P = (|x_I| + x_P) \vee (|y_I| + y_P) = |x| \vee |y| = [x_I \vee (-x_I)] \vee [y_I \vee (-y_I)] + x_P \vee y_P \geq [x_I \vee y_I] \vee [(-x_I) \wedge (-y_I)] + x_P \vee y_P = |x_I \vee y_I| + x_P \vee y_P = |(x \vee y)_I| + (x \vee y)_P = |x \vee y|$ . Further, we have  $|x| \vee |y| = [x_I \vee (-x_I)] \vee [y_I \vee (-y_I)] + x_P \vee y_P \geq [x_I \wedge y_I] \vee [(-x_I) \vee (-y_I)] + x_P \vee y_P \geq |x_I \wedge y_I| + x_P \wedge y_P = |(x \wedge y)_I| + (x \wedge y)_P = |x \wedge y|$ . ■

Let  $A$  be a decomposable  $l$ -monoid, and  $B \subseteq A$ . Then

$$B^\perp = \{x \in A : |x| \wedge |y| = 0 \text{ for each } y \in B\}$$

is called the *polar* of the set  $B$ .

**Remark 5.** In a decomposable  $l$ -monoid  $A$ , the  $l$ -semigroup  $P$  is the polar of the  $l$ -group  $I$ . In fact, if  $x \in I^\perp$ , then  $0 = |x| \wedge |x_I| = (|x_I| + x_P) \wedge |x_I| = |x_I| + x_P \wedge 0 = |x_I|$ . Thus  $x_I = 0$  and hence  $x = x_P \in P$ . If  $z \in P$ , then  $0 = z \wedge |y| = |z| \wedge |y|$  for each  $y \in I$ . Therefore,  $P = I^\perp$ .

**Theorem 17.** Let  $A$  be a decomposable  $l$ -monoid,  $B \subseteq A$ . Then  $B^\perp$  is an  $l$ -monoid and a convex sublattice of  $A$ .

**Proof.** Let  $x, y \in B^\perp$ . Then  $|x| \wedge |z| = 0, |y| \wedge |z| = 0$  for each  $z \in B$ . By Lemma 1,  $(|x| + |y|) \wedge |z| = 0$  for each  $z \in B$ . Clearly,  $0 \in B$ . In view of Theorem 16 (vi) and (vii), we obtain  $0 = (|x| + |y|) \wedge |z| \geq |x + y| \wedge |z| \geq 0, 0 = (|x| + |y|) \wedge |z| \geq |x \vee y| \wedge |z| \geq 0, 0 = (|x| + |y|) \wedge |z| \geq |x \wedge y| \wedge |z| \geq 0$  and hence  $(|x + y|) \wedge |z| = 0, |x \vee y| \wedge |z| = 0, |x \wedge y| \wedge |z| = 0$  for each  $z \in B$ . Therefore,  $x + y, x \vee y, x \wedge y \in B^\perp$ . Thus  $B^\perp$  is an  $l$ -monoid and a sublattice of  $A$ .

Let  $a, b \in B^\perp, u \in A$ , and  $a \geq u \geq b$ . Then  $a_I \geq u_I \geq b_I, a_P \geq u_P \geq b_P$  and hence  $a_I - b_I \geq u_I - b_I \geq 0$ . By Theorem 16 (ii),  $|a_I - b_I| \geq |u_I - b_I| \geq 0, |a_P| \geq |u_P| \geq 0$ . Since  $a \in B^\perp$ , we get  $0 = |a| \wedge |z| = (|a_I| + a_P) \wedge |z| \geq |a_I| \wedge |z| \geq 0$  for each  $z \in B$  and hence  $a_I \in B^\perp$ . Similarly,  $b_I \in B^\perp$ . Analogously,  $a_P, b_P \in B^\perp$ . In view of Lemma 1 and Theorem 16 (vi), we obtain  $0 = (|a_I| + |b_I|) \wedge |z| = (|a_I| + |-b_I|) \wedge |z| \geq (|a_I - b_I|) \wedge |z| \geq (|u_I - b_I|) \wedge |z| \geq 0$  for each  $z \in B$ . Then  $(|u_I - b_I|) \wedge |z| = 0$  for each  $z \in B$  and hence  $u_I - b_I \in B^\perp$ . Therefore,  $(u_I - b_I) + b_I = u_I \in B^\perp$ . Further, we have  $0 = |a_P| \wedge |z| \geq |u_P| \wedge |z| \geq 0$  for each  $z \in B$ . Thus  $|u_P| \wedge |z| = 0$  for each  $z \in B$  and hence  $u_P \in B^\perp$ . Therefore,  $u_I + u_P = u \in B^\perp$ . ■

**Theorem 18.** *Let  $A$  be a decomposable  $l$ -monoid,  $B, D \subseteq A$ . Then:*

- (i) *if  $B \subseteq D$ , then  $B^\perp \supseteq D^\perp$ ,*
- (ii)  *$B \subseteq B^{\perp\perp}$ ,*
- (iii)  *$B^\perp = B^{\perp\perp\perp}$ .*

The proofs of (i) and (ii) are obvious. Assertion (iii) follows from (i) and (ii). ■

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