

A SCHEME FOR CONGRUENCE SEMIDISTRIBUTIVITY

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Abstract

A diagrammatic statement is developed for the generalized semidistributive law in case of single algebras assuming that their congruences are permutable. Without permutable congruences, a diagrammatic statement is developed for the \wedge -semidistributive law.

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Some attempts show that instead of identities in congruence lattices, certain diagrammatic statements are reasonable to consider, see [2] and [5]. The aim of the present paper is to show that this phenomenon can be extended to lattice Horn sentences as well.

Definition 1. A lattice L is \wedge -semidistributive if it satisfies the following implication for all $\alpha, \beta, \gamma \in L$:

$$\alpha \wedge \beta = \alpha \wedge \gamma \quad \Rightarrow \quad \alpha \wedge (\beta \vee \gamma) = \alpha \wedge \beta.$$

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The \wedge -semidistributive law above is often denoted by SD_\wedge . More general (in fact, weaker) Horn sentences have been investigated in Geyer [4] and Czédli [3]. For $n \geq 2$ put $\mathbf{n} = \{0, 1, \dots, n-1\}$ and let $P_2(\mathbf{n})$ denote the set $\{S : S \subseteq \mathbf{n} \text{ and } |S| \geq 2\}$.

Definition 2. For $\emptyset \neq H \subseteq P_2(\mathbf{n})$ we define the *generalized meet semidistributive law* $SD_\wedge(n, H)$ for lattices as follows: for all $\alpha, \beta_0, \dots, \beta_{n-1}$

$$\alpha \wedge \beta_0 = \alpha \wedge \beta_1 = \dots = \alpha \wedge \beta_{n-1} \quad \Rightarrow \quad \alpha \wedge \beta_0 = \alpha \wedge \bigwedge_{I \in H} \bigvee_{i \in I} \beta_i.$$

As a particular case, when $H = \{S : S \subseteq \mathbf{n} \text{ and } |S| = 2\}$, $SD_\wedge(n, H)$ is denoted by $SD_\wedge(n, 2)$.

Notice that $SD_\wedge(n, 2)$ is the following lattice Horn sentence:

$$\alpha \wedge \beta_0 = \alpha \wedge \beta_1 = \dots = \alpha \wedge \beta_{n-1} \quad \Rightarrow \quad \alpha \wedge \bigwedge_{0 \leq i < j < n} (\beta_i \vee \beta_j) = \alpha \wedge \beta_0,$$

which was originally studied by Geyer [4], and $SD_\wedge(2, 2)$ is the \wedge -semidistributivity law defined in Definition 1. Czédli [3] has noticed that $SD_\wedge(n, 2)$ is strictly weakening in n , i. e. $SD_\wedge(n, 2)$ implies $SD_\wedge(n+1, 2)$ but not conversely.

Our goal is to study $SD_\wedge(n, H)$ in congruence lattices of *single* algebras. Although it is usual to consider lattice identities and Horn sentences in congruence lattices of all algebras of a variety, this is not our case. The reason is that, for an arbitrary variety \mathcal{V} , if $SD_\wedge(n, H)$ holds in $\{\text{Con}(A) : A \in \mathcal{V}\}$ then so does SD_\wedge . (This was proved by Czédli [3] and an anonymous referee of [3] who pointed out that both Kearnes and Szendrei [6] and Lipparini [7] contain implicitly the statement that if a lattice Horn sentence λ can be characterized by a weak Mal'cev condition and, for each nontrivial module variety \mathcal{M} , λ fails in $\text{Con } M$ for some $M \in \mathcal{M}$, then for an arbitrary variety \mathcal{V} , if λ holds in $\{\text{Con}(A) : A \in \mathcal{V}\}$, then so does SD_\wedge , cf. the last paragraph in [3].) In particular, for any variety \mathcal{V} and any $n \geq 2$, $SD_\wedge(n, 2)$ and SD_\wedge are equivalent for the class $\{\text{Con}(A) : A \in \mathcal{V}\}$. Hence $SD_\wedge(n, 2)$ does not deserve a separate study for *varieties*.

First, we consider congruence permutable algebras.

Theorem 1. *Let A be a congruence permutable algebra. Then $\text{Con}(A)$ satisfies $SD_\wedge(n, 2)$ if and only if A satisfies the scheme depicted in Figure 1 for $\alpha, \beta_0, \dots, \beta_{n-1} \in \text{Con}(A)$ and $x_0, \dots, x_k, y, z \in A$, where $k = \frac{n(n-1)}{2} - 1$ and δ stands for $\beta_0 \cap \beta_1 \cap \dots \cap \beta_{n-1}$.*

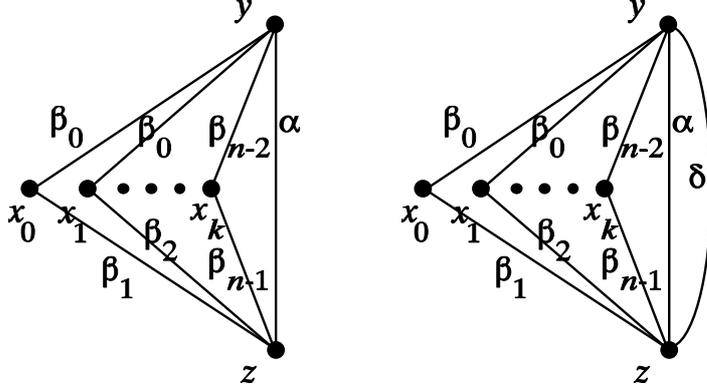


Figure 1

Proof. Suppose $SD_{\wedge}(n, 2)$ holds. Using the premise of $SD_{\wedge}(n, 2)$ we obtain

$$\alpha \cap \beta_0 = (\alpha \cap \beta_0) \cap \cdots \cap (\alpha \cap \beta_{n-1}) = \alpha \cap (\beta_0 \cap \cdots \cap \beta_{n-1}) \subseteq \delta,$$

whence $\text{Con}(A)$ satisfies the Horn sentence

$$\alpha \cap \beta_0 = \cdots = \alpha \cap \beta_{n-1} \quad \Rightarrow \quad \alpha \cap \bigcap_{0 \leq i < j < n} (\beta_i \vee \beta_j) \leq \delta.$$

This implies the scheme, for the situation on the left hand side in Figure 1 then gives

$$(y, z) \in \alpha \cap \bigcap_{0 \leq i < j < n} (\beta_i \circ \beta_j) \subseteq \alpha \cap \bigcap_{0 \leq i < j < n} (\beta_i \vee \beta_j) \subseteq \delta.$$

To show the converse, suppose that the scheme given by Figure 1 holds, $\alpha, \beta_0, \dots, \beta_{n-1} \in \text{Con}(A)$ with $\alpha \cap \beta_0 = \cdots = \alpha \cap \beta_{n-1}$, and suppose that $(y, z) \in \alpha \cap \bigcap_{0 \leq i < j < n} (\beta_i \vee \beta_j)$. Since $\beta_i \vee \beta_j = \beta_i \circ \beta_j$ by congruence permutability, there exist x_0, x_1, \dots, x_k of A such that for each j ($1 \leq j \leq k$) there exist u, v such that $(z, x_j) \in \beta_u$ and $(x_j, y) \in \beta_v$ (according to the left hand side of Figure 1). Then the scheme applies and we conclude $(y, z) \in \delta$. Since $\delta \subseteq \beta_0$, $(y, z) \in \beta_0$. Hence $(y, z) \in \alpha \cap \beta_0$. This proves the " \leq " part of $SD_{\wedge}(n, 2)$. The reverse part is simpler and does not need the scheme: $\alpha \supseteq \alpha \cap \beta_0$ and $\beta_i \vee \beta_j \supseteq \beta_i \supseteq \alpha \cap \beta_i = \alpha \cap \beta_0$ clearly give

$$\alpha \cap \bigcap_{0 \leq i < j < n} (\beta_i \vee \beta_j) \supseteq \alpha \cap \beta_0,$$

proving the theorem.

In the particular case when $n = 2$, we trivially conclude the following assertion:

Theorem 2. *Let A be a congruence permutable algebra. Then $\text{Con}(A)$ is \wedge -semidistributive if and only if A satisfies the so-called triangular scheme in Figure 2 for any $\alpha, \beta, \gamma \in \text{Con}(A)$ and $x, y, z \in A$.*

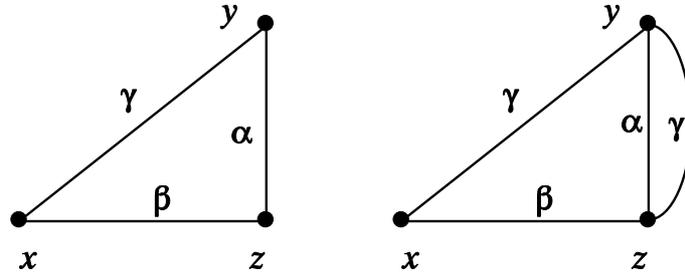


Figure 2

Proof. If $\text{Con}(A)$ is \wedge -semidistributive, then the premise of the Triangular Scheme gives $(y, z) \in \beta \cap \gamma \subseteq \gamma$ by Theorem 1. Conversely, if the Triangular Scheme holds for A then its premise, after interchanging the role of β and γ , implies $(y, z) \in \beta \cap \gamma$, so $SD_{\wedge}(2, 2)$, which is the usual \wedge -semidistributivity, holds in $\text{Con}(A)$ by Theorem 1.

One may observe that this scheme in Theorem 2 is the same as that in [2] characterising congruence distributivity in the congruence permutable case. This implies that: *in presence of congruence permutability, congruence \wedge -semidistributivity is equivalent to congruence distributivity.*

This follows also from another direction. Let A be congruence permutable and satisfying SD_{\wedge} . In this case A is congruence distributive, since otherwise its congruence lattice, being modular due to congruence permutability, contains M_3 but with the choice α, β, γ on Figure 3 we see that SD_{\wedge} fails.

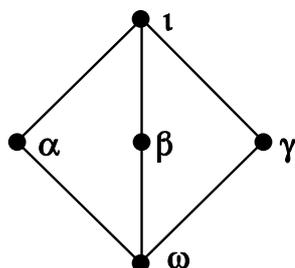


Figure 3

Remark. For $SD_{\wedge}(n, H)$, a similar scheme can be derived as in Theorem 1.

Without congruence permutability, for the case $SD_{\wedge}(2, 2) = SD_{\wedge}$, the following theorem can be stated:

Theorem 3. *Let A be an algebra. The congruence lattice $\text{Con}(A)$ is \wedge -semidistributive if and only if for each n , A satisfies the scheme in Figure 4 for $\alpha, \beta, \gamma \in \text{Con}(A)$ and $x, y, z \in A$, where $\Lambda_0 = \beta$ and $\Lambda_{m+1} = \Lambda_m \circ \gamma \circ \beta$.*

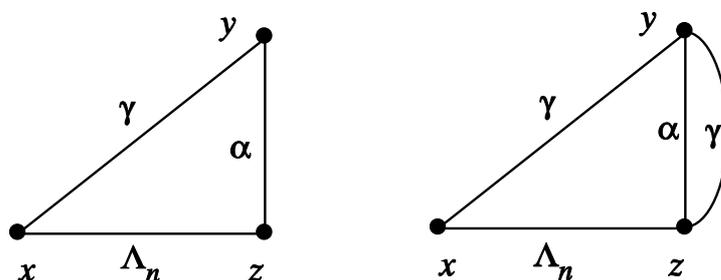


Figure 4

Proof. Suppose that $\text{Con}(A)$ is \wedge -semidistributive and $\alpha, \beta, \gamma \in \text{Con}(A)$ with $\alpha \cap \beta = \alpha \cap \gamma$. Let $x, y, z \in A$ and let $(x, y) \in \gamma$, $(y, z) \in \alpha$ and $(x, z) \in \Lambda_n$. Then

$$(y, z) \in \alpha \cap (\Lambda_n \circ \gamma) \subseteq \alpha \cap (\beta \vee \gamma) = \alpha \cap \beta = \alpha \cap \gamma$$

due to the \wedge -semidistributivity. Thus $(y, z) \in \gamma$, proving the validity of the scheme.

Conversely, let A satisfy the scheme for each $n \in \mathbf{N}_0$, let $\alpha, \beta, \gamma \in \text{Con}(A)$ with $\alpha \cap \beta = \alpha \cap \gamma$. Suppose $(z, y) \in \alpha \cap (\beta \vee \gamma)$. Then there exists $n \in \mathbf{N}_0$ such that $(z, y) \in \alpha \cap (\Lambda_n \circ \gamma)$ and hence $(x, y) \in \gamma$ and $(y, z) \in \alpha$ and $(x, z) \in \Lambda_n$ for some $x \in A$. Due to the scheme, we conclude $(x, y) \in \alpha \cap \gamma$, i.e. $\alpha \cap (\beta \vee \gamma) \subseteq \alpha \cap \gamma \subseteq \alpha \cap \beta$. The converse inclusion is trivial, thus $\text{Con}(A)$ is \wedge -semidistributive.

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