Discussiones Mathematicae
General Algebra and Applications 31 (2011) 5–23

BI-IDEALS IN *k*-REGULAR AND INTRA *k*-REGULAR SEMIRINGS

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Abstract

Here we introduce the k-bi-ideals in semirings and the intra k-regular semirings. An intra k-regular semiring S is a semiring whose additive reduct is a semilattice and for each $a \in S$ there exists $x \in S$ such that $a + xa^2x = xa^2x$. Also it is a semiring in which every k-ideal is semiprime. Our aim in this article is to characterize both the k-regular semirings and intra k-regular semirings using of k-bi-ideals.

Keywords and phrases: k-bi-ideals, k-ideals, semiprimary subsets, k-regular semirings, intra k-regular semirings.

2000 Mathematics Subject Classification: 16Y60.

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1. Introduction

In 1952, R.A. Good and D.R. Hughes [4] first defined the notion of bi-ideals of a semigroup. It is also a special case of the (m,n)-ideal introduced by S. Lajos [11]. Lajos characterized both regular and intra regular semigroups by bi-ideals [12] and by generalized bi-ideals [13]. Different classes of semigroups has been characterized using bi-ideals by many authors in [3, 6, 7, 8, 9, 14, 15, 16, 21].

In this article we introduce the notion of k-bi-ideals in a semiring and characterize the k-regular semirings by k-bi-ideals. Bourne [2] introduced the k-regular semirings as a generalization of regular rings. Later these semirings has been studied by Sen, Weinert, Bhuniya, Adhikari [1, 18, 19, 20]. For any semigroup F, the set $P_f(F)$ of all finite subsets of F is a semiring whose additive reduct is a semilattice, where addition and multiplication are defined by the set union and usual product of subsets of a semigroup respectively. The semiring $P_f(F)$ is a k-regular semiring if and only if F is regular [20]. Here we show that B is a k-bi-ideal of the semiring $P_f(F)$ if and only if $B = P_f(A)$ for some bi-ideal A of F. Thus it is of interest to characterize the k-regular semirings using k-bi-ideals.

Some elementary results together with prerequisites have been discussed in Section 2.

Section 3 is devoted to characterize the k-regular semirings by k-bi-ideals.

In section 4, we introduce intra k-regular semirings and characterize these semirings by k-bi-ideals. The semiring $P_f(F)$ is intra k-regular if and only if F is an intra regular semigroup. Also a semiring S is intra k-regular if and only if every k-ideal of S is semiprime. Several equivalent characterizations for the semirings which are both k-regular and intra k-regular has been given here in terms of k-bi-ideals.

2. Preliminaries

A semiring $(S, +, \cdot)$ is an algebra with two binary operations $+$ and $\cdot$ such that both the *additive reduct* $(S, +)$ and the *multiplicative reduct* $(S, \cdot)$ are semigroups and such that the following distributive laws hold:

$$x(y+z) = xy + xz \text{ and } (x+y)z = xz + yz.$$

A band is a semigroup in which every element is an idempotent. A commutative band is called a semilattice. Throughout this paper, unless otherwise stated, S is always a semiring whose additive reduct is a semilattice.

A nonempty subset L of a semiring S is called a *left ideal* of S if $L+L \subseteq L$ and $SL \subseteq L$. The *right ideals* are defined dually. A subset I of S is called an *ideal* of S if it is both a left and a right ideal of S. A nonempty subset A is called an *interior ideal* of S if $A + A \subseteq A$ and $SAS \subseteq A$. Henriksen [5] defined an *ideal (left, right)* I of a semiring S to be a k-ideal (left, right) if for $a, x \in S$,

$$a, a + x \in I \;\Rightarrow\; x \in I.$$

We define interior k-ideal similarly.

Later on the notion of k-subset of a semiring evolved. A nonempty subset A of S is called a k-subset of S if for $x \in S$,

$$a \in A,\ x + a \in A \text{ implies that } x \in A.$$

Let A be a non empty subset of S. Since the intersection of any family of k-subsets of S is a k-subset (provided the intersection is nonempty), the smallest k-subset of S which contains A exists. This smallest subset of S can be thought as the k-subset generated by A. This k-subset of S is called the k-closure of A in S and will be denoted by $\overline{A}$. For a nonempty subset A of S,

$$\overline{A} = \{x \in S \mid \exists\ a, b \in A \text{ such that } x + a = b\}.$$

If A and B be two subsets of S such that $A \subseteq B$ then it follows that $\overline{A} \subseteq \overline{B}$. A nonempty subset A of S is a k-subset of S if and only if $\overline{A} = A$. Thus an ideal(left, right) K of S is a k-ideal(left, right) if and only if $\overline{K} = K$.

Definition 2.1. A subsemiring A is called a k-bi-ideal of S if $ASA \subseteq A$ and $\overline{A} = A$.

A nonempty subset A is called a generalized k-bi-ideal of S if $A + A \subseteq A$, $ASA \subseteq A$ and $\overline{A} = A$.

Example 2.2. we consider the distributive lattice $(\mathbb{N}, \leq)$ where $\mathbb{N}$ is the set of all natural numbers and $\leq$ is the natural partial order on $\mathbb{N}$. Then $S = M_2(\mathbb{N})$, the set of all 2×2 matrices over $\mathbb{N}$ is a semiring under the addition and multiplication defined in the following way:

For

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \in S,$$

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} = \begin{pmatrix} x_1 \vee y_1 & x_2 \vee y_2 \\ x_3 \vee y_3 & x_4 \vee y_4 \end{pmatrix}$$

and

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} = \begin{pmatrix} (x_1 \wedge y_1) \vee (x_2 \wedge y_3) & (x_1 \wedge y_2) \vee (x_2 \wedge y_4) \\ (x_3 \wedge y_1) \vee (x_4 \wedge y_3) & (x_3 \wedge y_2) \vee (x_4 \wedge y_4) \end{pmatrix}.$$

Let $B = \{\begin{pmatrix} x & y \\ z & t \end{pmatrix} \in M_2(\mathbb{N}) \mid x,\ y \leq a,\ z \leq b,\ t \leq c,$ where $a, b, c \in \mathbb{N}$ such that $a \leq b \leq c\}$.

Then for all $X, Y \in B$ we have $X + Y, XY \in B$. Also $BSB \subseteq B$. Thus B is a bi-ideal of S.

Let $\begin{pmatrix} u & v \\ w & z \end{pmatrix} \in B$ and $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in S$ be such that

$$\begin{pmatrix} u & v \\ w & z \end{pmatrix} + \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in B.$$

Then we have

$$x_1 \vee u \leq a,$$

$$x_2 \vee v \leq a,$$

$$x_3 \vee w \leq b,$$

$$\text{and} \quad x_4 \vee z \leq c$$

and so $x_1, x_2 \leq a, x_3 \leq b$ and $x_4 \leq c$, whence $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in B$. Thus B is a k-bi-ideal of S.

Let F be a semigroup and $P_f(F)$ be the set of all finite subsets of F. Define addition and multiplication on $P_f(F)$ by:

$$U + V = U \cup V$$

$$\text{and} \quad UV = \{ab \mid a \in U, b \in V\}, \quad \text{for all} \quad U, V \in P_f(F).$$

Then $(P_f(F), +, \cdot)$ is a semiring whose additive reduct is a semilattice. In the following theorem we investigate the relation between the k-bi-ideals of the semiring $P_f(F)$ and the bi-ideals of the semigroup F.

Theorem 2.3. *Let F be a semigroup. Then B is a k-bi-ideal of $P_f(F)$ if and only if $B = P_f(A)$ for some bi-ideal A of F.*

Proof. Let A be a bi-ideal of F and $B = P_f(A)$. Then B is a bi-ideal of $P_f(F)$. Now let $U \in P_f(F)$ and $V_1, V_2 \in B$ such that $U + V_1 = V_2$. Then we have

$$U \cup V_1 = V_2 \Rightarrow U \subseteq V_2$$

$$\Rightarrow U \subseteq A \text{ and } U \text{ is finite}$$

$$\Rightarrow U \in B.$$

Thus B is a k-bi-ideal of $P_f(F)$.

Conversely, let B be a k-bi-ideal of $S = P_f(F)$. We consider $A = \cup_{U \in B} U$. Then $A \subseteq F$ and $B \subseteq P_f(A)$. Let $Y = \{y_1, y_2, y_3, ..., y_n\} \in P_f(A)$. Then for each $i = 1, 2, \cdots, n$, $y_i \in U$ for some $U \in B$.

Then we have

$$\{y_i\} + U = U \;\Rightarrow\; \{y_i\} \in B, \text{ since } B \text{ is a } k\text{-set of } S,$$

and so $Y = \{y_1\} + \{y_2\} + \{y_3\} + ... + \{y_n\} \in B$. Hence $B = P_f(A)$. Also for $a, b \in A$, we have

$$\begin{aligned} \{a\}, \{b\} \in B &\Rightarrow \{a\}\{b\} \in B \\ &\Rightarrow \{ab\} \in B = P_f(A) \\ &\Rightarrow ab \in A, \end{aligned}$$

whence A is a subsemigroup of F. Again for $s \in F$ we have

$$\{asb\} = \{a\}\{s\}\{b\} \in BSB \subseteq B \Rightarrow asb \in A.$$

Thus A is a bi-ideal of F such that $B = P_f(A)$. ■

Let $a \in S$. We denote $B[a] = \{\Sigma_{i=1}^{n} x_i | x_i \in \{a\} \cup \{a^2\} \cup aSa\}$. Then $B[a]$ is a subsemiring of S. Also for any $s \in S$ and $b,\ c \in \{a\} \cup \{a^2\} \cup aSa$ we have $bsc \in aSa$ which implies that $B[a]SB[a] \subseteq B[a]$ and so $B[a]$ is a bi-ideal of S. In the following lemma we describe the principal k-bi-ideal of S, which is generated by a.

Lemma 2.4. *Let S be a semiring and $a \in S$. Then the principal k-bi-ideal of S generated by a is given by:*

$$B_k(a) = \{u \in S \mid u + a + a^2 + asa = a + a^2 + asa, \textit{ for some } s \in S\}.$$

Proof. Let $u, v \in B_k(a)$. Then there exist $s, t \in S$ such that $u + a + a^2 + asa = a + a^2 + asa$ and $v + a + a^2 + ata = a + a^2 + ata$. Then we have

$$u + a + a^2 + asa = a + a^2 + asa$$

$$\Rightarrow uv + (a + a^2 + asa)v = (a + a^2 + asa)v$$

$$\Rightarrow uv + (a + a^2 + asa)(v + a + a^2 + ata) = (a + a^2 + asa)(v + a + a^2 + ata)$$

$$\Rightarrow uv + (a + a^2 + asa)(a + a^2 + ata) = (a + a^2 + asa)(a + a^2 + ata)$$

$$\Rightarrow uv + a + a^2 + axa = a + a^2 + axa, \text{ since } (S, +) \text{ is a semilattice,}$$

where $x = a + a^2 + at + sa + a^2t + sa^2 + sa^2t$ and so $uv \in B_k(a)$. Also $u + v \in B_k(a)$. Thus $B_k(a)$ is a subsemiring of S.

Similarly $usv \in B_k(a)$ for all $u, v \in B_k(a)$ and $s \in S$. Thus $B_k(a)$ is a bi-ideal of S. In fact, $B_k(a) = \overline{B[a]}$, the k-closure of $B[a]$. Hence $B_k(a)$ is a k-bi-ideal of S.

Let B be a k-bi-ideal of S such that $a \in B$. Let $u \in B_k(a)$. Then there exists $s \in S$ such that $u + a + a^2 + asa = a + a^2 + asa$. Now $a \in B$ implies that $a + a^2 + asa \in B$ and so $u \in B$. Hence $B_k(a) \subseteq B$. Thus $B_k(a)$ is the least k-bi-ideal of S, which contains a. ■

The following lemma can be proved similarly.

Lemma 2.5. *Let S be a semiring and $a \in S$.*

1. *The principal left k-ideal of S generated by a is given by $L_k(a) = \{u \in S \mid u + a + sa = a + sa, \text{ for some } s \in S\}$.*
2. *The principal k-ideal of S generated by a is given by $J_k(a) = \{u \in S \mid u + a + sa + as + sas = a + sa + as + sas, \text{ for some } s \in S\}$.*

The principal right k-ideal $R_k(a)$ of S generated by a is given dually.

3. Bi-ideals in k-regular semirings

A semigroup S is called a regular semigroup if for each $a \in S$ there exists $x \in S$ such that $a = axa$. In [17], Von Neumann defined a ring R to be regular if the multiplicative reduct $(R, \cdot)$ is a regular semigroup. Bourne [2] defined a semiring S to be regular if for each $a \in S$ there exist $x, y \in S$ such that $a + axa = aya$. If a semiring S happens to be a ring then the Von Neumann regularity and the Bourne regularity are equivalent.

This is not true in a semiring in general (for counter example we refer [18]). Adhikari, Sen and Weinert [1] renamed the Bourne regularity of a semiring as k-regularity to distinguish this from the notion of Von Neumann regularity.

Definition 3.1. A semiring S is called a k-regular semiring if for each $a \in S$ there exist $x, y \in S$ such that $a + axa = aya$.

Since $(S, +)$ is a semilattice, we have

$$\begin{aligned} a + axa = aya \ &\Rightarrow a + axa + (axa + aya) = aya + (axa + aya) \\ &\Rightarrow a + a(x + y)a = a(x + y)a. \end{aligned}$$

Thus, a semiring S is k-regular if and only if for all $a \in S$ there exists $x \in S$ such that

$$a + axa = axa.$$

Let S be a k-regular semiring and $a \in S$. Then there exists $x \in S$ such that $a + axa = axa$. Then we have

$$\begin{aligned} a + axa = axa \ &\Rightarrow \ a + ax(a + axa) = ax(a + axa) \\ &\Rightarrow \ a + axaxa = axaxa. \end{aligned}$$

Thus, a semiring S is k-regular if and only if for all $a \in S$ there exists $x \in S$ such that

$$a + axaxa = axaxa. \tag{3.1}$$

For examples and properties of k-regular semirings we refer [1, 18, 19, 20].

Now we give several equivalent characterizations of k-regularity in terms of k-bi-ideals.

Theorem 3.2. *For a semiring S the following conditions are equivalent:*

1. *S is k-regular;*
2. *$B = \overline{BSB}$ for every k-bi-ideal B of S;*
3. *$G = \overline{GSG}$ for every generalized k-bi-ideal G of S.*

Proof. It is clear that (3) $\Rightarrow$ (2); since each k-bi-ideal is a generalized k-bi-ideal. Hence we are to prove (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) only.

(1) $\Rightarrow$ (3): Let G be a generalized k-bi-ideal of S. Then $GSG \subseteq G$ implies that $\overline{GSG} \subseteq G$. Let $a \in G$. Since S is k-regular, there exists $x \in S$ such that $a + axa = axa$. Now $axa \in GSG$ implies that $a \in \overline{GSG}$ and so $G \subseteq \overline{GSG}$. Thus $G = \overline{GSG}$.

(2) $\Rightarrow$ (1): Let $a \in S$. Consider the k-bi-ideal $B_k(a)$. Then $a \in B_k(a) = \overline{B_k(a)SB_k(a)}$ implies that there exist $b_1,\ b_2,\ b_3,\ b_4 \in B_k(a)$ and $s_1,\ s_2 \in S$ such that

$$a + b_1s_1b_2 = b_3s_2b_4 \Rightarrow a + (b_1 + b_2 + b_3 + b_4)(s_1 + s_2)(b_1 + b_2 + b_3 + b_4)$$

$$= (b_1 + b_2 + b_3 + b_4)(s_1 + s_2)(b_1 + b_2 + b_3 + b_4) \Rightarrow\ a + bsb = bsb,$$

where $b = b_1 + b_2 + b_3 + b_4 \in B_k(a)$ and $s = s_1 + s_2 \in S$. Hence there exists $s \in S$ such that $b + a + a^2 + asa = a + a^2 + asa$. Then we have

$$a + bsb = bsb \Rightarrow\ a + (b + a + a^2 + asa)s(b + a + a^2 + asa)$$

$$= (b + a + a^2 + asa)s(b + a + a^2 + asa) \Rightarrow a + (a + a^2 + asa)s(a + a^2 + asa)$$

$$= (a + a^2 + asa)s(a + a^2 + asa) \Rightarrow\ a + ata = ata$$

for some $t \in S$. Thus S is k-regular. ∎

Theorem 3.3. *For a semiring S, the following conditions are equivalent:*

1. *S is k-regular;*
2. *$B \cap J = \overline{BJB}$ for every k-bi-ideal B and every k-ideal J of S;*
3. *$B \cap I = \overline{BIB}$ for every k-bi-ideal B and every interior k-ideal I of S;*
4. *$G \cap J = \overline{GJG}$ for every generalized k-bi-ideal G and every k-ideal J of S;*
5. *$G \cap I = \overline{GIG}$ for every generalized k-bi-ideal G and every interior k-ideal I of S.*

Proof. It is clear that (5) $\Rightarrow$ (4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (3) $\Rightarrow$ (2), since each k-ideal is an interior k-ideal and each k-bi-ideal is generalized k-bi-ideal. Hence we are to prove (1) $\Rightarrow$ (5) and (2) $\Rightarrow$ (1) only.

(1) $\Rightarrow$ (5): Let G be a generalized k-bi-ideal and I be an interior k-ideal of S. Then $GIG \subseteq G \cap I$ and so $\overline{GIG} \subseteq G \cap I$. Let $a \in G \cap I$. Then there exists $x \in S$ such that $a + axaxa = axaxa$, by (3.1). Now $xax \in SIS \subseteq I$ implies that $a(xax)a \in GIG$. This implies that $a \in \overline{GIG}$ and so $G \cap I \subseteq \overline{GIG}$. Thus $G \cap I = \overline{GIG}$.

(2) $\Rightarrow$ (1): Let $a \in S$. Then $a \in B_k(a) \cap J_k(a) = \overline{B_k(a)J_k(a)B_k(a)}$. Therefore there exists $b_1,\ b_2,\ b_3,\ b_4 \in B_k(a)$ and $c_1,\ c_2 \in J_k(a)$ such that $a + b_1c_1b_2 = b_3c_2b_4$. Then similarly to (2) $\Rightarrow$ (1) of Theorem 3.2, it follows that S is k-regular. ∎

Theorem 3.4. *For a semiring S the following conditions are equivalent:*

1. *S is k-regular;*
2. *$B \cap L \subseteq \overline{BL}$ for every k-bi-ideal B and every left k-ideal L of S;*
3. *$G \cap L \subseteq \overline{GL}$ for every generalized k-bi-ideal G and every left k-ideal L of S.*

Proof. It is clear that (3) $\Rightarrow$ (2). Hence we are to prove (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) only.

(1) $\Rightarrow$ (3): Let G be a generalized k-bi-ideal and L be a left k-ideal of S. Let $a \in G \cap L$. Then there exists $x \in S$ such that $a + axa = axa$. Now $a(xa) \in GL$ implies that $a \in \overline{GL}$. Thus $G \cap L \subseteq \overline{GL}$.

(2) $\Rightarrow$ (1): Let $a \in S$. Then $a \in B_k(a) \cap L_k(a) \subseteq \overline{B_k(a)L_k(a)}$. Then there exist $b_1,\ b_2 \in B_k(a)$ and $l_1,\ l_2 \in L_k(a)$ such that $a + b_1l_1 = b_2l_2$ which implies that $a + bl = bl$, where $b = b_1 + b_2 \in B_k(a)$ and $l = l_1 + l_2 \in L_k(a)$. Then there exist $x, y \in S$ such that $b + a + a^2 + axa = a + a^2 + axa$ and $l + a + ya = a + ya$. Then we have

$$a + bl = bl \Rightarrow\ a + (b + a + a^2 + axa)(l + a + ya)$$

$$= (b + a + a^2 + axa)(l + a + ya) \Rightarrow\ a + (a + a^2 + axa)(a + ya)$$

$$= (a + a^2 + axa)(a + ya) \Rightarrow\ a + a^2 + asa = a^2 + asa, \quad \text{where}\ \ s \in S$$

$$\Rightarrow\ a + a(a + a^2 + asa) + asa = a(a + a^2 + asa) + asa \Rightarrow\ a + ata = ata,$$

where $t \in S$ and hence S is k-regular. ■

Theorem 3.5. *For a semiring S, the following conditions are equivalent:*

1. *S is k-regular;*
2. *$R \cap B \cap L \subseteq \overline{RBL}$ for every right k-ideal R, every k-bi-ideal B and every left k-ideal L of S;*
3. *$R \cap G \cap L \subseteq \overline{RGL}$ for every right k-ideal R, every generalized k-bi-ideal G and every left k-ideal L of S.*

Proof. It is clear that (3) ⇒ (2). Hence we are to prove (1) ⇒ (3) and (2) ⇒ (1) only.

(1) ⇒ (3): Let R, G and L be a right k-ideal, generalized k-bi-ideal and left k-ideal of S respectively. Let $a \in R \cap G \cap L$. Then there exists $x \in S$ such that $a + axaxa = axaxa$. Now $xax \in G$ implies that $a(xax)a \in RGL$ whence $a \in \overline{RGL}$. Thus $R \cap G \cap L \subseteq \overline{RGL}$.

(2) ⇒ (1): Let $a \in S$. Then $a \in R_k(a) \cap B_k(a) \cap L_k(a) \subseteq \overline{R_k(a)B_k(a)L_k(a)}$ implies that there exist $r_1,\ r_2 \in R_k(a),\ b_1,\ b_2 \in B_k(a)$ and $l_1,\ l_2 \in L_k(a)$ such that $a + r_1b_1l_1 = r_2b_2l_2$ which implies that $a + rbl = rbl$, where $r = r_1 + r_2 \in R_k(a),\ l = l_1 + l_2 \in L_k(a)$ and $b = b_1 + b_2 \in B_k(a)$. Then there exist $x, y, z \in S$ such that $r + a + ax = a + ax,\ b + a + a^2 + aya = a + a^2 + aya$ and $l + a + za = a + za$. Then we have

$$a + (r + a + ax)(b + a + a^2 + aya)(l + a + za)$$

$$= (r + a + ax)(b + a + a^2 + aya)(l + a + za)$$

$$\Rightarrow\ a + (a + ax)(a + a^2 + aya)(a + za)$$

$$= (a + ax)(a + a^2 + aya)(a + za) \Rightarrow\ a + ata = ata,$$

where $t \in S$ and hence S is k-regular. ■

4. INTRA k-REGULAR SEMIRINGS

In this section we introduce intra-k-regular semirings and characterize these semirings using k-bi-ideals. Recall that a semigroup S is called intra-regular if $a \in Sa^2S$ for all $a \in S$. The intra k-regular semirings may be viewed as the semirings to which class the semiring $P_f(F)$ belongs when F is an intra regular semigroup.

Definition 4.1. A semiring S is called an intra k-regular semiring if for each $a \in S$,

$$a \in \overline{Sa^2S}.$$

It is easy to check that a semiring S is intra k-regular if and only if for each $a \in S$ there exists $x \in S$ such that

$$a + xa^2x = xa^2x. \tag{4.1}$$

Following proposition shows that intra k-regularity is a natural extension of the notion of intra-regularity in semigroups to the semirings whose additive reduct is a semilattice.

Proposition 4.2. *Let F be a semigroup. Then the semiring $P_f(F)$ is intra k-regular if and only if F is an intra-regular semigroup.*

Proof. Suppose F is an intra-regular semigroup. Let $A \in P_f(F)$. Then, for each $a \in A$, there exist $x_a, y_a \in F$ such that $a = x_a a^2 y_a$. Consider $X = \{x_a \mid a \in A\}$, $Y = \{y_a \mid a \in A\} \in P_f(F)$. Then we have

$$A \subseteq XA^2Y \;\Rightarrow\; A + XA^2Y = XA^2Y$$

and so $P_f(F)$ is intra k-regular.

Conversely, let $P_f(F)$ be an intra k-regular semiring. Let $a \in F$. Then $A = \{a\} \in P_f(F)$ and hence there exists $X \in P_f(F)$ such that

$$A + XA^2X = XA^2X \;\Rightarrow\; A \subseteq XA^2X.$$

This implies that there exist $x, y \in X$ such that $a = xa^2y$. Thus F is intra regular. ■

The following theorem shows that the k-ideals and the interior k-ideals coincides in an intra k-regular semiring.

Theorem 4.3. *Let S be an intra k-regular semiring and A be a non-empty subset of S. Then A is a k-ideal of S if and only if A is an interior k-ideal of S.*

Proof. Let A be an interior k-ideal of S. Let $u \in SA$. Then there exists $s \in S$ and $a \in A$ such that

$$u + sa = sa.$$

Since S is intra k-regular, there exists $x \in S$ such that $a + xa^2x = xa^2x$. Then we have

$$u + s(a + xa^2x) = s(a + xa^2x)$$
$$\Rightarrow u + sxa^2x = sxa^2x$$

and so $u \in \overline{SaS} \subseteq \overline{A} = A$. Hence $SA \subseteq A$. Similarly $AS \subseteq A$. Thus A is a k-ideal of S.

Converse follows directly since each k-ideal of S is interior k-ideal. ∎

Definition 4.4. Let S be a semiring. A subset P of S is called semiprime if for all $a \in S$,

$$a^2 \in P \text{ implies that } a \in P.$$

Following theorem characterizes the intra k-regular semirings as the semirings whose every k-ideal is semiprime.

Theorem 4.5. *For a semiring S, the following conditions are equivalent:*

1. *S is intra k-regular;*
2. *every k-ideal of S is semiprime;*
3. *every interior k-ideal of S is semiprime.*

Proof. Every k-ideal is an interior k-ideal, so it is clear that (3) $\Rightarrow$ (2). Thus we have to prove only (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ **(3)** : Let A be an interior k-ideal of S. Suppose $a \in S$ such that $a^2 \in A$. Since S is intra k-regular, there exists $x \in S$ such that $a + xa^2x = xa^2x$. Then $xa^2x \in A$ implies that $a \in \overline{A} = A$. Hence every interior k-ideal of S is semiprime.

(2) $\Rightarrow$ **(1)** : Let $a \in S$. Then $I_k(a^4)$ is semiprime. Hence we have

$$\begin{aligned} a^4 \in I_k(a^4) &\Rightarrow a^2 \in I_k(a^4) \\ &\Rightarrow a \in I_k(a^4). \end{aligned}$$

Then there exists $x \in S$ such that $a + a^4 + xa^4 + a^4x + xa^4x = a^4 + xa^4 + a^4x + xa^4x$ which implies that $a + ya^2y = ya^2y$ for some $y \in S$. Thus S is intra k-regular. ■

In the following theorem we characterize the intra k-regular semirings by their left and right k-ideals.

Theorem 4.6. *Let S be a semiring. Then the following conditions are equivalent:*

1. *S is intra k-regular;*
2. *$L \cap R \subseteq \overline{LR}$ for every left k-ideal L and every right k-ideal R of S.*

Proof. **(1)** $\Rightarrow$ **(2)** : Let R and L be a right and a left k-ideal of S respectively. Let $a \in L \cap R$. Then there exists $x \in S$ such that $a + xa^2x = xa^2x$. Now $(xa)(ax) \in LR$ and so $a \in \overline{LR}$. Thus $L \cap R \subseteq \overline{LR}$.

(2) $\Rightarrow$ **(1)** : Let $a \in S$. Then $a \in L_k(a) \cap R_k(a) \subseteq \overline{L_k(a)R_k(a)}$ implies that there exist $u_1, u_2 \in L_k(a)$ and $v_1, v_2 \in R_k(a)$ such that

$$a + u_1v_1 = u_2v_2 \;\Rightarrow\; a + uv = uv,$$

where $u = u_1 + u_2 \in L_k(a)$ and $v = v_1 + v_2 \in R_k(a)$. Now there exist $s_1, s_2 \in S$ such that $u + a + s_1a = a + s_1a$ and $v + a + as_2 = a + as_2$.

Then we have

$$a + (u + a + s_1 a)(v + a + as_2) = (u + a + s_1 a)(v + a + as_2)$$

$$\Rightarrow a + (a + s_1 a)(a + as_2) = (a + s_1 a)(a + as_2)$$

$$\Rightarrow a + a^2 + s_1 a^2 + a^2 s_2 + s_1 a^2 s_2 = a^2 + s_1 a^2 + a^2 s_2 + s_1 a^2 s_2$$

$$\Rightarrow a + a^2 + sa^2 + a^2 s + sa^2 s = a^2 + sa^2 + a^2 s + sa^2 s, \quad \text{where } s = s_1 + s_2.$$

This can be written as

$$a + a^2 + tat = a^2 + tat, \qquad \text{for some } t \in S$$

$$\Rightarrow a + a(a + a^2 + tat) + tat = a(a + a^2 + tat) + tat$$

$$\Rightarrow a + a(a^2 + tat) + tat = a(a^2 + tat) + tat$$

$$\Rightarrow a + rar = rar, \qquad \text{for some } r \in S$$

$$\Rightarrow a + r(a + a^2 + sa^2 + a^2 s + sa^2 s)r = r(a + a^2 + sa^2 + a^2 s + sa^2 s)r$$

$$\Rightarrow a + r(a^2 + sa^2 + a^2 s + sa^2 s)r = r(a^2 + sa^2 + a^2 s + sa^2 s)r$$

and so $a + xa^2x = xa^2x$ for some $x \in S$. Thus S is an intra k-regular semiring. ■

We left it to check to the readers that a semiring S is both k-regular and intra k-regular if and only if for each $a \in S$ there exists $z \in S$ such that

$$a + aza^2za = aza^2za. \tag{4.2}$$

Now we prove a series of theorems characterizing the semirings which are both k-regular and intra k-regular using k-bi-ideals.

Theorem 4.7. *For a semiring S, the following conditions are equivalent:*

1. *S is both k-regular and intra k-regular;*
2. *$B = \overline{B^2}$ for every k-bi-ideal B of S;*
3. *$G = \overline{G^2}$ for every generalized k-bi-ideal G of S.*

Proof. It is clear that (3) $\Rightarrow$ (2). Hence we are to prove (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) only.

(1) $\Rightarrow$ **(3)** : Let G be a generalized k-bi-ideal of S. Let $a \in G$. Since S is both regular and intra k-regular, there exists $x \in S$ such that

$$a + axa^2xa = axya^2xa.$$

Now $axa \in GSG \subseteq G$ shows that $axa^2xa \in G^2$ and so $a \in \overline{G^2}$. Thus $G \subseteq \overline{G^2}$.

Again let $b \in \overline{G^2}$. Then there exist $g_1,\ g_2,\ g_3,\ g_4 \in G$ such that $b + g_1g_2 = g_3g_4$. Also there exist $x \in S$ such that $b + bxb^2xb = bxb^2xb$. Then we have

$$b + (b + g_1g_2)xb^2x(b + g_1g_2) = (b + g_1g_2)xb^2x(b + g_1g_2)$$
$$\Rightarrow b + g_3g_4xb^2xg_3g_4 = g_3g_4xb^2xg_3g_4.$$

and so $b \in \overline{GSG} \subseteq \overline{G} = G$. Hence $\overline{G^2} \subseteq G$. Thus $G = \overline{G^2}$.

(2) $\Rightarrow$ **(1)** : Let $a \in S$. Then $a \in B_k(a) = \overline{B_k(a)^2}$ implies that there exists $b \in B_k(a)$ such that $a + b^2 = b^2$. Again $b \in B_k(a) = \overline{B_k(a)^2}$ implies that there exist $c \in B_k(a)$ such that $b + c^2 = c^2$. Then $a + c^4 = c^4$. Now there exists $x \in S$ such that $c + a + a^2 + axa = a + a^2 + axa$. Then, since $(S, +)$ is a semilattice, we have

$$a + c^4 = c^4$$
$$\Rightarrow a + (c + a + a^2 + axa)^4 = (c + a + a^2 + axa)^4$$
$$\Rightarrow a + (a + a^2 + axa)^4 = (a + a^2 + axa)^4,$$

which implies that there exist $s, t \in S$ such that

$$a + asa = asa \text{ and } a + ta^2t = ta^2t.$$

Thus S is both k-regular and intra k-regular. ■

Theorem 4.8. *For a semiring S, the following conditions are equivalent:*

1. *S is k-regular and intra k-regular;*
2. *$B \cap C \subseteq \overline{BC}$ for every k-bi-ideals B and C of S;*
3. *$B \cap G \subseteq \overline{BG}$ for every k-bi-ideal B and every generalized k-bi-ideal G of S;*
4. *$G \cap B \subseteq \overline{GB}$ for every generalized k-bi-ideal G and every k-bi-ideal B of S;*
5. *$G \cap H \subseteq \overline{GH}$ for every generalized k-bi-ideals G and H of S.*

Proof. It is clear that (5) ⇒ (4) ⇒ (2)and (5) ⇒ (3) ⇒ (2). So we have to prove (1) ⇒ (5) and (2) ⇒ (1) only.

(1) ⇒ **(5)** : Let G and H be two generalized k-bi-ideals of S and $a \in G \cap H$. Then there exists $x \in S$ such that

$$a + axa^2xa = axa^2xa.$$

Now $(axa)(axa) \in (GSG)(HSH) \subseteq GH$ implies that $a \in \overline{GH}$. Hence $G \cap H \subseteq \overline{GH}$.

(2) ⇒ **(1)** : Let B be a k-bi-ideal of S. Then $B = B \cap B \subseteq \overline{B^2}$. Since B is a k-subsemiring, $\overline{B^2} \subseteq B$. Thus we have $B = \overline{B^2}$, whence S is both k-regular and intra k-regular, by Theorem 4.7. ■

Theorem 4.9. *For a semiring S, the following conditions are equivalent:*

1. *S is k-regular and intra k-regular;*
2. *$B \cap L \subseteq \overline{BLB}$ for every k-bi-ideal B and every left k-ideal L of S;*

3. *$B \cap R \subseteq \overline{BRB}$ for every k-bi-ideal B and every right k-ideal R of S;*
4. *$B \cap C \subseteq \overline{BCB}$ for all k-bi-ideals B and C of S;*
5. *$B \cap G \subseteq \overline{BGB}$ for every k-bi-ideal B and every generalized k-bi-ideal G of S;*
6. *$G \cap L \subseteq \overline{GLG}$ for every generalized k-bi-ideal G and every left k-ideal L of S;*
7. *$G \cap R \subseteq \overline{GRG}$ for every generalized k-bi-ideal G and every right k-ideal R of S;*
8. *$G \cap B \subseteq \overline{GBG}$ for every generalized k-bi-ideal G and every k-bi-ideal B of S;*
9. *$G \cap H \subseteq \overline{GHG}$ for all generalized k-bi-ideals G and H of S.*

Proof. Every left k-ideal and right k-ideal is a k-bi-ideal. Hence it is clear that (9) $\Rightarrow$ (8) $\Rightarrow$ (7) $\Rightarrow$ (3), (9) $\Rightarrow$ (6) $\Rightarrow$ (2) and (9) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3). So it is sufficient to prove (1) $\Rightarrow$ (9), (3) $\Rightarrow$ (1) and (2) $\Rightarrow$ (1) only.

(1) $\Rightarrow$ **(9)** : Let G and H be two generalized k-bi-ideals of S and $a \in G \cap H$. Then there exists $x \in S$ such that $a + axa^2xa = axa^2xa$. Then we have

$$a + axa^2xa = axa^2xa \;\Rightarrow\; a + axa^2x(a + axa^2xa) = axa^2x(a + axa^2xa)$$
$$\Rightarrow\; a + axa^2xaxa^2xa = axa^2xaxa^2xa.$$

Now $axa^2xaxa^2xa = (axa)(axaxa)(axa) \in GHG$ implies that $a \in \overline{GHG}$. Hence $G \cap H \subseteq \overline{GHG}$.

(2) $\Rightarrow$ **(1)** : Let $a \in S$. Consider $B = B_k(a)$ and $L = L_k(a)$. Then $a \in B \cap L \subseteq \overline{BLB}$ implies that there exist $b \in B$ and $l \in L$ such that $a+blb = blb$. Now there exist $s, t \in S$ such that $b+a+a^2+asa = a+a^2+asa$ and $l + a + ta = a + ta$. We denote $u = a + a^2 + asa$ and $v = a + ta$. Then we have

$$a + (b + a + a^2 + asa)(l + v)(b + u)$$
$$= (b + a + a^2 + asa)(l + v)(b + u) \Rightarrow\; a + (a + a^2 + asa)vu$$

$= (a + a^2 + asa)vu \Rightarrow\ a + avu + (a^2 + asa)vu$

$= avu + (a^2 + asa)vu \Rightarrow\ a + (a + avu + (a^2 + asa)vu)vu + (a^2 + asa)vu$

$= (a + avu + (a^2 + asa)vu)vu + (a^2 + asa)vu \Rightarrow a + (avu + (a^2 + asa)vu)vu$

$+(a^2 + asa)vu = (avu + (a^2 + asa)vu)vu + (a^2 + asa)vu \Rightarrow a + axau = axau,$

where $x = avu + avut + (a^2 + asa)vu + (a^2 + asa)vut + (a^2 + asa) + (a^2 + asa)t$. Again this implies that

$$
\begin{aligned}
a + axau = axau\ &\Rightarrow\ a + ax(a + axau)u = ax(a + axau)u \\
&\Rightarrow\ a + a(xa)^2u^2 = a(xa)^2u^2.
\end{aligned}
$$

Proceeding in this way, we get $a + a(xa)^4u^4 = a(xa)^4u^4$. Then substituting the value of u, it can be rearranged in such a way that we have $y \in S$ such that

$$a + aya^2ya = aya^2ya.$$

Thus S is both k-regular and intra k-regular.

(3) $\Rightarrow$ **(1)** : Similar to (2) $\Rightarrow$ (1). ■

REFERENCES

[1] M.R. Adhikari, M.K. Sen and H.J. Weinert, *On k-regular semirings*, Bull.Cal.Math.Soc. **88** (1996), 141–144.

[2] S. Bourne, *The Jacobson radical of a semiring*, Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 163–170. doi:10.1073/pnas.37.3.163

[3] R. Chinram and K. Tinpun, *A note on minimal bi-ideals in ordered* Γ-*semigroups*, International Math. Forum **4** (1) (2009), 1–5.

[4] R.A. Good and D.R. Hughes, *Associated groups for a semigroup*, Bull. Amer. Math. Soc. **58** (1952), 624–625.

[5] M. Henricksen, Ideals in semirings with commutative addition, Amer. Math. Soc. Notices (1958), 321.

[6] A. Iampan, *On bi-ideals of semigroups*, Lobachevskii J. Math **29** (2) (2008), 68–72. doi:10.1134/S1995080208020042

[7] K.M. Kapp, *On bi-ideals and quasi-ideals in semigroups*, Publ. Math. Debrecan **16** (1969), 179–185.

[8] K.M. Kapp, *Bi-ideals in associative rings and semigroups*, Acta. Sci. Math. **33** (1972), 307–314.

[9] N. Kehayopulu, J.S. Ponizovskii and M. Tsingelis, *Bi-ideals in ordered semigroups and ordered groups*, *J. Math. Sci.* **112** (4) (2002), 4353–4354. doi:10.1023/A:1020347003781

[10] Y. Kemprasit, *Quasi-ideals and bi-ideals in semigroups and rings*, Proceedings of the International Conference on Algebra and its Applicatipns (2002), 30–46.

[11] S. Lajos, *On (m, n)-ideals of semigroups*, Abstract of Second Hunger. Math. Congress I (1960), 42–44.

[12] S. Lajos, *On the bi-ideals in semigroups*, Proc. Japan. Acad. **45** (1969), 710–712. doi:10.3792/pja/1195520625

[13] S. Lajos, *On generalized bi-ideals in semigroups*, Coll. Math. Soc. Janos Bolyai, Algebraic Theory of semigroups (G. Polak, Ed.), North-Holland **20** (1979), 335–340.

[14] S. Lajos and F. Szaaz, *Bi-ideals in associative rings*, Acta. Sci. Math. **32** (1971), 185–193.

[15] S. Li and Y. He, *On semigroups whose bi-ideals are strongly prime*, International J. Algebra **1** (6) (2007), 263–268.

[16] M. M. Miccoli, Bi-ideals in regular semigroups and orthogroups, Acta. Math. Hung. **47** (1-2) (1986), 3-6. doi:10.1007/BF01949118

[17] J.V. Neumann, *On regular rings*, Proc. Nat. Acad. Sci. USA **22** (1936), 707–713. doi:10.1073/pnas.22.12.707

[18] M.K. Sen and A.K. Bhuniya, *Completely k-regular semirings*, Bull. Cal. Math. Soc. **97** (2005), 455–466.

[19] M.K. Sen and A.K. Bhuniya, *On Additive Idempotent k-Clifford Semirings*, Southeast Asian Bulletin of Mathematics **32** (2008), 1149–1159.

[20] M.K. Sen and A.K. Bhuniya, On semirings whose additive reduct is semilattice, Communicated.

[21] X.Z. Xu and J.Y. Ma, *A note on minimal bi-ideals in ordered semigroups*, SEA Bull. Math. **27** (2003), 149–154.

Received 6 April 2009
Revised 18 January 2010