

## A NOTE ON GOOD PSEUDO BL-ALGEBRAS

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### Abstract

Pseudo BL-algebras are a noncommutative extension of BL-algebras. In this paper we study good pseudo BL-algebras and consider some classes of these algebras.

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## 1. Introduction

Hájek [9] introduced BL-algebras in 1998. MV-algebras introduced by Chang [1] are contained in the class of BL-algebras. A noncommutative extension of MV-algebras, called pseudo MV-algebras, were introduced by Georgescu and Iorgulescu [6]. A concept of pseudo BL-algebras were firstly introduced by Georgescu and Iorgulescu in 2000 as noncommutative generalization of BL-algebras and pseudo MV-algebras. The basic properties of pseudo BL-algebras were given in [2] and [3]. The pseudo BL-algebras correspond to a pseudo-basic fuzzy logic (see [10] and [11]).

In [8], there were characterized some classes of pseudo BL-algebras. In this paper we give some interesting facts about good pseudo BL-algebras.

We study bipartite good pseudo BL-algebras and some connections between a good pseudo BL-algebra  $A$  and the set  $M(A)$  of elements  $a \in A$  such that  $a = (a^-)^\sim = (a^\sim)^-$ .

## 2. Preliminaries

**Definition 2.1.** An algebra  $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$  of type  $(2, 2, 2, 2, 2, 0, 0)$  is called a *pseudo BL-algebra* if it satisfies the following axioms for any  $a, b, c \in A$ :

- (C1)  $(A, \vee, \wedge, 0, 1)$  is a bounded lattice,
- (C2)  $(A, \odot, 1)$  is a monoid,
- (C3)  $a \odot b \leq c \Leftrightarrow a \leq b \rightarrow c \Leftrightarrow b \leq a \rightsquigarrow c$ ,
- (C4)  $a \wedge b = (a \rightarrow b) \odot a = a \odot (a \rightsquigarrow b)$ ,
- (C5)  $(a \rightarrow b) \vee (b \rightarrow a) = (a \rightsquigarrow b) \vee (b \rightsquigarrow a) = 1$ .

Throughout this paper  $A$  will denote a pseudo BL-algebra. For any  $a \in A$  and  $n = 0, 1, \dots$ , we put  $a^0 = 1$  and  $a^{n+1} = a^n \odot a$ .

**Proposition 2.2** ([2]). *The following properties hold in  $A$  for all  $a, b, c \in A$ :*

- (i)  $a \leq b \Leftrightarrow a \rightarrow b = 1$ ,
- (ii)  $b \leq a \rightarrow b$  and  $b \leq a \rightsquigarrow b$ ,
- (iii)  $a \odot b \leq a$  and  $a \odot b \leq b$ ,
- (iv)  $a \rightarrow (b \rightarrow c) = a \odot b \rightarrow c$  and  $a \rightsquigarrow (b \rightsquigarrow c) = b \odot a \rightsquigarrow c$ ,
- (v)  $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$  and  $(b \vee c) \odot a = (b \odot a) \vee (c \odot a)$ ,
- (vi)  $a \leq b \Leftrightarrow a \odot c \leq b \odot c$ .

We define  $a^- := a \rightarrow 0$  and  $a^\sim := a \rightsquigarrow 0$ . We have

**Proposition 2.3** ([2]). *The following properties hold in  $A$  for all  $a, b, c \in A$ :*

- (i)  $a \leq (a^-)^\sim$  and  $a \leq (a^\sim)^-$ ,
- (ii)  $a^- \odot a = a \odot a^\sim = 0$ ,
- (iii)  $(a \odot b)^- = a \rightarrow b^-$  and  $(a \odot b)^\sim = b \rightsquigarrow a^\sim$ ,
- (iv)  $a \rightsquigarrow b \leq b^\sim \rightarrow a^\sim$  and  $a \rightarrow b \leq b^- \rightsquigarrow a^-$ ,
- (v)  $(a \vee b)^- = a^- \wedge b^-$  and  $(a \vee b)^\sim = a^\sim \wedge b^\sim$ ,
- (vi)  $(a \wedge b)^- = a^- \vee b^-$  and  $(a \wedge b)^\sim = a^\sim \vee b^\sim$ ,
- (vii)  $((a^-)^\sim)^- = a^-$  and  $((a^\sim)^-)^\sim = a^\sim$ ,
- (viii)  $a \rightarrow b^\sim = b \rightsquigarrow a^-$ ,
- (ix)  $a \leq b$  implies  $b^- \leq a^-$  and  $b^\sim \leq a^\sim$ .

**Definition 2.4.** A nonempty subset  $F$  of  $A$  is called a *filter* if it satisfies the following two conditions:

- (F1) If  $a \in F$  and  $a \leq b$ , then  $b \in F$ ,
- (F2) If  $a, b \in F$ , then  $a \odot b \in F$ .

A filter  $F$  is called *proper* if  $F \neq A$ . A proper filter  $F$  is called *maximal* or an *ultrafilter* if  $F$  is not contained in any other proper filter.

Let  $\text{Max } A$  denote the set of all ultrafilters of  $A$ . Denote  $\mathcal{M}(A) = \bigcap \text{Max } A$ . For every filter  $F$  of  $A$  we define sets

$$F_\sim^* = \{a \in A : a \leq x^\sim \text{ for some } x \in F\},$$

$$F_-^* = \{a \in A : a \leq x^- \text{ for some } x \in F\}.$$

**Proposition 2.5** ([8]).

- (a)  $F_\sim^* = \{a \in A : a^- \in F\}$ ,
- (b)  $F_-^* = \{a \in A : a^\sim \in F\}$ .

**Definition 2.6.**  $A$  is called:

- (1) *bipartite* if  $A = F \cup F_{\sim}^* = F \cup F_-^*$  for some ultrafilter  $F$  of  $A$ .
- (2) *strongly bipartite* if  $A = F \cup F_{\sim}^* = F \cup F_-^*$  for all  $F \in \text{Max } A$ .

**Proposition 2.7** ([13]). *Let  $F$  be a proper filter of  $A$ . Then the following conditions are equivalent:*

- (i)  $A = F \cup F_{\sim}^* = F \cup F_-^*$ ,
- (ii)  $F_-^* = F_{\sim}^* = A - F$ ,
- (iii)  $\forall a \in A (a \in F \text{ or } (a^- \in F \text{ and } a_{\sim} \in F))$ .

Let  $S(A) := \{a \vee a_{\sim} : a \in A\} \cup \{a \vee a^- : a \in A\}$ .

**Proposition 2.8** ([8]).  $S(A) = \{a \in A : a \geq a_{\sim} \text{ or } a \geq a^-\}$ .

**Proposition 2.9** ([13]).  $\mathcal{M}(A) \subseteq S(A)$ .

**Proposition 2.10** ([13]). *The following conditions are equivalent:*

- (i)  $A$  is strongly bipartite,
- (ii)  $\forall F \in \text{Max } A \ A = F \cup F_{\sim}^* = F \cup F_-^*$ ,
- (iii)  $\forall F \in \text{Max } A \ S(A) \subseteq F$ ,
- (iv)  $S(A) = \mathcal{M}(A)$ .

In the sequel, we need to recall some facts about pseudo MV-algebras, which are the noncommutative generalizations of MV-algebras.

**Definition 2.11.** A *pseudo MV-algebra* is an algebra  $(M; \oplus, ^-, \sim, 0, 1)$  of type  $(2, 1, 1, 0, 0)$ , which satisfies the following conditions for all  $a, b, c \in M$ :

- (A1)  $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ ,
- (A2)  $a \oplus 0 = 0 \oplus a = a$ ,

- (A3)  $a \oplus 1 = 1 \oplus a = 1$ ,
- (A4)  $1^\sim = 0; 1^- = 0$ ,
- (A5)  $(a^\sim \oplus b^\sim)^- = (a^- \oplus b^-)^\sim$ ,
- (A6)  $a \oplus a^\sim \cdot b = b \oplus b^\sim \cdot a = a \cdot b^- \oplus b = b \cdot a^- \oplus a$ ,
- (A7)  $a \cdot (a^- \oplus b) = (a \oplus b^\sim) \cdot a$ ,
- (A8)  $(a^-)^\sim = a$ .

where  $a \cdot b = (b^- \oplus a^-)^\sim$  and the operation  $\cdot$  has a priority to the operation  $\oplus$ .

Recall that in a pseudo MV-algebra  $M$  the following conditions hold:

- (i)  $(a^\sim)^- = a$ ,
- (ii)  $a \cdot b = (b^\sim \oplus a^\sim)^-$ ,
- (iii)  $0^- = 1$ .

**Definition 2.12.** The nonempty subset  $I \subseteq M$  is called *an ideal* of a pseudo MV-algebra  $M$  if the following conditions hold for all  $a, b \in M$ :

- (I1) If  $a \in I$ ,  $b \in M$  and  $b \leq a$ , then  $b \in I$ ;
- (I2) If  $a, b \in I$ , then  $a \oplus b \in I$ .

**Definition 2.13.** An ideal  $I$  of  $M$  is called *proper* if  $I \neq M$ . A proper ideal  $I$  of  $M$  is called *maximal* if  $I$  is not contained in any other proper ideal of  $M$ .

The set of all maximal ideals of a pseudo MV-algebra  $M$  is denoted by  $\mathbf{Max}M$  and the intersection of all maximal ideals of  $M$  by  $\text{Rad}M$ .

Set  $T(M) = \{a \wedge a^- : a \in M\}$ . We have

**Proposition 2.14** ([5]).  $\text{Rad}M \subseteq T(M)$ .

Let  $I$  be an ideal of a pseudo MV-algebra  $M$ . We set

$$I^- = \{a^- : a \in I\},$$

$$I^\sim = \{a^\sim : a \in I\}.$$

A pseudo MV-algebra  $M$  is called *bipartite* if there exists a maximal ideal  $I$  of  $M$  such that  $M = I \cup I^- = I \cup I^\sim$ . If  $M = I \cup I^- = I \cup I^\sim$  for all  $I \in \mathbf{Max}M$ , then  $M$  is called *strongly bipartite*.

**Proposition 2.15** ([4]). *The following conditions are equivalent for pseudo MV-algebra  $M$  :*

- (i)  $M$  is strongly bipartite,
- (ii) for all  $I \in \mathbf{Max}M$ ,  $M = I \cup I^- = I \cup I^\sim$ ,
- (iii)  $T(M) = \text{Rad}M$ .

### 3. GOOD PSEUDO BL-ALGEBRAS

**Definition 3.1.** A *good* pseudo BL-algebra is a pseudo BL-algebra which satisfies the following identity:

$$(a^-)^\sim = (a^\sim)^-.$$

From this place to the end of this paper,  $A$  will denote a good pseudo BL-algebra.

We consider the subset

$$M(A) = \{a \in A : a = (a^-)^\sim = (a^\sim)^-\}$$

of  $A$ .

For any  $a, b \in A$ , we define

$$a \oplus b := (b^- \odot a^-)^\sim.$$

**Proposition 3.2** ([8]). *The following properties hold in  $A$  :*

- (i)  $0, 1 \in M(A)$ ,
- (ii)  $a^- \in M(A)$  and  $a^\sim \in M(A)$  for any  $a \in A$ ,

- (iii) If  $a, b \in M(A)$ , then  $a \oplus b = b^\sim \rightarrow a = a^- \rightsquigarrow b$ ,
- (iv) If  $a, b \in M(A)$ , then  $a \oplus b^- = b \rightarrow a, a \oplus b^\sim = a^- \rightsquigarrow b^\sim, a^- \oplus b = b^\sim \rightarrow a^-$  and  $a^\sim \oplus b = a \rightsquigarrow b$ .

**Proposition 3.3** ([8]). *The structure  $(M(A), \oplus, \sim, ^-, 0, 1)$  is a pseudo MV-algebra. The order on  $A$  agrees with the one of  $M(A)$ , defined by  $a \leq_{M(A)} b$  iff  $a^\sim \oplus b = 1$ .*

Following [8] we define two maps:  $\varphi_1 : A \rightarrow M(A)$  by  $\varphi_1(a) = a^-$  and  $\varphi_2 : A \rightarrow M(A)$  by  $\varphi_2(a) = a^\sim$ .

Let  $X \subseteq A$ . Write  $X^- = \varphi_1(X)$  and  $X^\sim = \varphi_2(X)$ . It is obvious that

$$X^- = \{a^- : a \in X\},$$

$$X^\sim = \{a^\sim : a \in X\}.$$

Set

$$X_\sim = \{a : a^\sim \in X\},$$

$$X_- = \{a : a^- \in X\}.$$

If  $X \subseteq M(A)$ , then  $\varphi_1^{-1}(X) = X_-$  and  $\varphi_2^{-1}(X) = X_\sim$ .

Following [8] we have

**Proposition 3.4.** *If  $F$  is a filter of  $A$  and  $I$  is an ideal of  $M(A)$ , then:*

- (i)  $F^-$  and  $F^\sim$  are ideals of  $M(A)$ ;
- (ii)  $I_-$  and  $I_\sim$  are filters of  $A$ ;
- (iii) if  $I$  is proper, then  $I_-$  and  $I_\sim$  are proper filters of  $A$ ;
- (iv) if  $F$  is proper, then  $F^-$  and  $F^\sim$  are proper ideals of  $M(A)$ ;

- (v)  $F \subseteq (F^-)_-$  and  $F \subseteq (F^\sim)_\sim$ ;
- (vi) if  $F$  is an ultrafilter, then  $(F^-)_- = (F^\sim)_\sim = F$ ;
- (vii)  $(I_\sim)^\sim = (I_-)^- = I$ ;
- (viii) if  $I$  is maximal, then  $I_-$  and  $I_\sim$  are ultrafilters of  $A$ ;
- (ix) if  $F$  is an ultrafilter, then  $F^-, F^\sim$  are maximal ideals of  $M(A)$ .

**Proposition 3.5.** *Let  $F$  be a filter of  $A$ . Then  $F^- = F_-^*$  and  $F^\sim = F_\sim^*$ .*

**Proof.** Let  $b \in F^-$ . Then  $b = a^-$ , where  $a \in F$ . Obviously,  $b^\sim = (a^-)^\sim$ . Since  $a \leq (a^-)^\sim$ ,  $a \in F$  and  $F$  is a filter, we have  $b^\sim \in F$  and hence  $b \in F_-^*$ .

Conversely, let  $b \in F_-^*$ . Then  $b^\sim \in F$ . So we have  $(b^\sim)^- \in F^-$ . Since  $b \leq (b^\sim)^-$ ,  $(b^\sim)^- \in F^-$  and  $F^-$  is an ideal, we have  $b \in F^-$ .

Similarly we can show that  $F^\sim = F_\sim^*$ . ■

From Propositions 2.5 and 3.5 we obtain

**Corollary 3.6.** *Let  $F$  be a filter of  $A$ . Then  $F^- = F_\sim$  and  $F^\sim = F_-$ .*

**Proposition 3.7.** *Let  $I$  be an ideal of  $M(A)$ . Then  $I^- = M(A) \cap I_\sim$  and  $I^\sim = M(A) \cap I_-$ .*

**Proof.** Let  $b \in I^-$ . Then  $b = a^-$ , where  $a \in I$ . Hence  $b^\sim = (a^-)^\sim$ . Since  $I \subseteq M(A)$  and  $a \in I$ , we have  $b^\sim = a$ . Therefore  $b^\sim \in I$ . Consequently  $b \in I_\sim$ . By Proposition 3.2 (ii),  $b = a^- \in M(A)$ . We obtain that  $b \in M(A) \cap I_\sim$ .

Conversely, let  $b \in M(A) \cap I_\sim$ . Then  $b \in M(A)$  and  $b \in I_\sim$ , i.e.,  $b \in M(A)$  and  $b^\sim \in I$ . Hence  $b = (b^\sim)^- \in I^-$ .

Similarly we can prove that  $I^\sim = M(A) \cap I_-$ . ■

**Proposition 3.8.**  $(\text{Rad}M(A))_- = (\text{Rad}M(A))_\sim = \mathcal{M}(A)$ .

**Proof.** Let us notice that:



$$\begin{aligned}
a \notin \mathcal{M}(A) &\Leftrightarrow a \notin \bigcap_{F \in \text{Max } A} F \Leftrightarrow \exists_{F \in \text{Max } A} a \notin F \Leftrightarrow \\
&\Leftrightarrow \exists_{F \in \text{Max } A} a \notin (F^\sim)^\sim \Leftrightarrow \exists_{F \in \text{Max } A} a^\sim \notin F^\sim \Leftrightarrow \\
&\Leftrightarrow \exists_{I = F^\sim \in \text{Max}(M(A))} a^\sim \notin I \Leftrightarrow a^\sim \notin \text{Rad}M(A) \Leftrightarrow \\
&\Leftrightarrow a \notin (\text{Rad}M(A))^\sim.
\end{aligned}$$

Similarly, we can prove that  $(\text{Rad}M(A))_- = \mathcal{M}(A)$ . ■

**Proposition 3.9.**  $\text{Rad}M(A) = (\mathcal{M}(A))^- = (\mathcal{M}(A))^\sim$ .

**Proof.**  $\text{Rad}M(A)$  is an ideal. By Proposition 3.4 (vii)  $\text{Rad}M(A) = ((\text{Rad}M(A))_-)^\sim$ . From Proposition 3.8 we obtain  $\text{Rad}M(A) = (\mathcal{M}(A))^-$ . Similarly,  $\text{Rad}M(A) = (\mathcal{M}(A))^\sim$ . ■

**Corollary 3.10.**

- (i)  $((\mathcal{M}(A))^-)_- = ((\mathcal{M}(A))^\sim)^\sim = \mathcal{M}(A)$ ,
- (ii)  $((\text{Rad}M(A))_-)^\sim = ((\text{Rad}M(A))^\sim)^\sim = \text{Rad}M(A)$ .

**Proof.** By Propositions 3.8 and 3.9  $((\mathcal{M}(A))^-)_- = (\text{Rad}M(A))_- = \mathcal{M}(A)$  and  $((\mathcal{M}(A))^\sim)^\sim = (\text{Rad}M(A))^\sim = \mathcal{M}(A)$ .

- (ii) Follows from Proposition 3.4 (vii). ■

**Proposition 3.11.** *If  $M(A)$  is bipartite by  $I$ , then  $I_- = I_\sim$ .*

**Proof.** By assumption,  $M(A) = I \cup I^\sim = I \cup I^-$ . Hence  $I^- = M(A) - I = I^\sim$ .

Let  $a \in I_\sim$ , then  $a^\sim \in I$ , which implies  $(a^\sim)^- = (a^-)^\sim \in I^- = I^\sim$ . Hence  $(a^-)^\sim = b^\sim$  for some  $b \in I$ . Since  $b \in M(A)$ , we conclude that  $b = (b^\sim)^- = [(a^-)^\sim]^- = a^-$ . Therefore  $a^- \in I$ . Thus  $a \in I_-$ . We have  $I_\sim \subseteq I_-$ . Similarly we can show that  $I_- \subseteq I_\sim$ . Consequently,  $I_- = I_\sim$ . ■

**Proposition 3.12.** *If  $A$  is bipartite by  $F$ , then  $F^- = F^\sim$ .*

**Proof.** Let  $F$  be an ultrafilter such that  $A = F \cup F_-^* = F \cup F_{\sim}^*$ . By Proposition 2.7,  $F_{\sim}^* = F_-^* = A - F$ . Then from Proposition 3.5 we have  $F^- = F_{\sim}$ . ■

**Theorem 3.13.** *A good pseudo BL-algebra  $A$  is bipartite iff  $M(A)$  is a bipartite pseudo MV-algebra.*

**Proof.** Let  $A$  be bipartite, i.e. there exists an ultrafilter  $F$  such that  $A = F \cup F_-^* = F \cup F_{\sim}^*$ . Then we have  $M(A) = (F \cap M(A)) \cup F^-$ .

By Propositions 3.4 and 3.7,  $F \cap M(A) = (F^-)_- \cap M(A) = (F^-)_{\sim}$ .

So we obtain,  $M(A) = (F^-)_{\sim} \cup F^-$  and by Proposition 3.4 (ix),  $F^-$  is a maximal ideal of  $M(A)$ . From Propositions 3.4, 3.7 and 3.12 we have  $F \cap M(A) = (F_{\sim})_{\sim} \cap M(A) = (F_{\sim})^- = (F^-)^-$ . Then we have  $M(A) = (F \cap M(A)) \cup F^- = (F^-)^- \cup F^-$ , thus  $M(A)$  is bipartite.

Conversely, let  $M(A) = I \cup I_{\sim} = I \cup I^-$ , where  $I$  is a maximal ideal of  $M(A)$ . Now we prove that

$$(1) \quad \forall_{a \in A} [a \in I_- \text{ or } (a_{\sim} \in I_- \text{ and } a^- \in I_-)]$$

holds. Suppose  $a \notin I_- = I_{\sim}$  (see Proposition 3.11) we have  $a_{\sim} \notin I$ . Hence  $a_{\sim} \in I^-$ . Then  $a_{\sim} \in I_-$ , by Proposition 3.7. Thus (1) satisfied.  $I_-$  is proper due to Proposition 3.4 (iii). Applying Proposition 2.7 we get  $A = I_- \cup (I_-)_{\sim}^* = I_- \cup (I_-)_-^*$  where, by Proposition 3.4 (viii),  $I_-$  is an ultrafilter of  $A$ . ■

**Corollary 3.14.**

- (i) *If  $M(A)$  is a strongly bipartite pseudo MV-algebra, then  $I_- = I_{\sim}$  for any maximal ideal  $I$  of  $M(A)$ .*
- (ii) *If  $A$  is strongly bipartite pseudo BL-algebra, then  $F^- = F_{\sim}$  for any ultrafilter  $F$  of  $A$ .*

**Proof.** By Propositions 3.11 and 3.12. ■

**Theorem 3.15.** *A good pseudo BL-algebra  $A$  is strongly bipartite iff  $M(A)$  is a strongly bipartite pseudo MV-algebra.*

**Proof.** Let  $A$  be a strongly bipartite pseudo BL-algebra and suppose that  $M(A)$  is not strongly bipartite. Then there exists a maximal ideal  $I$  of  $M(A)$  such that  $M(A) \neq I \cup I^-$  or  $M(A) \neq I \cup I^\sim$ . Without loss of generality we can assume that there is  $a_0 \in M(A) - (I \cup I^-)$ . Let  $F = I_-$ . By Proposition 3.4 (viii),  $F$  is an ultrafilter of  $A$ . From Proposition 3.4 (viii) and Corollary 3.14 we have  $I = (I_-)^- = (I_-)^\sim$ . Observe that

$$(2) \quad a \in M(A) - I \Rightarrow a^- \notin I_-.$$

Indeed, suppose that  $a \in M(A) - I$  and  $a^- \in I_-$ . Then  $a = (a^-)^\sim \in (I_-)^\sim = I$ , a contradiction. Thus (2) holds. Since  $a_0 \in M(A) - I$ , we conclude that  $a_0^- \notin I_-$ . It is easy to see that  $a_0^\sim \notin I$ . Applying (2) yields  $a_0 = (a_0^\sim)^- \notin I_-$ . Consequently,  $a_0 \notin F$  and  $a_0^- \notin F$ . By Propositions 2.7 and 2.10,  $A$  is not strongly bipartite. A contradiction.

Conversely, let  $M(A)$  be a strongly bipartite pseudo MV-algebra and  $A$  is not bipartite. Then there exists an ultrafilter  $F$  of  $A$  such that

$$\exists_{a \in A} [a \notin F \text{ and } (a^- \notin F \text{ or } a^\sim \notin F)].$$

Suppose that  $b, b^- \notin F$ . Let  $I = F^-$ . Then  $I$  is a maximal ideal of  $M(A)$ , by Proposition 3.4 (ix). From Proposition 3.2 we see that  $b^- \in M(A)$ . Observe that  $b^- \notin I$ . Indeed,  $b \notin F = (F^-)^-$  and hence  $b^- \notin F^- = I$ . Since  $I_- = (F^-)_- = F$  (see Proposition 3.4) and  $b^- \notin F$ , we have  $b^- \notin I_-$  and hence  $b^- \notin M(A) \cap I_- = I^\sim$ . Thus  $b^- \in M(A) - (I \cup I^\sim)$ . Therefore  $M(A) \neq I \cup I^\sim$ . It is a contradiction. ■

**Corollary 3.16.** *Let  $A$  be strongly bipartite. Then:*

- (a)  $T(M(A))_- = (T(M(A)))^\sim = S(A)$ ,
- (b)  $(S(A))^- = (S(A))^\sim = T(M(A))$ .

**Proof.** (a) By Theorem 3.15,  $M(A)$  is a strongly bipartite pseudo MV-algebra and hence  $T(M(A)) = \text{Rad}M(A)$  (see Proposition 2.15). Applying Propositions 3.8 and 2.10 we obtain

$$(T(M(A)))_- = (\text{Rad}M(A))_- = \mathcal{M}(A) = S(A).$$

Similarly,  $(T(M(A)))_{\sim} = S(A)$ .

(b) From the proof of (a) and by Proposition 3.4 (vii) we have

$$(S(A))^{-} = ((\text{Rad}M(A))_{-})^{-} = \text{Rad}M(A) = T(M(A))$$

and similarly,  $(S(A))_{\sim} = T(M(A))$ . ■

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