

NOTE ON ALGEBRAIC INTERIOR SYSTEMS

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Abstract

We get an interrelation between an algebraic closure system and its conjugated interior system. We introduce the concept of algebraic interior system and we get its representation.

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Let A be a non-void set. Denote by $\text{Exp}A$ the power set of A , i.e., the system of all subsets of A . A subsystem $\mathcal{S} \subseteq \text{Exp}A$ is called an *interior system over A* if \mathcal{S} is closed under arbitrary unions, i.e., if for each $\mathcal{N} \subseteq \mathcal{S}$ also $\bigcup \mathcal{N} \subseteq \mathcal{S}$. Of course, we have $\emptyset \in \mathcal{S}$ for every interior system because it is a union of an empty family of subsets of \mathcal{S} . A dual concept is the so-called *closure system over A* ; it is a subsystem $\mathcal{T} \subseteq \text{Exp}A$ which is closed under arbitrary intersections, i.e., for each subsystem $\mathcal{M} \subseteq \mathcal{T}$ we have $\bigcap \mathcal{M} \subseteq \mathcal{T}$.

A mapping $I : \text{Exp}A \rightarrow \text{Exp}A$ is called an *interior operator* on A if for every $X, Y \subseteq A$ we have:

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- (i) $I(X) \subseteq X$,
- (ii) $X \subseteq Y$ implies $I(X) \subseteq I(Y)$,
- (iii) $I(I(X)) = I(X)$.

Dually, a mapping $C : \text{Exp}A \rightarrow \text{Exp}A$ is called a *closure operator* on A if for every $X, Y \subseteq A$ we have:

- (i)' $X \subseteq C(X)$,
- (ii)' $X \subseteq Y$ implies $C(X) \subseteq C(Y)$,
- (iii)' $C(C(X)) = C(X)$.

The connection between an interior system and an interior operator is established by the following well-known

Lemma 1. *Let \mathcal{S} be an interior system and I an interior operator on A .*

- (a) *Define $I_{\mathcal{S}}(X) = \bigcup\{B \in \mathcal{S} : B \subseteq X\}$. Then $I_{\mathcal{S}}$ is an interior operator on A .*
- (b) *Define $\mathcal{S}_I = \{I(X) : X \in \text{Exp}A\}$. Then \mathcal{S}_I is an interior system over A .*

For the proof, see e.g., [3].

We will call $I_{\mathcal{S}}$ or \mathcal{S}_I the **assigned interior operator** or **system** to the interior system \mathcal{S} or the interior operator I .

Of course, analogous results are well known also for closure systems and closure operators. Moreover, there are connections between closure operators (systems) and interior operators (systems) as follows:

Lemma 2. *Let C be a closure operator and I an interior operator on a set A . Define*

$$I_C(X) = A \setminus C(A \setminus X) \quad \text{and} \quad C_I(X) = A \setminus I(A \setminus X).$$

Then I_C or C_I is an interior or a closure operator on A , respectively.

The proof is evident.

The interior operator I_C will be called to be *conjugated* to the closure operator C ; C_I is a *conjugated* closure operator to the interior operator I .

Of course, if \mathcal{S} or \mathcal{T} be an interior or a closure system over A , respectively, then

$$\mathcal{S}_T = \{A \setminus B; B \in \mathcal{T}\}$$

is an interior operator and

$$\mathcal{T}_S = \{A \setminus B; B \in \mathcal{S}\}$$

is a closure system over A . This follows immediately by Lemma 2. The system \mathcal{S}_T is called dual to \mathcal{T} and \mathcal{T}_S is called dual to \mathcal{S} .

A closure system \mathcal{T} is called *algebraic* (see e.g., [2]) if it is closed under unions of chains, i.e., for every chain $\{B_\lambda; \lambda \in \Lambda\}$ (that is $B_{\lambda_1} \subseteq B_{\lambda_2}$ for $\lambda_1 \leq \lambda_2$) of members of \mathcal{T} also $\bigcup\{B_\lambda; \lambda \in \Lambda\} \in \mathcal{T}$. It was proved by G. Birkhoff and O. Frink [1] that for every algebraic closure system \mathcal{T} over A there exists an algebra $\mathcal{A} = (A, F)$ on this support A such that $\mathcal{T} = \text{Sub}\mathcal{A}$ (the system are all subalgebras of \mathcal{A}). Moreover, for the assigned closure operator C_T and $X \subseteq A$ we have $C_T(X) = [X]$, the subalgebra of \mathcal{A} generated by X . We are going to relate the conjugate interior system.

Theorem 1. *Let \mathcal{S} be an interior system over A and I_S its assigned interior operator. Then \mathcal{S} is conjugated to an algebraic closure system if and only if*

$$(*) \quad I_S(A \setminus X) = \bigcap \{I_S(A \setminus Y); Y \subseteq X \text{ and } Y \text{ finite}\}.$$

Proof. Let \mathcal{S} be an interior system over A conjugated to a closure system \mathcal{T}_S . By Lemma 2 we have $C_{I_S}(X) = A \setminus I_S(A \setminus X)$, i.e., $A \setminus \bigcap \{I_S(A \setminus Y); Y \subseteq X \text{ and } Y \text{ finite}\} = \bigcup \{A \setminus I_S(A \setminus Y); Y \subseteq X \text{ and } Y \text{ finite}\} = \bigcup \{C_{T_S}(Y); Y \subseteq X \text{ and } Y \text{ finite}\}.$

Hence, I_S satisfies $(*)$ if and only if C_{T_S} satisfies

$$C_{T_S}(X) = \bigcup \{C_{T_S}(Y); Y \subseteq X \text{ and } Y \text{ finite}\}.$$

By Lemma 6.3 in [2], \mathcal{T}_S is an algebraic closure system. ■

Example. Let $\mathcal{A} = (A, F)$ be an algebra. Denote by $Sub_0\mathcal{A} = \{\emptyset\} \cup Sub\mathcal{A}$. For every $X \subseteq A$ define $I(X) = \bigcup\{B \in Sub_0\mathcal{A}; B \subseteq X\}$. It is an easy exercise to verify that I is an interior operator on A .

The foregoing example motivated us to introduce the following concept.

Let \mathcal{T} be a closure system over a set A and $\mathcal{T}_0 = \{\emptyset\} \cup \mathcal{T}$. The interior system \mathcal{S}_I assigned to the interior operator

$$I(X) = \bigcup\{B \in \mathcal{T}_0; B \subseteq X\}$$

will be called *algebraic*.

Let \mathcal{S} be an interior system over A . A subsystem $\mathcal{B} \subseteq \mathcal{S}$ is called a *base* of \mathcal{S} if for each $D \in \mathcal{S}$ there exist $B_\lambda \in \mathcal{B} (\lambda \in \Lambda)$ such that

$$D = \bigcup\{B_\lambda; \lambda \in \Lambda\}.$$

Hence, an interior system \mathcal{S} is algebraic if it has a base which is a closure system.

Theorem 2. *An interior system \mathcal{S} over A is algebraic if and only if there exists an algebra $\mathcal{A} = (A, F)$ such that $Sub_0\mathcal{A}$ is a base of \mathcal{S} .*

Proof. We need only to find an algebra $\mathcal{A} = (A, F)$ such that $Sub_0\mathcal{A}$ is a base of \mathcal{S} .

Let \mathcal{S} be an algebraic interior system over A and the closure system \mathcal{C} over A is its base. Denote by \mathcal{B} the system of all \bigcup -irreducible members of \mathcal{C} . Of course, \mathcal{B} is also a base of \mathcal{S} . We use the Birkhoff-Frink construction of operations $f_{\bar{a}}$ (see [1]). Let $Y \subseteq A$ be a finite subset, say $Y = \{a_0, \dots, a_{n-1}\}$. Let C be the closure operator on A assigned to the closure system \mathcal{C} and let $a \in C(Y)$. Denote by $\bar{a} = \langle a_0, \dots, a_{n-1} \rangle$ and define

$$f_{\bar{a}}(b_0, \dots, b_{n-1}) = \begin{cases} a & \text{if } \langle b_0, \dots, b_{n-1} \rangle = \bar{a} \\ b_0 & \text{otherwise.} \end{cases}$$

If $B \in \mathcal{B}$ and $B \neq \emptyset$ then $B \in Sub_0\mathcal{A}$, where $\mathcal{A} = (A, F)$ is an algebra on the support A whose all operations are $f_{\bar{a}}$ for all possible choices of \bar{a} and arbitrary integers n . If $B \in \mathcal{B}$ and $B \neq \emptyset$ then it is routine way (see [1]) to show that B is closed under every $f_{\bar{a}}$ and hence $B \in Sub_0\mathcal{A}$. Thus for each subset $X \subseteq A$, $I_S(X)$ is a union of elementals of $Sub_0\mathcal{A}$.

On the other hand, suppose $B \in \text{Sub}_0\mathcal{A}$. If $B = \emptyset$ then $B \in \mathcal{S}$. If $B \neq \emptyset$ then B is a subalgebra of \mathcal{A} . Suppose $H \subseteq B$, H finite, e.g. $H = \{a_0, \dots, a_{n-1}\}$. If $a \in C(H)$ then $a = f_{\bar{a}}(a_0, \dots, a_{n-1})$ for $\bar{a} = \langle a_0, \dots, a_{n-1} \rangle$ and hence $a \in B$ thus $C(H) \subseteq B$. However, $B = \bigcup \{C(H); H \subseteq B \text{ and } H \text{ finite}\} \in \mathcal{S}$ since \mathcal{S} is an interior system and C is its base. We have shown that $\text{Sub}_0\mathcal{A}$ is a base of \mathcal{S} . ■

Corollary 1. *For any algebraic interior system \mathcal{S} there exists an algebraic closure system \mathcal{C} which is a base of \mathcal{S} .*

Corollary 2. *For any algebraic interior system \mathcal{S} over A there exists an algebra $\mathcal{A} = (A, F)$ such that for each $X \subseteq A$ either $I_{\mathcal{S}}(X) = \emptyset$ or $I_{\mathcal{S}}(X)$ is a union of subalgebras of \mathcal{A} .*

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