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# ON THE MAXIMAL SUBSEMIGROUPS OF THE SEMIGROUP OF ALL MONOTONE TRANSFORMATIONS

Iliya Gyudzhenov and Ilinka Dimitrova*

*South-West University "Neofit Rilski"*
*Faculty of Mathematics and Natural Science*
*2700 Blagoevgrad, Bulgaria*

**e-mail:** iliadg@aix.swu.bg
**e-mail:** ilinka_dimitrova@yahoo.com

**Abstract**

In this paper we consider the semigroup $M_n$ of all monotone transformations on the chain $X_n$ under the operation of composition of transformations. First we give a presentation of the semigroup $M_n$ and some propositions connected with its structure. Also, we give a description and some properties of the class $\tilde{J}_{n-1}$ of all monotone transformations with rank $n-1$. After that we characterize the maximal subsemigroups of the semigroup $M_n$ and the subsemigroups of $M_n$ which are maximal in $\tilde{J}_{n-1}$.

**Keywords:** transformation semigroup, maximal subsemigroups, idempotent, isotone, antitone and monotone transformations, Green's equivalences.

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## 1. Introduction

The maximal subsemigroups of some classes of semigroups have been the object of study by many semigroups theoreticians. In particular, Todorov and Kračolova [8] have determined the maximal subsemigroups of transformations of fixed rank of finite chains. Nichols [5] and Reilly [6] considered the maximal inverse subsemigroups of $T_n$. Yang Xiuliang [9] obtained a complete classification of the maximal inverse subsemigroups of finite symmetric inverse semigroups, Taijie You and Yang Xiuliang [7] have determined the maximal idempotent-generated subsemigroups of finite singular semigroups.

The semigroup $O_n$ of all isotone transformations on the finite chain $X_n$ is also of great interest. Yang Xiuliang [10] obtained a complete classification of the maximal subsemigroups of the semigroup $O_n$. Yang Xiuliang and Lu Chunghan [11] received the maximal idempotent-generated and regular subsemigroups of the semigroup $O_n$ and etc. Recently Fernandes, Gomes and Jesus [2] gave a presentations for the monoid of all order-preserving and order-reversing transformations on a chain with $n$ elements.

In this paper, we consider the semigroup $M_n$ of all monotone transformations on $X_n$. This semigroup consists of all isotone transformations together with all antitone transformations. We present two subsets of the class $\tilde{J}_{n-1}$ of all monotone transformations with defect 1, which generate the semigroup $M_n$. Here, we characterize all maximal subsemigroups of $M_n$. We also determine all subsemigroups of $M_n$ which are maximal in the class $\tilde{J}_{n-1}$. Here, we show that the maximal subsemigroups of the semigroup $M_n$ are closely connected with the maximal subsemigroups of the semigroup $O_n$.

## 2. Preliminary results

For convenience, the following well-known definition and propositions will be used throughout the paper.

**Definition 1 .** Let $A \subseteq X_n$ and let $\pi$ be an equivalence relation on $X_n$. If $\mid \bar{x} \cap A \mid = 1$ for all $\bar{x} \in X_n/\pi$, then $A$ is called a cross-section of $\pi$, denoted by $A \# \pi$.

**Proposition 1** ([8]) **.** *Let $\alpha, \beta \in J_k$. Then there hold*

1) $\alpha\beta \in J_k \Longleftrightarrow X_n\alpha \# \pi_\beta$.
2) *If* $X_n\alpha \# \pi_\beta$ *and* $X_n\beta \# \pi_\alpha$ *then* $X_n\alpha\beta \# \pi_{\alpha\beta}$.

**Notation 1 .** Let $U$ be a subset of $\mathcal{T}_n$. Then the set of all idempotents of the semigroup $\mathcal{T}_n$, belonging to the set $U$, is denoted by $E(U)$.

**Proposition 2** ([3])**.** *Let $\alpha$ and $\beta$ be two elements of the class $J_k \subseteq \mathcal{T}_n$ $(1 \leq k \leq n-1)$. Then the following are equivalent:*

1) $\alpha\beta \in J_k$.

2) $\alpha\beta \in R_\alpha \cap L_\beta$.

3) $L_\alpha \cap R_\beta \subseteq E(J_k)$.

4) $L_\alpha R_\beta = J_k$.

From Proposition 1 and Proposition 2, we have the following two corollaries.

**Corollary 1 .** *If $\alpha$ and $\beta$ are two transformations of the class $J_k$, then*

$$X_n\alpha \# \pi_\beta \Longleftrightarrow L_\alpha \cap R_\beta \subseteq E_k.$$

**Corollary 2 .** *Let $\alpha, \beta_1, \ldots, \beta_p$ $(2 \leq p \in I\!N)$ be transformations of the class $J_k$ $(1 \leq k \leq n-1)$. Then $\alpha = \beta_1\beta_2 \cdots \beta_p$ iff $R_{\beta_1} = R_\alpha$, $L_{\beta_p} = L_\alpha$ and $X_n\beta_i \# \pi_{\beta_{i+1}}$ for all $i = 1, \ldots, p-1$.*

The next proposition determines the number of the idempotents of $R_\alpha$ (respectively of $L_\alpha$) for any $\alpha \in \mathcal{T}_n$.

**Proposition 3** ([1])**.** *Let $1 \leq k \leq n-1$. Then for all $\alpha \in J_k$, there hold*

1) $|E(R_\alpha)| = |\bar{x}_1||\bar{x}_2| \cdots |\bar{x}_k|$ *for* $X_n/\pi_\alpha = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k\}$.

2) $|E(L_\alpha)| = (a_2-a_1)(a_3-a_2) \cdots (a_k-a_{k-1})$ *for* $X_n\alpha = \{a_1, a_2, \ldots, a_k\}$.

*Here, we consider the elements of the set $X_n\alpha$ as cardinals.*

## 3. STRUCTURE OF THE SEMIGROUP $M_n$

We call a transformation $\alpha$ on $X_n$ isotone if $x \leq y \implies x\alpha \leq y\alpha$ and antitone if $x \leq y \implies x\alpha \geq y\alpha$. A transformation $\alpha$ is called monotone if it is isotone or antitone. The product of two isotone transformations or two antitone transformations is an isotone transformation and the product of an isotone transformation with an antitone transformation, or vice-versa, is an antitone transformation.

We denote by $O_n$ the semigroup of all isotone transformations with defect $\geq 1$ under the operation of composition of transformations and by $Q_n$ the set of all antitone transformations on $X_n$ with defect $\geq 1$. The set $M_n := O_n \cup Q_n$ forms a semigroup under the operation of composition of transformations, which is called the semigroup of all monotone transformations.

Since $Q_n$ is a subset of the semigroup $\mathcal{T}_n$, each $\mathcal{L}$ - class $L_\alpha$ of $Q_n$ is uniquely determined by the image $X_n\alpha$ and each $\mathcal{R}$ - class $R_\alpha$ of $Q_n$ is uniquely determined by the kernel $\pi_\alpha$. From the definition of the antitone transformation, it follows that each $\mathcal{H}$ - class $H_\alpha$ of $Q_n$ consists of exactly one transformation, namely $\alpha$ (i.e. the $\mathcal{H}$ - classes of $Q_n$ are trivial). The $\mathcal{H}$ - classes of the semigroup $O_n$ are also trivial and since $M_n = O_n \cup Q_n$, it follows that each $\mathcal{H}$ - class of $M_n$ consists of exactly two transformations with the same image and the same kernel - one isotone and one antitone

**Definition 2 .** Let $S$ be a subsemigroup of the semigroup $M_n$. We call an $\mathcal{H}$ - class $H$ of the semigroup $S$ full if $|S \cap H| = 2$.

The semigroup $S$ is called $\mathcal{H}$ - full if all $\mathcal{H}$ - classes of $S$ are full.

**Definition 3 .** For $1 \leq k \leq n-1$ we put

1) $\hat{J}_k := \{\alpha \in O_n : |X_n\alpha| = k\}$.
2) $\hat{I}_k := \cup\{\hat{J}_i : 1 \leq i \leq k\}$.
3) $\check{J}_k := \{\alpha \in Q_n : |X_n\alpha| = k\}$.
4) $\check{I}_k := \cup\{\check{J}_i : 1 \leq i \leq k\}$.
5) $\tilde{J}_k := \{\alpha \in M_n : |X_n\alpha| = k\}$.
6) $\tilde{I}_k := \cup\{\tilde{J}_i : 1 \leq i \leq k\}$.

Clearly, $\tilde{I}_k$ forms an ideal of $M_n$. It is easy to verify that $\tilde{J}_k = \hat{J}_k \cup \check{J}_k$ and $\tilde{I}_k = \hat{I}_k \cup \check{I}_k$.

In this paper, we pay particular attention to the $\mathcal{J}$ - class $\tilde{J}_{n-1}$ at the top of the semigroup $M_n$.

**Remark 1 .** The $\mathcal{R}$ -, $\mathcal{L}$ - and $\mathcal{H}$ - classes in the class $\tilde{J}_{n-1}$ have the following form:

$$R_i := \{\alpha \in \tilde{J}_{n-1} \subseteq M_n : \ i\alpha = (i+1)\alpha\} \qquad \text{for } \ 1 \le i \le n-1,$$

$$L_k := \{\alpha \in \tilde{J}_{n-1} \subseteq M_n : X_n\alpha = X_n \setminus \{k\}\} \ \text{ for } \ 1 \le k \le n,$$

$$H_i^k := R_i \cap L_k.$$

Each $\mathcal{H}$ - class $H_i^k$ of the semigroup $M_n$ contains exactly two transformations, one isotone and one antitone, which are denoted by $\alpha_i^k$ and $\gamma_i^k$, respectively. We write $\varepsilon_i^k$ for the isotone transformation, if it is an idempotent. Only the classes $H_i^i$ and $H_i^{i+1}$ $(1 \le i \le n-1)$ each contains an idempotent. Each $\mathcal{R}$ - class $R_i$ for some $1 \le i \le n-1$, and each $\mathcal{L}$ - class $L_k$ for some $2 \le k \le n-1$ contains only two idempotents. The $\mathcal{L}$ - classes $L_1$ and $L_n$ each contains only one idempotent.

**Notation 2 .** We denote by $E_k := E(M_n) \cap \tilde{J}_k$ the set of all idempotents of the class $\tilde{J}_k$ for all $1 \le k \le n-1$.

**Corollary 3 .** *For all $\alpha, \beta \in \tilde{J}_{n-1}$ there hold:*

1) *Let $\alpha \in L_k$ for some $1 \le k \le n$. Then $X_n\alpha \# \pi_\beta$ iff $\beta \in \{R_{k-1}, R_k\}$.*
2) *Let $\alpha \in R_i$ for some $1 \le i \le n-1$. Then $X_n\beta \# \pi_\alpha$ iff $\beta \in \{L_i, L_{i+1}\}$.*

**Remark 2 .** Let $\alpha, \beta \in M_n$ and $X_n\alpha \# \pi_\beta$. Then $X_n\gamma \# \pi_\delta$ for all $\gamma \in H_\alpha$ and all $\delta \in H_\beta$, since the transformations of one $\mathcal{H}$ - class have the same image and the same kernel.

In the next two lemmas, we will show the connection between the class of all monotone transformations $\tilde{J}_{n-1}$ and its subclasses $\hat{J}_{n-1}$ and $\check{J}_{n-1}$.

**Lemma 1 .** *Let $1 \le k \le n-1$ and let $\gamma$ be an antitone transformation with rank $k$. Then the set $\hat{J}_k \cup \{\gamma\}$ generates $\tilde{J}_k$.*

***Proof.*** The class $R_\gamma$ certainly contains at least one idempotent $\varepsilon$ (see Proposition 3). Hence for each transformation $\beta \in L_\varepsilon$ we have $R_\gamma \cap L_\beta = \varepsilon \subseteq E_k$. By Proposition 2, we have that the product $\beta\gamma$ belongs to $\tilde{J}_k$ and $\beta\gamma \in R_\beta \cap L_\gamma$. Since $\beta$ is an isotone transformation and $\gamma$ is an antitone transformation, it follows that $\beta\gamma$ is antitone. Therefore, $L_\varepsilon\gamma = L_\gamma \subseteq \tilde{J}_k$.

Moreover, we certainly have that the class $L_\gamma$ contains at least one idempotent $\delta$ (see Proposition 3). Thus from Proposition 2, we obtain $L_\gamma R_\delta = \tilde{J}_k$, and so $\langle \hat{J}_k, \gamma\rangle \supseteq \tilde{J}_k$. ∎

**Lemma 2.** *Let $1 \le k \le n-1$. Then the class $\check{J}_k$ generates the class $\tilde{J}_k$, i.e. $\tilde{J}_k \subseteq \langle \check{J}_k\rangle$.*

***Proof.*** Let $\alpha \in \hat{J}_k$ be an isotone transformation with rank $k$. For each transformation $\alpha$ of this type we can find antitone transformations $\beta_1, \beta_2 \in \check{J}_k$, such that $\pi_{\beta_1} = \pi_\alpha$ and $X_n\beta_2 = X_n\alpha$. We choose the image of the transformation $\beta_1$ and the kernel of $\beta_2$, such that the condition $X_n\beta_1 \# \pi_{\beta_2}$ will hold. Then by Corollary 2 we have $\alpha = \beta_1\beta_2$, i.e. each isotone transformation $\alpha \in \hat{J}_k$ can be represented as a product of two antitone transformations $\beta_1, \beta_2 \in \check{J}_k$. Hence $\langle \check{J}_k\rangle \supseteq \hat{J}_k \cup \check{J}_k = \tilde{J}_k$. ∎

**Theorem 1.** *$M_n = \langle E_{n-1}, \gamma\rangle = \langle \check{J}_{n-1}\rangle$ for any $\gamma \in \check{J}_{n-1}$.*

***Proof.*** The set of all idempotents of the class $\hat{J}_{n-1}$ generates the semigroup $O_n$, i.e. $\langle E_{n-1}\rangle = O_n$ (see [4]). By Lemma 1 we have that $\hat{J}_{n-1} \cup \{\gamma\}$ generates $\tilde{J}_{n-1}$, for any $\gamma \in \check{J}_{n-1}$. From Corollary 1, we have that $X_n\alpha \# \pi_\beta$ iff $L_\alpha \cap R_\beta \subseteq E_{n-1}$ for all $\alpha, \beta \in \tilde{J}_{n-1}$. We also know that only the classes $H_i^i$ and $H_i^{i+1}$ $(1 \le i \le n-1)$ of the class $\tilde{J}_{n-1}$ each contains an idempotent. Therefore, there are an isotone transformation $\alpha \in \tilde{J}_{n-1}$ and an antitone transformation $\beta \in \tilde{J}_{n-1}$ such that $X_n\alpha$ is not a cross-section of $\pi_\beta$, and so the product $\alpha\beta = \gamma_1$ belongs to the class $\check{J}_{n-2}$ (see Proposition 1). Hence from Lemma 1, we have that $\langle \hat{J}_{n-2}, \gamma_1\rangle = \tilde{J}_{n-2}$. Continuing in this way, we find $\langle \hat{J}_{n-k}, \gamma_{k-1}\rangle = \tilde{J}_{n-k}$ for all $2 \le k \le n-1$ and $\gamma_{k-1} \in \check{J}_{n-k}$, i.e. we obtain the ideal $\tilde{I}_{n-2}$. Since $\tilde{I}_{n-2} \cup \tilde{J}_{n-1} = M_n$, we deduce that the set $E_{n-1} \cup \{\gamma\}$ generates the semigroup $M_n$.

From Lemma 2 we have that the set $\check{J}_{n-1}$ of all antitone transformations with defect 1 generates the class $\tilde{J}_{n-1}$. Since the set $E_{n-1} \cup \{\gamma\}$, where $\gamma \in \check{J}_{n-1}$, is a subset of the class $\tilde{J}_{n-1}$ we obtain that $\check{J}_{n-1}$ generates the semigroup $M_n$. ∎

## 4. THE MAXIMAL SUBSEMIGROUPS OF THE SEMIGROUP $M_n$

In this section, we give a complete classification of the maximal subsemigroups of the semigroup $M_n$ of all monotone transformations. Here, we can use the classification of the maximal subsemigroups of $O_n$ given by Yang Xiuliang ([10]).

Any maximal subsemigroup $S$ of $O_n$ has the form $S = \hat{I}_{n-2} \cup U$ for some $U \subseteq \hat{J}_{n-1}$. Now, we will study the elements of $S \setminus \tilde{I}_{n-2}$ for any maximal subsemigroup $S$ of $M_n$.

**Lemma 3 .** *Let $S$ be a maximal subsemigroup of $M_n$. Then $S \cap E_{n-1} \neq \emptyset$.*

***Proof.*** Let $U := S \setminus \tilde{I}_{n-2}$ and let us assume that $U \cap E_{n-1} = \emptyset$, i.e. no one of the idempotents

$$\varepsilon \in \left(H_i^i \cup H_i^{i+1}\right) \cap O_n \quad \text{for} \quad i = 1, \ldots, n-1$$

belongs to the semigroup $S$. On the other hand, the product of any two antitone transformations is an isotone transformation. Hence the semigroup $S$ certainly contains isotone transformations. If $\alpha_i^k \in S \cap H_i^k$ is such a transformation, then by Corollary 2 and Corollary 3 we get the following equations:

$$\varepsilon_i^i = \alpha_i^k \ \alpha_{k-1}^i = \alpha_i^k \ \alpha_k^i, \qquad \varepsilon_i^{i+1} = \alpha_i^k \ \alpha_{k-1}^{i+1} = \alpha_i^k \ \alpha_k^{i+1}. \tag{1}$$

Therefore, from the assumption that $\alpha_i^k \in U$ and $\varepsilon_i^i, \varepsilon_i^{i+1} \notin U$, it follows that

$$\left\{\alpha_{k-1}^i,\ \alpha_{k-1}^{i+1},\ \alpha_k^i, \alpha_k^{i+1}\right\} \subseteq \hat{J}_{n-1} \setminus U. \tag{2}$$

We have that $S$ is a maximal subsemigroup of the semigroup $M_n$, and so $\langle S, \alpha\rangle = M_n$ for each transformation $\alpha \in M_n \setminus S$. In particular, we have $\langle S, \varepsilon_i^i\rangle = M_n$. The transformation $\alpha_k^i \in M_n \setminus S$ is expressible as a product of transformations in the following way:

$$\alpha_k^i = \beta \varepsilon_i^i \gamma \in \hat{J}_{n-1} \tag{3}$$

for some $\beta, \gamma \in S$. From Corollary 2 we have

$$R_\beta = R_k, \quad L_\gamma = L_i, \quad X_n\beta \ \# \ \pi_{\varepsilon_i^i}, \quad X_n\varepsilon_i^i \ \# \ \pi_\gamma. \tag{4}$$

From the last requirements and Corollary 3, it follows that

$$\beta \in H_k^i \cup H_k^{i+1} \subseteq \tilde{J}_{n-1} \quad \text{and} \quad \gamma \in H_{i-1}^i \cup H_i^i \subseteq \tilde{J}_{n-1}. \tag{5}$$

Moreover, since $\alpha_k^i$ and $\varepsilon_i^i$ are isotone transformations, it follows that the transformations $\beta$ and $\gamma$ are both isotone or both antitone transformations. Let us now consider separately each of these two cases.

Let $\beta$ and $\gamma$ be isotone transformations. We know that each $\mathcal{H}$ - class of the class $\tilde{J}_{n-1}$ contains exactly one isotone and one antitone transformation. In this case from (2) and (5) we have:

$$\beta \in H_k^i \cup H_k^{i+1} = \left\{\alpha_k^i\right\} \cup \left\{\alpha_k^{i+1}\right\} \subseteq \hat{J}_{n-1} \setminus S$$

$$\gamma \in H_{i-1}^i \cup H_i^i = \left\{\alpha_{i-1}^i\right\} \cup \left\{\alpha_i^i\right\} \subseteq \hat{J}_{n-1} \setminus S,$$

i.e. $\beta$ and $\gamma$ do not belong to the semigroup $S$.

Now let $\beta$ and $\gamma$ be antitone transformations. Then from $\gamma \in H_{i-1}^i \cup H_i^i$, it follows that $\gamma^2$ is an idempotent and from the assumption it does not belong to $S$. Therefore, the transformation $\gamma$ does not belong to $S$.

Thus we deduce that the transformations $\beta$ and $\gamma$ do not belong to the semigroup $S$. This contradicts the assumption that $\beta, \gamma \in S$.

The argument above hold for any representation of the transformation $\alpha_k^i$, since the first and the last element in that product have to satisfy the conditions (3) and (4). Therefore, $U \cap E_{n-1} \neq \emptyset$. ■

For the proof of the main result, which characterizes the maximal subsemigroups of the semigroup $M_n$, we need the following lemmas.

**Lemma 4.** *If the maximal subsemigroup $S$ of $M_n$ contains an $\mathcal{H}$ - class which is not full then $S$ contains an idempotent $\varepsilon$ such that $H_\varepsilon$ is not full.*

***Proof.*** Let $H_i^k = \{\alpha_i^k,\ \gamma_i^k\}$ be one $\mathcal{H}$ - class of the semigroup $S$ which is not full and let $\alpha_i^k \in S$ and $\gamma_i^k \notin S$. From Lemma 3, we have that the semigroup $S$ contains an idempotent $\varepsilon$. We will prove that $H_\varepsilon \subseteq S$ is not full. From Corollary 2 and Corollary 3 we obtain:

$$\gamma_i^k = \alpha_i^k\ \gamma_{k-1}^k = \alpha_i^k\ \gamma_k^k = \gamma_i^i\ \alpha_i^k = \gamma_i^{i+1}\ \alpha_i^k \notin S.$$

This shows that the antitone transformations $\gamma_{k-1}^k,\ \gamma_k^k,\ \gamma_i^i,\ \gamma_i^{i+1}$ do not belong to $S$, since $\alpha_i^k \in S$ and $\gamma_i^k \notin S$. The respective $\mathcal{H}$ - classes of these transformations contain the idempotents $\varepsilon_{k-1}^k,\ \varepsilon_k^k,\ \varepsilon_i^i,\ \varepsilon_i^{i+1}$. If at least one of these idempotents belongs to $S$ then the semigroup $S$ contains an idempotent whose $\mathcal{H}$ - class is not full.

Assume that all these idempotents do not belong to $S$. Then from Corollary 2 and Corollary 3 we have:

$$\varepsilon_i^i = \alpha_i^k\ \alpha_k^i = \alpha_i^k\ \alpha_{k-1}^i \implies \alpha_k^i,\ \alpha_{k-1}^i \in \hat{J}_{n-1} \setminus S.$$

Since $S$ is a maximal subsemigroup and $\varepsilon_k^k \notin S$, it follows that $\langle S, \varepsilon_k^k \rangle = M_n$. On the other hand, again by Corollary 2 and Corollary 3 we have:

$$\alpha_k^i = \varepsilon_k^k\ \alpha_{k-1}^i,$$

where $\alpha_k^i, \varepsilon_k^k$ and $\alpha_{k-1}^i$ do not belong to $S$ (as we mentioned above). This shows that the transformation $\alpha_k^i \notin \langle S, \varepsilon_k^k \rangle$, i.e. $\langle S, \varepsilon_k^k \rangle \neq M_n$. Hence the assumption that all idempotents $\varepsilon_{k-1}^k$, $\varepsilon_k^k$, $\varepsilon_i^i$, $\varepsilon_i^{i+1}$ do not belong to $S$ contradicts the condition that $S$ is a maximal subsemigroup of $M_n$.
Analogously, one can obtain the same result for the converse case, when the antitone transformation $\gamma_i^k$ of the class $H_i^k$ belongs to $S$ and the isotone transformation $\alpha_i^k$ does not belong to $S$. ■

**Lemma 5.** *Let $S$ be a maximal subsemigroup of the semigroup $M_n$. Then either $S \setminus \tilde{I}_{n-2} = \hat{J}_{n-1}$ or $S$ is $\mathcal{H}$ - full.*

***Proof.*** Let $U = S \setminus \tilde{I}_{n-2}$ and let $S$ be a maximal subsemigroup of the semigroup $M_n$. Then from Lemma 3, it follows that $S$ contains at least one idempotent. Let us assume that $S$ is not an $\mathcal{H}$ - full semigroup, i.e. there is at least one element of $U$, whose $\mathcal{H}$ - class is not full. From Lemma 4, we have that $U$ contains an idempotent whose $\mathcal{H}$ - class is not full.

Let the idempotent $\varepsilon_i^i \in H_i^i$ belong to $S$ and let the antitone transformation $\gamma_i^i$ not belong to $S$. We will show that all $\mathcal{H}$ - classes of the semigroup $S$, which contain idempotents, are not full.

From the condition $\gamma_i^i \notin S$ and the equations (see Corollary 2, Corollary 3)

$$\gamma_i^i = \varepsilon_i^i\ \gamma_{i-1}^i = \gamma_i^{i+1}\ \varepsilon_i^i\ ,$$

it follows that the antitone transformations $\gamma_{i-1}^i, \gamma_i^{i+1} \in \check{J}_{n-1}$ do not belong to $S$. For this two transformations, we can construct the following product:

$$\gamma_{i-1}^i = \alpha \gamma_i^{i+1} \beta, \tag{6}$$

and by Corollary 2, it follows that

$$R_\alpha = R_{i-1}, \quad L_\beta = L_i, \quad X_n\alpha \# \pi_{\gamma_i^{i+1}}, \quad X_n\gamma_i^{i+1} \# \pi_\beta,$$

i.e. $\alpha \in H_{i-1}^i \cup H_{i-1}^{i+1}$ and $\beta \in H_i^i \cup H_{i+1}^i$ (see Corollary 3).

Obviously, the last conditions completely define the $\mathcal{R}$ - and $\mathcal{L}$ - classes of the transformations $\alpha$ and $\beta$. Moreover, they are isotone or antitone at the same time.

Since $S$ is maximal in $M_n$ and $\gamma_{i-1}^i, \gamma_i^{i+1} \notin S$ then $\langle S, \gamma_i^{i+1}\rangle = M_n$ and thus in the equation (6) the transformations $\alpha$ and $\beta$ must belong to $S$. Now we will consider the cases, when they are isotone and when they are antitone transformations.

If $\alpha$ and $\beta$ are antitone transformations then we have:

$$\alpha = \gamma_{i-1}^{i+1} \in H_{i-1}^{i+1} \text{ and } \beta = \gamma_i^i \in H_i^i \text{ or } \beta = \gamma_{i+1}^i \in H_{i+1}^i.$$

Since the transformation $\gamma_{i-1}^i$ does not belong to the semigroup $S$ from the equation $\gamma_{i-1}^i = \gamma_{i-1}^{i+1}\ \varepsilon_i^i \notin S$ it follows that the transformation $\gamma_{i-1}^{i+1} = \alpha$ does not belong to $S$. This contradicts the condition $\alpha \in S$. Therefore, the equation (6) if $\alpha$ and $\beta$ are antitone transformations contradicts the maximality of $S$.

If $\alpha$ and $\beta$ are isotone transformations then we have:

$$\alpha = \varepsilon_{i-1}^i \in H_{i-1}^i \text{ or } \alpha = \alpha_{i-1}^{i+1} \in H_{i-1}^{i+1}$$

and

$$\beta = \varepsilon_i^i \in H_i^i \text{ or } \beta = \alpha_{i+1}^i \in H_{i+1}^i.$$

From equation (6) we obtain:

$$\gamma_{i-1}^i = \varepsilon_{i-1}^i\gamma_i^{i+1}\varepsilon_i^i = \varepsilon_{i-1}^i\gamma_i^{i+1}\alpha_{i+1}^i = \alpha_{i-1}^{i+1}\gamma_i^{i+1}\varepsilon_i^i = \alpha_{i-1}^{i+1}\gamma_i^{i+1}\alpha_{i+1}^i$$

and since $\varepsilon_i^i \in S$, so in order for at least one of these equations to hold it is enough that at least one of the transformations either $\varepsilon_{i-1}^i$ or $\alpha_{i-1}^{i+1}$ belongs to $S$. If $\alpha_{i-1}^{i+1} \in S$, then $\varepsilon_{i-1}^i = \alpha_{i-1}^{i+1}\ \varepsilon_i^i \in S$. Hence in both cases the idempotent $\varepsilon_{i-1}^i$ belong to $S$. Its $\mathcal{H}$ - class is not full in $S$, since the antitone transformation $\gamma_{i-1}^i$ does not belong to $S$.

Since the transformation $\gamma_{i-1}^i$ does not belong to the semigroup $S$, from the equation $\gamma_{i-1}^i = \gamma_{i-1}^{i-1}\ \varepsilon_{i-1}^i \notin S$, it follows that the antitone transformation $\gamma_{i-1}^{i-1}$ does not belong to $S$.

Further, from the conditions $\gamma^i_{i-1}, \gamma^{i-1}_{i-1} \notin S$ and $\langle S, \gamma^i_{i-1} \rangle = M_n$ for the transformation $\gamma^{i-1}_{i-1}$ we have

$$\gamma^{i-1}_{i-1} = \alpha \gamma^i_{i-1} \beta,$$

and by Corollary 2 we have:

$$R_\alpha = R_{i-1}, \quad L_\beta = L_{i-1}, \quad X_n \alpha \; \# \; \pi_{\gamma^i_{i-1}}, \quad X_n \gamma^i_{i-1} \; \# \; \pi_\beta,$$

i.e. $\alpha \in H^{i-1}_{i-1} \cup H^i_{i-1}$ and $\beta \in H^{i-1}_{i-1} \cup H^{i-1}_i$ (see Corollary 3).

Since $S$ is maximal in $M_n$, the transformations $\alpha$ and $\beta$ belong to $S$. They can not be antitone, since the antitone transformations $\gamma^i_{i-1}$ and $\gamma^{i-1}_{i-1}$, of the respective $\mathcal{H}$ - classes, do not belong to $S$.

Therefore, $\alpha$ and $\beta$ are isotone transformations and thus

$$\alpha = \varepsilon^{i-1}_{i-1} \in H^{i-1}_{i-1}, \quad \alpha = \varepsilon^i_{i-1} \in H^i_{i-1}$$

and

$$\beta = \varepsilon^{i-1}_{i-1} \in H^{i-1}_{i-1}, \quad \beta = \alpha^{i-1}_i \in H^{i-1}_i.$$

Hence we obtain that the idempotent $\varepsilon^{i-1}_{i-1}$ belongs to $S$. Its $\mathcal{H}$ - class is not full, since the antitone transformation $\gamma^{i-1}_{i-1}$ does not belong to $S$.

Thus the assumption that the $\mathcal{H}$ - class of the idempotent $\varepsilon^i_i$ is not full, implies the same deduction for the $\mathcal{H}$ - class of the idempotent $\varepsilon^{i-1}_{i-1}$. Continuing by induction on the indices $k \leq i$ of the idempotents $\varepsilon^k_k$, we obtain that all idempotents

$$\varepsilon^1_1, \; \varepsilon^2_1, \; \varepsilon^2_2, \; \varepsilon^3_2, \; \ldots, \; \varepsilon^{i-1}_{i-1}, \; \varepsilon^i_{i-1}, \; \varepsilon^i_i$$

belong to $S$ and moreover, their $\mathcal{H}$ - classes are not full.

Analogously, starting from the idempotent $\varepsilon^i_i$ and the conditions $\langle S, \gamma^i_{i-1} \rangle = M_n$, and $\gamma^{i+1}_i \notin S$ first we obtain that the idempotent $\varepsilon^{i+1}_i$ belongs to $S$ and its $\mathcal{H}$ - class is not full and then we obtain the same result for the idempotent $\varepsilon^{i+1}_{i+1}$.

Continuing by induction on the indices $k \geq i$ of the idempotents $\varepsilon^k_k$ we deduce that all idempotents

$$\varepsilon^{i+1}_i, \; \varepsilon^{i+1}_{i+1}, \; \varepsilon^{i+2}_{i+1}, \; \ldots, \; \varepsilon^{n-1}_{n-2}, \; \varepsilon^{n-1}_{n-1}, \; \varepsilon^n_{n-1}$$

belong to $S$ and their $\mathcal{H}$ - classes are not full.

Hence if $S$ contains an $\mathcal{H}$ - class which is not full then all idempotents of the class $\tilde{J}_{n-1}$ belong to $S$ and their $\mathcal{H}$ - classes are not full.

By Howie ([4]), $\langle E_{n-1}\rangle = \hat{J}_{n-1}$ and thus $\hat{J}_{n-1} \subseteq S$. Let us assume that $S$ contains at least one antitone transformation $\gamma \in \check{J}_{n-1}$. From Theorem 1 we have $\langle \hat{J}_{n-1}, \gamma\rangle = M_n$, and so $M_n = S$. Therefore, $S \cap \check{J}_{n-1} = \emptyset$, since $S$ is a maximal subsemigroup of $M_n$ and $S = \tilde{I}_{n-2} \cup \hat{J}_{n-1}$, i.e. $U = \hat{J}_{n-1}$.

We have shown that if $S$ contains an $\mathcal{H}$ - class which is not full then $S$ contains all idempotents of the class $\tilde{J}_{n-1}$ and thus the set $U$ coincides with the class $\hat{J}_{n-1}$. If $S$ contains an antitone transformation $\gamma \in \check{J}_{n-1}$ then $S$ does not contain all idempotents of the class $\tilde{J}_{n-1}$, since $\langle \hat{J}_{n-1}, \gamma\rangle = M_n$. Thus from the argument above, it follows that $S$ is an $\mathcal{H}$ - full semigroup. ∎

**Lemma 6.** *Let $S'$ be a subsemigroup of $O_n$ and $\gamma \in \check{J}_{n-1}$ with $\gamma^2 \in S \cap E_{n-1}$. Then $S = \langle S', \gamma\rangle$ is an $\mathcal{H}$ - full subsemigroup of $M_n$ and $S \cap O_n = S'$.*

***Proof.*** Let $U' = S' \setminus \hat{I}_{n-2}$ and let $L_k(U')$ (respectively $R_i(U')$) be the set of all elements of the class $L_k$ (respectively $R_i$), that belong to the semigroup $S'$, i.e. $L_k(U') = L_k \cap S'$ for all $1 \le k \le n$ and $R_i(U') = R_i \cap S'$ for all $1 \le i \le n-1$. From $\gamma^2 = \varepsilon$ we have that $H_\gamma = H_\varepsilon$, i.e. $L_\gamma = L_\varepsilon$ and $R_\gamma = R_\varepsilon$. Let $\gamma^2 = \varepsilon \in L_k(U') \cap R_i(U')$. Then for each $\alpha \in L_k(U')$ we have $\alpha\gamma = \delta \in R_\alpha \cap L_\gamma \subseteq \check{J}_{n-1}$, since $\varepsilon \in L_\alpha \cap R_\gamma$ (see Proposition 2). The transformation $\delta$ is antitone, since $\alpha$ is isotone and $\gamma$ is antitone. We also have that $L_\alpha = L_\varepsilon = L_\delta$. This shows that $H_\delta = H_\alpha$. Therefore, for each $\alpha \in L_k(U')$ we have $\alpha\gamma = \delta \in H_\alpha \cap Q_n$.

If we denote by $L_k(U)$ the $\mathcal{L}$ - class of all elements of the class $L_k(U')$ together with the respective antitone transformations, then from Proposition 2 and $L_k(U) \cap R_i(U') = \varepsilon \in E(U')$, we have that $L_k(U)R_i(U')$ $= U$, where $U$ is $\mathcal{H}$ - full and its isotone transformations are the same as those of the set $U'$.

In the set $U$ there are an isotone transformation $\alpha$ and an antitone transformation $\beta$ such that $X_n\alpha$ is not a cross-section of $\pi_\beta$. Then the product $\alpha\beta = \gamma_1$ belongs to the class $\check{J}_{n-2}$ (see Proposition 1) and from Lemma 1 we have that $\langle \hat{J}_{n-2}, \gamma_1\rangle = \tilde{J}_{n-2}$. Continuing in this way, we find $\langle \hat{J}_{n-k}, \gamma_{k-1}\rangle = \tilde{J}_{n-k}$ for all $2 \le k \le n-1$, i.e. we obtain the ideal $\tilde{I}_{n-2}$. Hence the semigroup $S = \langle S', \gamma\rangle = \tilde{I}_{n-2} \cup U$ is $\mathcal{H}$ - full.

We have shown that for each $\alpha \in S'$ the class $H_\alpha \subseteq S$ contains exactly two transformations - one isotone and one antitone. Now we will show that the product of any two of the given antitone transformations belongs to the semigroup $S'$, i.e. $S \cap O_n = S'$.

We have $S' \subseteq O_n$ and $S' \subseteq S$, since $S = \langle S', \gamma \rangle$. This shows that $S' \subseteq S \cap O_n$.

Now we will show that $S \cap O_n \subseteq S'$. Let $\alpha \in S \cap O_n$, i.e. $\alpha$ is an isotone transformation of the semigroup $S = \tilde{I}_{n-2} \cup U$. If $\alpha \in \tilde{I}_{n-2} \cap O_n = \hat{I}_{n-2}$ then $\alpha \in S'$, since $S' = \hat{I}_{n-2} \cup U'$.

We have that $S = \langle S', \gamma \rangle$, thus $S'\gamma$, $\gamma S' \subseteq S$ and $S'\gamma$, $\gamma S' \subseteq Q_n$. Then for each isotone transformation $\alpha$ we have $\alpha \in S'$ or $\alpha = \gamma_1\gamma_2$, where $\gamma_1, \gamma_2 \in S \cap Q_n$.

Let $\alpha \in U \cap O_n \subseteq \hat{J}_{n-1}$. Assume that there are antitone transformations $\gamma_1, \gamma_2 \in U$ such that $\alpha = \gamma_1\gamma_2$. Then from Proposition 1 we have $X_n\gamma_1 \# \pi_{\gamma_2}$. The semigroup $S$ is $\mathcal{H}$ - full, and so all $\mathcal{H}$ - classes contain one isotone and one antitone transformation. Therefore, there are isotone transformations $\alpha_1, \alpha_2 \in U'$ such that $H_{\alpha_1} = H_{\gamma_1}$ and $H_{\alpha_2} = H_{\gamma_2}$. This shows that

$$H_{\alpha_j} = H_{\gamma_j} \Longrightarrow \begin{cases} L_{\alpha_j} = L_{\gamma_j} \implies X_n\alpha_j = X_n\gamma_j \\ R_{\alpha_j} = R_{\gamma_j} \implies \pi_{\alpha_j} = \pi_{\gamma_j} \end{cases}, \quad j = 1, 2.$$

From $X_n\gamma_1 \# \pi_{\gamma_2}$, it follows that $X_n\alpha_1 \# \pi_{\alpha_2}$ and $\alpha_1\alpha_2 = \alpha \in U' \subseteq S'$. Consequently, we have $S \cap O_n \subseteq S'$ and thus $S \cap O_n = S'$. ■

By the definition of an $\mathcal{H}$ - full semigroup we get:

**Corollary 4.** *If $S' = O_n$ and $\gamma \in \check{J}_{n-1}$ with $\gamma^2 \in S \cap E_{n-1}$ then $\langle O_n, \gamma \rangle = M_n$.*

Now, we will study the connection between the maximal subsemigroups of the semigroups $O_n$ and $M_n$.

**Lemma 7.** *Let $S'$ be a maximal subsemigroup of the semigroup $O_n$ and let $\gamma \in \check{J}_{n-1}$ be an antitone transformation. Then $\langle S', \gamma \rangle$ is a maximal subsemigroup of $M_n$.*

***Proof.*** 1) Let $S = \langle S', \gamma \rangle$ and $\alpha \in M_n \setminus S$. We will show that $\langle S, \alpha \rangle = M_n$, i.e. for each transformation $\beta \in M_n \setminus S$ there exist transformations $\eta$ and $\delta$ in $S$, such that $\beta = \eta\alpha\delta$. Since we know the transformations $\alpha$ and $\beta$, we can completely define the transformations $\eta$ and $\delta$ by the following conditions (see Corollary 2):

$$R_\eta = R_\beta, \quad L_\delta = L_\beta \text{ and } X_n\alpha \# \pi_\delta, \quad X_n\eta \# \pi_\alpha. \tag{7}$$

The transformation $\alpha$ belongs to the semigroup $M_n$ and hence it can be isotone or antitone. We will consider each of these cases.

a) Let $\alpha$ be an isotone transformation, i.e. $\alpha \in (M_n \setminus S) \cap O_n$. Since $S'$ is a maximal subsemigroup of the semigroup $O_n$ we have $\langle S', \alpha \rangle = O_n$. Therefore, for each transformation $\beta' \in O_n \setminus S'$ there exist transformations $\eta', \delta' \in S'$, such that $\beta' = \eta'\alpha\delta'$.

The transformations $\eta'$ and $\delta'$ belong to the semigroup $S' \subseteq S$, but $S$ is an $\mathcal{H}$ - full semigroup (by Lemma 6) and thus the antitone transformations $\eta''$ and $\delta''$ of the $\mathcal{H}$ - classes of $\eta'$ and $\delta'$ also belong to $S$.

Hence both transformations $\eta''$ and $\delta''$ also satisfy the conditions in (7). Then we have

$$\beta' = \eta'\alpha\delta' = \eta''\alpha\delta''$$

for each isotone transformation $\beta' \in M_n \setminus S$ and

$$\beta'' = \eta''\alpha\delta' = \eta'\alpha\delta''$$

for each antitone transformation $\beta'' \in M_n \setminus S$. Hence $\langle S, \alpha \rangle = M_n$ for each isotone transformation $\alpha \in (M_n \setminus S) \cap \hat{J}_{n-1}$.

b) Let $\alpha$ be an antitone transformation, i.e. $\alpha \in (M_n \setminus S) \cap \check{J}_{n-1}$. Since $S'$ is a maximal subsemigroup of the semigroup $O_n$ by the results of Yang Xiuliang ([10]) each $\mathcal{L}$ - and each $\mathcal{R}$ - class of $S'$ contains an idempotent. Therefore, either $L_\alpha$ or $R_\alpha$ contains an idempotent $\varepsilon \in S$.

Since $S$ is an $\mathcal{H}$ - full semigroup, the antitone transformation $\eta''$ of the $\mathcal{H}$ - class of $\varepsilon$ also belongs to $S$. The transformation $\alpha$ does not belong to $S$ and thus the isotone transformation $\alpha' \in H_\alpha$ also does not belong to $S$.If $L_\varepsilon = L_\alpha$ then from the conditions $R_\alpha = R_{\alpha'}$ and $L_\alpha = L_\varepsilon = L_{\eta''}$ we have $\alpha\eta'' = \alpha' \in \langle S, \alpha\rangle$. If $R_\varepsilon = R_\alpha$ then from the conditions $R_\alpha = R_\varepsilon = R_{\eta''}$ and $L_\alpha = L_{\alpha'}$ we have $\eta''\alpha = \alpha' \in \langle S, \alpha\rangle$.

From the case a), $\langle S, \alpha'\rangle = M_n$ and so $\langle S, \alpha\rangle = M_n$ for each antitone transformation $\alpha \in (M_n \setminus S) \cap \check{J}_{n-1}$. ■

**Lemma 8.** *Let $S$ be a maximal subsemigroup of the semigroup $M_n$ and $S' = S \cap O_n$. Then either $S' = O_n$ or $S'$ is maximal in $O_n$.*

***Proof.*** Let $S = \tilde{I}_{n-2} \cup U$ for some $U \subseteq \tilde{J}_{n-1}$. If $S$ is not an $\mathcal{H}$ - full semigroup then $S$ does not contain any antitone transformations with defect 1 and $U = \hat{J}_{n-1}$ (see Lemma 5). Therefore, for the semigroup $S'$ we have $S' = (\hat{I}_{n-2} \cup \hat{J}_{n-1}) \cap O_n = \hat{I}_{n-2} \cup \hat{J}_{n-1} = O_n$.

If $S$ contains at least one antitone transformation, then $U \neq \hat{J}_{n-1}$ and it is an $\mathcal{H}$ - full semigroup, also by Lemma 5.

Now we will show that in this case the semigroup $S' = S \cap O_n$ is a maximal subsemigroup of the semigroup $O_n$. Assume that $S'$ is not a maximal subsemigroup of $O_n$. Then there is a semigroup $T'$ such that $S' \subset T' \subset O_n$. Since $S$ is maximal in $M_n$, it contains an idempotent $\varepsilon \in S'$, and so $\varepsilon \in T'$. The semigroup $S$ is $\mathcal{H}$ - full and so it contains the antitone transformation $\gamma \in H_\varepsilon$. From Lemma 6, it follows that $\langle T', \gamma\rangle = T$ is an $\mathcal{H}$ - full semigroup for which $S \subset T \subset M_n$. This contradicts the condition that $S$ is a maximal subsemigroup of $M_n$. Therefore, we obtain that $S'$ is a maximal subsemigroup of $O_n$. ■

Now we are able to give a description of all maximal subsemigroups of $M_n$.

**Theorem 2.** *Let $S$ be a subsemigroup of the semigroup $M_n$. Then $S$ is a maximal subsemigroup iff one of the following statements holds:*

1) $S := \tilde{I}_{n-2} \cup \hat{J}_{n-1}$.

2) $S := \langle S', \gamma\rangle$ *for some maximal subsemigroup $S'$ of $O_n$ and some* $\gamma \in \check{J}_{n-1}$.

***Proof.*** 1) Let $\gamma$ be an antitone transformation which belongs to $(M_n \setminus S) \cap \check{J}_{n-1}$. By Lemma 1, we have that $\hat{J}_{n-1} \cup \gamma$ generates the class $\tilde{J}_{n-1}$, and $\tilde{I}_{n-2} \cup \tilde{J}_{n-1} = M_n$. Therefore, $\langle S, \gamma \rangle = M_n$ for each $\gamma \in \check{J}_{n-1}$, i.e. $S$ is a maximal subsemigroup of the semigroup $M_n$.

2) The semigroup $S'$ can be only one of the semigroups given in [10], i.e., $S' = \hat{I}_{n-2} \cup U'$ for some $U' \subseteq \hat{J}_{n-1}$. Hence $S$ is maximal by Lemma 7.

Conversely, let $S$ be a maximal subsemigroup of the semigroup $M_n$. By Lemma 8 the semigroup $S' = S \cap O_n$ coincides with the semigroup $O_n$, i.e., $S' = O_n$ or $S'$ is a maximal subsemigroup of $O_n$.

Let $S' = O_n$ and let $\gamma \in \check{J}_{n-1}$ be an antitone transformation, then from Corollary 4, it follows that $\langle O_n, \gamma \rangle = M_n$. Hence $S \subseteq O_n \cup \check{I}_{n-2} = \tilde{I}_{n-2} \cup \hat{J}_{n-1}$, but since $S$ is maximal, we have $S = \tilde{I}_{n-2} \cup \hat{J}_{n-1}$. Therefore, the semigroup $S$ is of type 1).

Let $S'$ be a maximal subsemigroup of the semigroup $O_n$. Then by Lemma 7 the semigroup $\langle S', \gamma \rangle$, where $\gamma \in \check{J}_{n-1}$, is maximal in $M_n$. The semigroup $S$ contains all elements of the semigroup $S'$ and the respective antitone transformations. Therefore, $S \subseteq \langle S', \gamma \rangle$, and so $S = \langle S', \gamma \rangle$, since $S$ is maximal. Consequently, we conclude that $S$ is of type 2). ■

In the last part of this section we consider the class $\tilde{J}_{n-1}$ and subsemigroups of $M_n$ which are contained in $\tilde{J}_{n-1}$. We are interested in subsemigroups $S$ of $M_n$ with $S \subseteq \tilde{J}_{n-1}$ for which $\langle S, \gamma \rangle$ is not a subset of the class $\tilde{J}_{n-1}$ for any $\gamma \in \tilde{J}_{n-1} \setminus S$. Such semigroups are called maximal in $\tilde{J}_{n-1}$.

Let us put:

1) $S_1(i) := H_i^{i+1} \cup H_i^i$ for $1 \le i \le n-1$.

2) $S_2(i) := H_{i-1}^i \cup H_i^i$ for $2 \le i \le n-1$.

3) $S_2(1) := H_1^1$.

4) $S_2(n) := H_{n-1}^n$.

**Theorem 3.** *A subsemigroup $S$ of $M_n$ is maximal in $\tilde{J}_{n-1}$ iff either $S = S_1(i)$ for some $1 \le i \le n-1$ or $S = S_2(i)$ for some $1 \le i \le n$.*

***Proof.*** Let $S = S_1(i)$ for some $1 \leq i \leq n-1$. Then we have $S = \{\alpha_i^i,\ \gamma_i^i,\ \alpha_i^{i+1},\ \gamma_i^{i+1}\}$ contains two isotone transformations and two antitone transformations. Moreover, the isotone transformations $\alpha_i^i$ and $\alpha_i^{i+1}$ are idempotents. It is easy to verify that the product of any two elements $\alpha, \beta$ of the set $S$ also belongs to $S$. Therefore, $S$ is a semigroup and since the classes $H_i^i$ and $H_i^{i+1}$ belong to $\tilde{J}_{n-1}$, we have that $S$ is contained in the class $\tilde{J}_{n-1}$.

Now we will show that $S$ is maximal in $\tilde{J}_{n-1}$.

Let us assume that $S$ is not maximal, i.e. there is a transformation $\beta \in \tilde{J}_{n-1} \setminus S$ such that $\langle S, \beta \rangle \subseteq \tilde{J}_{n-1}$. Then for each transformation $\alpha \in S$ we have $\beta\alpha, \alpha\beta \in \tilde{J}_{n-1}$, i.e. $X_n\beta \ \# \ \pi_\alpha$ and $X_n\alpha \ \# \ \pi_\beta$ (see Proposition 1). From Corollary 3 we have that $L_\beta = L_i$ or $L_\beta = L_{i+1}$, since $R_\alpha = R_i$. If $\alpha \in H_i^i$ then $L_\alpha = L_i$ and from Corollary 3, it follows that $R_\beta = R_{i-1}$ or $R_\beta = R_i$. If $\alpha \in H_i^{i+1}$ then $L_\alpha = L_{i+1}$ and so $R_\beta = R_i$ or $R_\beta = R_{i+1}$. Since $X_n\alpha \ \# \ \pi_\beta$ has to be satisfied for all $\alpha \in S$, it follows that $R_\beta = R_i$.

Finally, we obtain that $\beta \in L_i \cap R_i = H_i^i$ or $\beta \in L_{i+1} \cap R_i = H_i^{i+1}$, i.e. $\beta \in S$. This contradicts the condition $\beta \in \tilde{J}_{n-1} \setminus S$ and so $S$ is maximal in $\tilde{J}_{n-1}$.

The proof of the case $S = S_2(i)$ is similar.

Conversely, let $S$ be maximal in $\tilde{J}_{n-1}$. Then $\beta\alpha, \alpha\beta \in \tilde{J}_{n-1}$ for all $\alpha, \beta \in S$. This implies $X_n\alpha \ \# \ \pi_\beta$ and $X_n\beta \ \# \ \pi_\alpha$ (see Proposition 1). Obviously, $\alpha\alpha \in S \subseteq \tilde{J}_{n-1}$ and thus $\alpha\alpha \in R_\alpha \cap L_\alpha$ for all $\alpha \in S$. This shows that $\alpha\alpha = \alpha \in E_{n-1}$ or $\alpha\alpha = \alpha \in Q_n$ and $H_\alpha$ contains an idempotent. Therefore, we have $\alpha \in O_n \cap (H_i^i \cup H_i^{i+1})$ or $\alpha \in Q_n \cap (H_i^i \cup H_i^{i+1})$ for some $1 \leq i \leq n-1$.

Let $\alpha \in H_i^i$. Then from the conditions $X_n\alpha \ \# \ \pi_\beta$ and $X_n\beta \ \# \ \pi_\alpha$, it follows $\beta \in \{R_{i-1}, R_i\}$ and $\beta \in \{L_{i+1}, L_i\}$ for all $\beta \in S$ (see Corollary 3). Hence we get

$$\beta \in \left\{H_{i-1}^i, H_i^i, H_i^{i+1}\right\}.$$

If $L_\beta = L_{i+1}$ and $R_\gamma = R_{i-1}$ for all $\beta, \gamma \in S$ then $X_n\beta$ is not a cross-section of $\pi_\gamma$. Therefore, we obtain $S = H_i^i \cup H_i^{i+1} = S_1(i)$ or $S = H_{i-1}^i \cup H_i^i = S_2(i)$ for all $1 \leq i \leq n-1$. In the second case, if $i = 1$ then the $\mathcal{H}$ - class $H_{i-1}^i = H_0^1$ does not exist and $S = H_1^1 = S_2(1)$.

Now let $\alpha \in H_i^{i+1}$. Then in the same way we obtain that $S = H_i^i \cup H_i^{i+1} = S_1(i)$ or $S = H_i^{i+1} \cup H_{i+1}^{i+1} = S_2(i+1)$ for all $1 \leq i \leq n-1$. In the second case, if $i = n-1$ then the $\mathcal{H}$ - class $H_{i+1}^{i+1} = H_n^n$ does not exist and $S = H_{n-1}^n = S_2(n)$. ■

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