

## ON A PERIOD OF ELEMENTS OF PSEUDO-BCI-ALGEBRAS

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### Abstract

The notions of a period of an element of a pseudo-BCI-algebra and a periodic pseudo-BCI-algebra are defined. Some of their properties and characterizations are given.

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### 1. INTRODUCTION

In 1966 K. Iséki introduced the notion of BCI-algebra (see [10]). BCI-algebras have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. The name of BCI-algebra originates from the combinatories B, C, I in combinatory logic.

The concept of pseudo-BCI-algebra has been introduced in [1] as an extension of BCI-algebras. Pseudo-BCI-algebras are algebraic models of some extension of a noncommutative version of the BCI-logic (see [5] for details). These algebras have also connections with other algebras of logic such as pseudo-BCK-algebras, pseudo-BL-algebras and pseudo-MV-algebras introduced by G. Georgescu and A. Iorgulescu in [6, 7] and [8], respectively. More about those algebras the reader can find in [9].

In this paper we define the notion of a period of an element of a pseudo-BCI-algebra. Some of its properties are also given. Finally, we study the concept of a periodic pseudo-BCI-algebra proving some of its interesting characterization. All necessary material needed in the sequel is presented in Section 2 making our exposition self-contained.

## 2. PRELIMINARIES

A *pseudo-BCI-algebra* is a structure  $\mathcal{X} = (X; \leq, \rightarrow, \rightsquigarrow, 1)$ , where  $\leq$  is binary relation on a set  $X$ ,  $\rightarrow$  and  $\rightsquigarrow$  are binary operations on  $X$  and  $1$  is an element of  $X$  such that for all  $x, y, z \in X$ , we have

- (a1)  $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z), \quad x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z),$
- (a2)  $x \leq (x \rightarrow y) \rightsquigarrow y, \quad x \leq (x \rightsquigarrow y) \rightarrow y,$
- (a3)  $x \leq x,$
- (a4) if  $x \leq y$  and  $y \leq x$ , then  $x = y$ ,
- (a5)  $x \leq y$  iff  $x \rightarrow y = 1$  iff  $x \rightsquigarrow y = 1$ .

It is obvious that any pseudo-BCI-algebra  $(X; \leq, \rightarrow, \rightsquigarrow, 1)$  can be regarded as a universal algebra  $(X; \rightarrow, \rightsquigarrow, 1)$  of type  $(2, 2, 0)$ . Note that every pseudo-BCI-algebra satisfying  $x \rightarrow y = x \rightsquigarrow y$  for all  $x, y \in X$  is a BCI-algebra.

Every pseudo-BCI-algebra satisfying  $x \leq 1$  for all  $x \in X$  is a pseudo-BCK-algebra. A pseudo-BCI-algebra which is not a pseudo-BCK-algebra will be called *proper*.

Any pseudo-BCI-algebra  $\mathcal{X} = (X; \leq, \rightarrow, \rightsquigarrow, 1)$  satisfies the following, for all  $x, y, z \in X$ ,

- (b1) if  $1 \leq x$ , then  $x = 1$ ,
- (b2) if  $x \leq y$ , then  $y \rightarrow z \leq x \rightarrow z$  and  $y \rightsquigarrow z \leq x \rightsquigarrow z$ ,
- (b3) if  $x \leq y$  and  $y \leq z$ , then  $x \leq z$ ,
- (b4)  $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z),$
- (b5)  $x \leq y \rightarrow z$  iff  $y \leq x \rightsquigarrow z$ ,
- (b6)  $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y), \quad x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y),$
- (b7) if  $x \leq y$ , then  $z \rightarrow x \leq z \rightarrow y$  and  $z \rightsquigarrow x \leq z \rightsquigarrow y$ ,
- (b8)  $1 \rightarrow x = 1 \rightsquigarrow x = x,$
- (b9)  $((x \rightarrow y) \rightsquigarrow y) \rightarrow y = x \rightarrow y, \quad ((x \rightsquigarrow y) \rightarrow y) \rightsquigarrow y = x \rightsquigarrow y,$
- (b10)  $x \rightarrow y \leq (y \rightarrow x) \rightsquigarrow 1, \quad x \rightsquigarrow y \leq (y \rightsquigarrow x) \rightarrow 1,$
- (b11)  $(x \rightarrow y) \rightarrow 1 = (x \rightarrow 1) \rightsquigarrow (y \rightsquigarrow 1), \quad (x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \rightarrow (y \rightarrow 1),$
- (b12)  $x \rightarrow 1 = x \rightsquigarrow 1.$

If  $(X; \leq, \rightarrow, \rightsquigarrow, 1)$  is a pseudo-BCI-algebra, then, by (a3), (a4), (b3) and (b1),  $(X; \leq)$  is a poset with 1 as a maximal element.

**Example 2.1** ([3]). Let  $X = \{a, b, c, d, e, f, 1\}$  and define binary operations  $\rightarrow$  and  $\rightsquigarrow$  on  $X$  by the following tables:

$\rightarrow$	$a$	$b$	$c$	$d$	$e$	$f$	$1$	$\rightsquigarrow$	$a$	$b$	$c$	$d$	$e$	$f$	$1$
$a$	1	$d$	$e$	$b$	$c$	$a$	$a$	$a$	1	$c$	$b$	$e$	$d$	$a$	$a$
$b$	$c$	1	$a$	$e$	$d$	$b$	$b$	$b$	$d$	1	$e$	$a$	$c$	$b$	$b$
$c$	$e$	$a$	1	$c$	$b$	$d$	$d$	$c$	$b$	$e$	1	$c$	$a$	$d$	$d$
$d$	$b$	$e$	$d$	1	$a$	$c$	$c$	$d$	$e$	$a$	$d$	1	$b$	$c$	$c$
$e$	$d$	$c$	$b$	$a$	1	$e$	$e$	$e$	$c$	$d$	$a$	$b$	1	$e$	$e$
$f$	$a$	$b$	$c$	$d$	$e$	1	1	$f$	$a$	$b$	$c$	$d$	$e$	1	1
1	$a$	$b$	$c$	$d$	$e$	$f$	1	1	$a$	$b$	$c$	$d$	$e$	$f$	1

Then  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  is a (proper) pseudo-BCI-algebra. Observe that it is not a pseudo-BCK-algebra because  $a \not\leq 1$ .

**Example 2.2** ([11]). Let  $Y_1 = (-\infty, 0]$  and let  $\leq$  be the usual order on  $Y_1$ . Define binary operations  $\rightarrow$  and  $\rightsquigarrow$  on  $Y_1$  by

$$x \rightarrow y = \begin{cases} 0 & \text{if } x \leq y, \\ \frac{2y}{\pi} \arctan(\ln(\frac{y}{x})) & \text{if } y < x, \end{cases}$$

$$x \rightsquigarrow y = \begin{cases} 0 & \text{if } x \leq y, \\ ye^{-\tan(\frac{\pi x}{2y})} & \text{if } y < x \end{cases}$$

for all  $x, y \in Y_1$ . Then  $\mathcal{Y}_1 = (Y_1; \leq, \rightarrow, \rightsquigarrow, 0)$  is a pseudo-BCK-algebra, and hence it is a nonproper pseudo-BCI-algebra.

**Example 2.3** ([4]). Let  $Y_2 = \mathbb{R}^2$  and define binary operations  $\rightarrow$  and  $\rightsquigarrow$  and a binary relation  $\leq$  on  $Y_2$  by

$$(x_1, y_1) \rightarrow (x_2, y_2) = (x_2 - x_1, (y_2 - y_1)e^{-x_1}),$$

$$(x_1, y_1) \rightsquigarrow (x_2, y_2) = (x_2 - x_1, y_2 - y_1e^{x_2 - x_1}),$$

$$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow (x_1, y_1) \rightarrow (x_2, y_2) = (0, 0) = (x_1, y_1) \rightsquigarrow (x_2, y_2)$$

for all  $(x_1, y_1), (x_2, y_2) \in Y_2$ . Then  $\mathcal{Y}_2 = (Y_2; \leq, \rightarrow, \rightsquigarrow, (0, 0))$  is a proper pseudo-BCI-algebra. Notice that  $\mathcal{Y}_2$  is not a pseudo-BCK-algebra because there exists  $(x, y) = (1, 1) \in Y_2$  such that  $(x, y) \not\leq (0, 0)$ .

**Example 2.4** ([4]). Let  $\mathcal{Y}$  be the direct product of pseudo-BCI-algebras  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$  from Examples 2.2 and 2.3, respectively. Then  $\mathcal{Y}$  is a proper pseudo-BCI-algebra, where  $Y = (-\infty, 0] \times \mathbb{R}^2$  and binary operations  $\rightarrow$  and  $\rightsquigarrow$  and binary relation  $\leq$  are defined on  $Y$  by

$$(x_1, y_1, z_1) \rightarrow (x_2, y_2, z_2) = \begin{cases} (0, y_2 - y_1, (z_2 - z_1)e^{-y_1}) & \text{if } x_1 \leq x_2, \\ (\frac{2x_2}{\pi} \arctan(\ln(\frac{x_2}{x_1})), y_2 - y_1, (z_2 - z_1)e^{-y_1}) & \text{if } x_2 < x_1, \end{cases}$$

$$(x_1, y_1, z_1) \rightsquigarrow (x_2, y_2, z_2) = \begin{cases} (0, y_2 - y_1, z_2 - z_1 e^{y_2 - y_1}) & \text{if } x_1 \leq x_2, \\ (x_2 e^{-\tan(\frac{\pi x_1}{2x_2})}, y_2 - y_1, z_2 - z_1 e^{y_2 - y_1}) & \text{if } x_2 < x_1, \end{cases}$$

$$(x_1, y_1, z_1) \leq (x_2, y_2, z_2) \Leftrightarrow x_1 \leq x_2 \text{ and } y_1 = y_2 \text{ and } z_1 = z_2.$$

Notice that  $\mathcal{Y}$  is not a pseudo-BCK-algebra because there exists  $(x, y, z) = (0, 1, 1) \in Y$  such that  $(x, y, z) \not\leq (0, 0, 0)$ .

Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. Define

$$\begin{aligned} x \rightarrow^0 y &= y, \\ x \rightarrow^n y &= x \rightarrow (x \rightarrow^{n-1} y), \end{aligned}$$

where  $x, y \in X$  and  $n = 1, 2, \dots$ . Similarly we define  $x \rightsquigarrow^n y$  for any  $n = 0, 1, 2, \dots$

**Proposition 2.5.** *Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. The following are equivalent for any  $x, y \in X$  and  $n = 0, 1, 2, \dots$ ,*

- (i)  $x \rightarrow^n y = 1$ ,
- (ii)  $x \rightsquigarrow^n y = 1$ .

**Proof.** It follows by (a5) and (b4). ■

For any pseudo-BCI-algebra  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  the set

$$K(X) = \{x \in X : x \leq 1\}$$

is a subalgebra of  $\mathcal{X}$  (called pseudo-BCK-part of  $\mathcal{X}$ , see [1]). Then  $(K(X); \rightarrow, \rightsquigarrow, 1)$  is a pseudo-BCK-algebra. Note that if  $\mathcal{X}$  is a pseudo-BCK-algebra, then  $X = K(X)$ .

It is easily seen that for the pseudo-BCI-algebras  $\mathcal{X}$ ,  $\mathcal{Y}_1$ ,  $\mathcal{Y}_2$  and  $\mathcal{Y}$  from Examples 2.1, 2.2, 2.3 and 2.4, respectively, we have  $K(X) = \{f, 1\}$ ,  $K(Y_1) = Y_1$ ,  $K(Y_2) = \{(0, 0)\}$  and  $K(Y) = \{(x, 0, 0) : x \leq 0\}$ .

An element  $a$  of a pseudo-BCI-algebra  $\mathcal{X}$  is called a *maximal element* of  $\mathcal{X}$  if for every  $x \in X$  the following holds

$$\text{if } a \leq x, \text{ then } x = a.$$

We will denote by  $M(X)$  the set of all maximal elements of  $\mathcal{X}$ . Obviously,  $1 \in M(X)$ . Notice that  $M(X) \cap K(X) = \{1\}$ . Indeed, if  $a \in M(X) \cap K(X)$ , then  $a \leq 1$  and, by above implication,  $a = 1$ . Moreover, observe that 1 is the only maximal element of a pseudo-BCK-algebra. Therefore, for a pseudo-BCK-algebra  $\mathcal{X}$ ,  $M(X) = \{1\}$ . In [2] there is shown that  $M(X) = \{x \in X : x = (x \rightarrow 1) \rightarrow 1\}$ . Moreover we have the following simple lemma.

**Lemma 2.6.** *Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra and  $x, y \in X$ . If  $x \leq y$ , then  $x \rightarrow 1 = x \rightsquigarrow 1 = y \rightarrow 1 = y \rightsquigarrow 1$ .*

Observe that for the pseudo-BCI-algebras  $\mathcal{X}$ ,  $\mathcal{Y}_1$ ,  $\mathcal{Y}_2$  and  $\mathcal{Y}$  from Examples 2.1, 2.2, 2.3 and 2.4, respectively, we have  $M(X) = \{a, b, c, d, e, 1\}$ ,  $M(Y_1) = \{0\}$ ,  $M(Y_2) = Y_2$  and  $M(Y) = \{(0, y, z) : y, z \in \mathbb{R}\}$ .

Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. Then  $\mathcal{X}$  is *p-semisimple* if it satisfies for all  $x \in X$ ,

$$\text{if } x \leq 1, \text{ then } x = 1.$$

Note that if  $\mathcal{X}$  is a p-semisimple pseudo-BCI-algebra, then  $K(X) = \{1\}$ . Hence, if  $\mathcal{X}$  is a p-semisimple pseudo-BCK-algebra, then  $X = \{1\}$ . Moreover, as it is proved in [4],  $M(X)$  is a p-semisimple pseudo-BCI-subalgebra of  $\mathcal{X}$  and  $\mathcal{X}$  is p-semisimple if and only if  $X = M(X)$ .

**Proposition 2.7** ([4]). *Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. The following are equivalent:*

- (i)  $\mathcal{X}$  is p-semisimple,
- (ii) for all  $x, y \in X$ ,  $(x \rightarrow 1) \rightsquigarrow y = (y \rightsquigarrow 1) \rightarrow x$ ,
- (iii) for all  $x \in X$ ,  $x = (x \rightarrow 1) \rightarrow 1$ .

It is not difficult to see that the pseudo-BCI-algebras  $\mathcal{X}$ ,  $\mathcal{Y}_1$  and  $\mathcal{Y}$  from Examples 2.1, 2.2 and 2.4, respectively, are not p-semisimple, and the pseudo-BCI-algebra  $\mathcal{Y}_2$  from Example 2.3 is a p-semisimple algebra.

**Theorem 2.8.** *Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. The following are equivalent:*

- (i)  $\mathcal{X}$  is  $p$ -semisimple,
- (ii)  $X = \{x \rightarrow 1 : x \in X\}$ .

**Proof.** (i) $\Rightarrow$ (ii) Take  $y \in X$ . Since  $\mathcal{X}$  is  $p$ -semisimple,  $y = (y \rightarrow 1) \rightarrow 1$ . Putting  $x = y \rightarrow 1 \in X$ , we get  $y = x \rightarrow 1$ .

(ii) $\Rightarrow$ (i) Take  $a \in X$ . We show that  $a$  is a maximal element of  $\mathcal{X}$ , that is,  $X = M(X)$ . Suppose that  $a = x \rightarrow 1$  for some  $x \in X$ . Let  $y \in X$  be such that  $a \leq y$ . Then  $(x \rightarrow 1) \rightarrow y = 1$  and, by (b4), (b9), (b11) and (b12), we have

$$\begin{aligned} y \rightarrow a &= y \rightarrow (x \rightarrow 1) = y \rightarrow (((x \rightarrow 1) \rightarrow 1) \rightsquigarrow 1) \\ &= ((x \rightarrow 1) \rightarrow 1) \rightsquigarrow (y \rightarrow 1) = ((x \rightarrow 1) \rightarrow y) \rightarrow 1 \\ &= 1 \rightarrow 1 = 1. \end{aligned}$$

Hence  $y \leq a$ . So,  $y = a$ , that is,  $a \in M(X)$  and  $\mathcal{X}$  is  $p$ -semisimple. ■

For  $p$ -semisimple pseudo-BCI-algebras we have the following useful fact.

**Theorem 2.9** [4]. *A pseudo-BCI-algebra  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  is  $p$ -semisimple if and only if  $(X; \cdot, {}^{-1}, 1)$  is a group, where  $x \cdot y = (x \rightarrow 1) \rightsquigarrow y = (y \rightsquigarrow 1) \rightarrow x$ ,  $x^{-1} = x \rightarrow 1 = x \rightsquigarrow 1$ ,  $x \rightarrow y = y \cdot x^{-1}$  and  $x \rightsquigarrow y = x^{-1} \cdot y$  for any  $x, y \in X$ .*

Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. We say that a subset  $D$  of  $X$  is a *deductive system* of  $\mathcal{X}$  if it satisfies: (i)  $1 \in D$ , (ii) for all  $x, y \in X$ , if  $x \in D$  and  $x \rightarrow y \in D$ , then  $y \in D$ . Under this definition,  $\{1\}$  and  $X$  are the simplest examples of deductive systems. Note that the condition (ii) can be replaced by (ii') for all  $x, y \in X$ , if  $x \in D$  and  $x \rightsquigarrow y \in D$ , then  $y \in D$ . It can be easily proved that for any  $x, y \in X$ , if  $x \in D$  and  $x \leq y$ , then  $y \in D$ . A deductive system  $D$  of a pseudo-BCI-algebra  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  is called *closed* if  $D$  is closed under operations  $\rightarrow$  and  $\rightsquigarrow$ , that is, if  $D$  is a subalgebra of  $\mathcal{X}$ . It is not difficult to show (see [3]) that a deductive system  $D$  of a pseudo-BCI-algebra  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  is closed if and only if for any  $x \in D$ ,  $x \rightarrow 1 = x \rightsquigarrow 1 \in D$ . Obviously, the pseudo-BCK-part  $K(X)$  is a closed deductive system of  $\mathcal{X}$ .

**Proposition 2.10** ([2]). *Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra and  $M(X)$  be finite. Then every deductive system of  $\mathcal{X}$  is closed.*

Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. It is obvious that the intersection of arbitrary number of deductive systems is a deductive system. Hence, for any  $A \subseteq X$  there exists the least deductive system containing  $A$ . Denote it by  $D(A)$  and call it the deductive system *generated* by  $A$ . In particular, if  $A = \{a_1, \dots, a_n\}$ , then we write  $D(a_1, \dots, a_n)$  instead of  $D(\{a_1, \dots, a_n\})$ . It is also obvious that  $D(\emptyset) = \{1\}$ .

**Proposition 2.11** ([3]). *Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. For any  $a \in X$ ,*

$$\begin{aligned} D(a) &= \{1\} \cup \{x \in X : a \rightarrow^n x = 1 \text{ for some } n \in \mathbb{N}\} \\ &= \{1\} \cup \{x \in X : a \rightsquigarrow^n x = 1 \text{ for some } n \in \mathbb{N}\}. \end{aligned}$$

### 3. PERIOD OF ELEMENTS

**Proposition 3.1.** *Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. The following hold for any  $x, y, z \in X$  and  $m, n = 0, 1, 2, \dots$ ,*

- (i)  $x \rightarrow^n 1 = x \rightsquigarrow^n 1$ ,
- (ii)  $x \rightarrow^n x = x \rightarrow^{n-1} 1, \quad x \rightsquigarrow^n x = x \rightsquigarrow^{n-1} 1$ ,
- (iii)  $(x \rightarrow 1) \rightarrow^n 1 = (x \rightarrow^n 1) \rightarrow 1, \quad (x \rightsquigarrow 1) \rightsquigarrow^n 1 = (x \rightsquigarrow^n 1) \rightsquigarrow 1$ ,
- (iv)  $x \rightarrow (y \rightsquigarrow^n z) = y \rightsquigarrow^n (x \rightarrow z), \quad x \rightsquigarrow (y \rightarrow^n z) = y \rightarrow^n (x \rightsquigarrow z)$ ,
- (v)  $x \rightarrow^m (y \rightsquigarrow^n z) = y \rightsquigarrow^n (x \rightarrow^m z)$ ,
- (vi)  $x \rightarrow^n 1 = ((x \rightarrow 1) \rightarrow 1) \rightarrow^n 1, \quad x \rightsquigarrow^n 1 = ((x \rightsquigarrow 1) \rightsquigarrow 1) \rightsquigarrow^n 1$ .

**Proof.** (i) Follows from (b4) and (b12).

(ii) Obvious.

(iii) We prove first equation by induction. The proof of second equation is analogous. For  $n = 0$  it is obvious. Assume it for  $n = k$ :

$$(x \rightarrow 1) \rightarrow^k 1 = (x \rightarrow^k 1) \rightarrow 1.$$

We have, by definition, assumption, (i), (b11) and (b12),

$$\begin{aligned} (x \rightarrow 1) \rightarrow^{k+1} 1 &= (x \rightarrow 1) \rightarrow ((x \rightarrow 1) \rightarrow^k 1) = (x \rightarrow 1) \rightarrow ((x \rightarrow^k 1) \rightarrow 1) \\ &= (x \rightsquigarrow 1) \rightarrow ((x \rightsquigarrow^k 1) \rightsquigarrow 1) = (x \rightsquigarrow (x \rightsquigarrow^k 1)) \rightsquigarrow 1 \\ &= (x \rightsquigarrow^{k+1} 1) \rightsquigarrow 1 = (x \rightarrow^{k+1} 1) \rightarrow 1. \end{aligned}$$

So, the equation holds for any  $n = 0, 1, 2, \dots$

(iv) We prove first equation by induction. The proof of second equation is analogous. For  $n = 0$  it is obvious. Assume it for  $n = k$ , that is,

$$x \rightarrow (y \rightsquigarrow^k z) = y \rightsquigarrow^k (x \rightarrow z).$$

We have, by definition, assumption and (b4),

$$\begin{aligned} x \rightarrow (y \rightsquigarrow^{k+1} z) &= x \rightarrow (y \rightsquigarrow (y \rightsquigarrow^k z)) = y \rightsquigarrow (x \rightarrow (y \rightsquigarrow^k z)) \\ &= y \rightsquigarrow (y \rightsquigarrow^k (x \rightarrow z)) = y \rightsquigarrow^{k+1} (x \rightarrow z). \end{aligned}$$

Hence, the equation holds for any  $n = 0, 1, 2, \dots$

(v) We get it easily by (iv).

(vi) We prove first equation by induction. The proof of second equation is analogous. For  $n = 0$  it is obvious. Assume it for  $n = k$ , that is,

$$x \rightarrow^k 1 = ((x \rightarrow 1) \rightarrow 1) \rightarrow^k 1.$$

We have, by definition, assumption, (i), (iv), (b9) and (b12),

$$\begin{aligned} ((x \rightarrow 1) \rightarrow 1) \rightarrow^{k+1} 1 &= ((x \rightarrow 1) \rightarrow 1) \rightarrow (((x \rightarrow 1) \rightarrow 1) \rightarrow^k 1) \\ &= ((x \rightarrow 1) \rightarrow 1) \rightarrow (x \rightarrow^k 1) \\ &= ((x \rightarrow 1) \rightarrow 1) \rightarrow (x \rightsquigarrow^k 1) \\ &= x \rightsquigarrow^k (((x \rightarrow 1) \rightarrow 1) \rightarrow 1) \\ &= x \rightsquigarrow^k (x \rightarrow 1) \\ &= x \rightarrow (x \rightsquigarrow^k 1) \\ &= x \rightarrow (x \rightarrow^k 1) \\ &= x \rightarrow^{k+1} 1. \end{aligned}$$

Hence, the equation holds for any  $n = 0, 1, 2, \dots$  ■

Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. For any  $x \in X$ , if there exists the least natural number  $n$  such that  $x \rightarrow^n 1 = 1$ , then  $n$  is called a *period of  $x$*  denoted  $p(x)$ . If, for any natural number  $n$ ,  $x \rightarrow^n 1 \neq 1$ , then a period of  $x$  is called to be infinite and denoted  $p(x) = \infty$ . Obviously,  $p(1) = 1$ .

**Proposition 3.2.** *Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. Then  $p(x) = p(x \rightarrow 1) = p(x \rightsquigarrow 1)$  for all  $x \in X$ .*

**Proof.** Obviously,  $p(x \rightarrow 1) = p(x \rightsquigarrow 1)$ . For any  $x \in X$ , by Proposition 3.1(iii,v), we have

$$x \rightarrow^k 1 = ((x \rightarrow 1) \rightarrow 1) \rightarrow^k 1 = ((x \rightarrow 1) \rightarrow^k 1) \rightarrow 1.$$

Since  $(x \rightarrow 1) \rightarrow^k 1$  is a maximal element, we have that  $x \rightarrow^k 1 = 1$  if and only if  $((x \rightarrow 1) \rightarrow^k 1) \rightarrow 1 = 1$ . Thus,  $p(x) = p(x \rightarrow 1)$ . ■

**Proposition 3.3.** *Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra and  $x, y \in X$ . If  $x \leq y$ , then  $p(x) = p(y)$ .*



**Proof.** Let  $x, y \in X$ . By Lemma 2.6 and Proposition 3.2, if  $x \leq y$ , then  $x \rightarrow 1 = y \rightarrow 1$  and  $p(x) = p(x \rightarrow 1) = p(y \rightarrow 1) = p(y)$ . ■

**Theorem 3.4.** Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a  $p$ -semisimple pseudo-BCI-algebra and  $(X; \cdot, ^{-1}, 1)$  be a group related with  $\mathcal{X}$ . Then  $p(x) = o(x)$  for any  $x \in X$ , where  $o(x)$  means an order of an element  $x$  in a group  $(X; \cdot, ^{-1}, 1)$ .

**Proof.** Let  $x \in X$ . Since  $x \rightarrow y = y \cdot x^{-1}$ , it is not difficult to see that  $(x \rightarrow 1) \rightarrow^k 1 = x^k$  for any  $k = 0, 1, 2, \dots$ . Then,

$$(x \rightarrow 1) \rightarrow^k 1 = 1 \text{ iff } x^k = 1.$$

So,  $p(x \rightarrow 1) = o(x)$ . Thus, by Proposition 3.2,  $p(x) = o(x)$ . ■

**Corollary 3.5.** Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a  $p$ -semisimple pseudo-BCI-algebra. Then the following hold for any  $x, y \in X$ ,

- (i)  $p(x \rightarrow y) = p(y \rightarrow x)$ ,  $p(x \rightsquigarrow y) = p(y \rightsquigarrow x)$ ,
- (ii)  $p(x \rightarrow y) = p(x \rightsquigarrow y)$ .

Now we prove that identities from Corollary 3.5 hold also for arbitrary pseudo-BCI-algebras.

**Theorem 3.6.** Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. Then the following hold for any  $x, y \in X$ ,

- (i)  $p(x \rightarrow y) = p(y \rightarrow x)$ ,  $p(x \rightsquigarrow y) = p(y \rightsquigarrow x)$ ,
- (ii)  $p(x \rightarrow y) = p(x \rightsquigarrow y)$ .

**Proof.** (i) We show the first equation. The proof of the second one is analogous. Let  $x, y \in X$ . Then  $x \rightarrow 1, y \rightarrow 1 \in M(X)$ . By Proposition 3.2, (b11), (b12) and Corollary 3.5 we have

$$\begin{aligned} p(x \rightarrow y) &= p((x \rightarrow y) \rightarrow 1) = p((x \rightarrow 1) \rightsquigarrow (y \rightarrow 1)) \\ &= p((y \rightarrow 1) \rightsquigarrow (x \rightarrow 1)) = p((y \rightarrow x) \rightarrow 1) \\ &= p(y \rightarrow x). \end{aligned}$$

(ii) Similarly we have

$$\begin{aligned} p(x \rightarrow y) &= p((x \rightarrow y) \rightarrow 1) = p((x \rightarrow 1) \rightsquigarrow (y \rightarrow 1)) \\ &= p((x \rightsquigarrow 1) \rightarrow (y \rightsquigarrow 1)) = p((x \rightsquigarrow y) \rightarrow 1) \\ &= p(x \rightsquigarrow y) \end{aligned}$$

for any  $x, y \in X$ . ■

Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra and  $x \in X$ . It is not difficult to see that

$$p(x) = 1 \text{ iff } x \leq 1.$$

Hence we have the following proposition.

**Proposition 3.7.** *Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. Then it is a pseudo-BCK-algebra if and only if  $p(x) = 1$  for any  $x \in X$ .*

**Corollary 3.8.** *Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. Then it is proper if and only if there exists  $x \in X$  such that  $p(x) > 1$ .*

**Corollary 3.9.** *Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. Then it is  $p$ -semisimple if and only if  $p(x) > 1$  for any  $x \in X \setminus \{1\}$ .*

A pseudo-BCI-algebra  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  is called *periodic* if  $p(x) < \infty$  for any  $x \in X$ . It is immediately seen that every pseudo-BCK-algebra is periodic.

Now give an interesting characterization of periodic pseudo-BCI-algebras.

**Theorem 3.10.** *A pseudo-BCI-algebra  $\mathcal{X}$  is periodic if and only if every deductive system of  $\mathcal{X}$  is closed.*

**Proof.** Assume that  $\mathcal{X}$  is periodic and  $D$  is a deductive system of  $\mathcal{X}$ . Let  $x \in D$ . Then there exists a natural number  $n$  such that  $x \rightarrow^n 1 = 1$ . Since  $x, x \rightarrow^n 1 \in D$  and  $D$  is a deductive system, we have  $x \rightarrow 1 \in D$ , that is,  $D$  is closed.

Conversely, for any  $x \in X$ , a deductive system  $D(x)$  is closed. Hence,  $x \rightarrow 1 \in D(x)$ . So, there exists a natural number  $n$  such that  $x \rightarrow^n (x \rightarrow 1) = 1$ , that is,  $p(x) < \infty$ . Thus  $\mathcal{X}$  is periodic. ■

By Proposition 2.10 we have the following.

**Corollary 3.11.** *Let  $\mathcal{X}$  be a pseudo-BCI-algebra. If  $M(X)$  is finite, then  $\mathcal{X}$  is periodic.*

**Corollary 3.12.** *Every finite pseudo-BCI-algebra is periodic.*

**Example 3.13.** The pseudo-BCI-algebra  $\mathcal{X}$  from Example 2.1 is periodic because it is finite and the pseudo-BCI-algebra  $\mathcal{Y}$  from Example 2.4 is not periodic because a deductive system  $D = \{(x, y, y) : x \leq 0, y \in \mathbb{R}\}$  is not closed.

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