

ON BALANCED ORDER RELATIONS AND THE NORMAL HULL OF COMPLETELY SIMPLE SEMIRINGS

SUNIL K. MAITY

Department of Mathematics, University of Burdwan
Golapbag, Burdwan – 713104
West Bengal, India

e-mail: skmaity@math.buruniv.ac.in

Abstract

In [1] the authors proved that a semiring S is a completely simple semiring if and only if S is isomorphic to a Rees matrix semiring over a skew-ring R with sandwich matrix P and index sets I and Λ which are bands under multiplication. In this paper we characterize all the balanced order relations on completely simple semirings. Also we study the normal hull of a completely simple semiring.

Keywords: skew-ring, Rees matrix semiring, balanced order relation, essential extension, normal extension, normal ideal, normal hull.

2010 Mathematics Subject Classification: 16A78, 20M07, 20M10.

1. INTRODUCTION

Recall that a semiring $(S, +, \cdot)$ is a type $(2, 2)$ -algebra whose semigroup reducts $(S, +)$ and (S, \cdot) are connected by ring like distributivity, that is, $a(b+c) = ab+ac$ and $(b+c)a = ba+ca$ for all $a, b, c \in S$. A semiring $(S, +, \cdot)$ is called a skew-ring if its additive reduct $(S, +)$ is a group. In a skew-ring $(R, +, \cdot)$, a normal subgroup K of $(R, +)$ is called a skew-ideal of R if $a \in K$ implies $ca, ac \in K$ for all $c \in R$.

A semigroup (S, \cdot) is said to be a regular semigroup if for each element $a \in S$, there exists an element $x \in S$ such that $axa = a$. A semigroup (S, \cdot) is called a completely regular semigroup if for each element $a \in S$, there exists an element $x \in S$ such that $axa = a$ and $ax = xa$. We call a semiring $(S, +, \cdot)$ additively regular if for every element $a \in S$ there exists an element $x \in S$ such that

$a + x + a = a$. If $(S, +, \cdot)$ is a semiring, we denote Green's relations on the semigroup $(S, +)$ by \mathcal{L}^+ , \mathcal{R}^+ , \mathcal{J}^+ , \mathcal{D}^+ and \mathcal{H}^+ .

A semiring S is called a completely regular semiring [2] if for every element $a \in S$ there exists an element $x \in S$ such that the following conditions are satisfied:

$$(i) \ a = a + x + a, \quad (ii) \ a + x = x + a \quad \text{and} \quad (iii) \ a(a + x) = a + x.$$

In fact, conditions (i) and (ii) state that $a \in S$ is a completely regular element in the additive reduct $(S, +)$ of the semiring $(S, +, \cdot)$. Condition (iii) is an extra condition which makes the element a in $(S, +, \cdot)$ to be completely regular. A completely regular semiring $(S, +, \cdot)$ is called completely simple if $\mathcal{J}^+ = S \times S$. Clearly, from definition it follows that every completely simple semiring $(S, +, \cdot)$ is a semiring with completely simple additive semigroup $(S, +)$. Construction of a semiring $(S, +, \cdot)$ whose additive reduct $(S, +)$ is a completely simple semigroup has been studied by M.P. Grillet [3]. We know that a semigroup (S, \cdot) is completely simple if and only if S is isomorphic to a Rees matrix semigroup over a group with sandwich matrix P and index sets I and Λ . In [1] we extended this important result from semigroups to semirings.

Theorem 1.1 [1]. *Let R be skew-ring, (I, \cdot) and (Λ, \cdot) are bands such that $I \cap \Lambda = \{o\}$. Let $P = (p_{\lambda,i})$ be a matrix over $R, i \in I, \lambda \in \Lambda$ and assume*

$$(1.1) \ p_{\lambda,o} = p_{o,i} = 0;$$

$$(1.2) \ p_{\lambda\mu,kj} = p_{\lambda\mu,ij} - p_{\nu\mu,ij} + p_{\nu\mu,kj};$$

$$(1.3) \ p_{\mu\lambda,jk} = p_{\mu\lambda,ji} - p_{\mu\nu,ji} + p_{\mu\nu,jk};$$

$$(1.4) \ ap_{\lambda,i} = p_{\lambda,i}a = 0;$$

$$(1.5) \ ab + p_{o\mu,io} = p_{o\mu,io} + ab;$$

$$(1.6) \ ab + p_{\lambda o,oj} = p_{\lambda o,oj} + ab \text{ for all } i, j, k \in I; \lambda, \mu, \nu \in \Lambda \text{ and } a, b \in R.$$

Let \mathcal{M} consist of the elements of $I \times R \times \Lambda$ and define operations $'+'$ and $'\cdot'$ on \mathcal{M} by

$$(1.7) \quad (i, a, \lambda) + (j, b, \mu) = (i, a + p_{\lambda,j} + b, \mu)$$

$$(1.8) \quad (i, a, \lambda) \cdot (j, b, \mu) = (ij, -p_{\lambda\mu,ij} + ab, \lambda\mu).$$

Then $(\mathcal{M}, +, \cdot)$ is a completely simple semiring. Conversely every completely simple semiring is isomorphic to such a semiring.

The semiring constructed above is denoted by $\mathcal{M}(I, R, \Lambda; P)$ and is called Rees matrix semiring.

Corollary 1.2 [1]. *Let $\mathcal{M}(I, R, \Lambda; P)$ be a Rees matrix semiring. Then $ab + p_{\lambda, i} = p_{\lambda, i} + ab$ for all $i \in I, \lambda \in \Lambda$ and $a, b \in R$.*

In Section 2, we characterize all balanced order relations on a completely simple semiring. For a completely simple semiring, normal extension and essential extension are defined and their properties are studied in Section 3. The notion of normal hull $\Phi(S)$ of a completely simple semiring S is defined in Section 4. Finally, we show that $\Phi(S)$ admits a natural Rees matrix semigroup representation if S is given a Rees matrix semiring representation.

2. BALANCED ORDER RELATIONS ON REES MATRIX SEMIRINGS

Let $(G, +)$ be a group with identity 0, I and Λ be index sets such that $I \cap \Lambda = \{o\}$ and $P = (p_{\lambda, i})_{\lambda \in \Lambda, i \in I}$ be a matrix over G such that $p_{\lambda, o} = p_{o, i} = 0$ for all $\lambda \in \Lambda$ and $i \in I$. On $S = I \times R \times \Lambda$ define addition by

$$(i, a, \lambda) + (j, b, \mu) = (i, a + p_{\lambda, j} + b, \mu) \quad \text{for all } (i, a, \lambda), (j, b, \mu) \in S.$$

Then straightforward calculations show that $(S, +)$ is a semigroup and is denoted by $S = \mathcal{M}(I, G, \Lambda; P)$.

Definition 2.1. A relation ϱ on a semigroup (S, \cdot) is said to be stable if for $a, b \in S$; $a \varrho b$ implies $ac \varrho bc, ca \varrho cb$ for all $c \in S$; i.e., ϱ is stable under left and right multiplication. An order relation on a semigroup (S, \cdot) [a semiring $(S, +, \cdot)$] is a partial order relation on S which is stable in (S, \cdot) [as well as in $(S, +, \cdot)$].

Lemma 2.2. *Let $S = \mathcal{M}(I, G, \Lambda; P)$ be a semigroup and ϱ be a stable relation on S . Then*

- (i) $(i, a, \lambda) \varrho (j, b, \lambda)$ for some $\lambda \in \Lambda$ implies $(i, a, \mu) \varrho (j, b, \mu)$ for all $\mu \in \Lambda$.
- (ii) $(i, a, \lambda) \varrho (i, b, \mu)$ for some $i \in I$ implies $(j, a, \lambda) \varrho (j, b, \mu)$ for all $j \in I$.

Proof. (i) Since ϱ is a stable relation on S and $(i, a, \lambda) \varrho (j, b, \lambda)$, then we have $(i, a, \lambda) + (o, 0, \mu) \varrho (j, b, \lambda) + (o, 0, \mu)$, i.e., $(i, a, \mu) \varrho (j, b, \mu)$ for all $\mu \in \Lambda$.

(ii) This part follows in a similar way as (i) of this theorem. ■

Corollary 2.3. *Let $S = \mathcal{M}(I, G, \Lambda; P)$ be a semigroup and ϱ be a stable relation on S . Then $(i, a, \lambda) \varrho (j, b, \lambda)$ if and only if $(o, a, o) \varrho (o, b, o)$ if and only if $(o, a - b, o) \varrho (o, 0, o)$.*

Remark 2.4. Let $S = \mathcal{M}(I, G, \Lambda; P)$ be a semigroup and ϱ be a stable relation on S . Then $U_\varrho = \{a \in G : (o, a, o) \varrho (o, 0, o)\}$ satisfies

$$\begin{aligned} U_\varrho &= \{a \in G : (i, a, \lambda) \varrho (i, 0, \lambda) \text{ for some } i \in I \text{ and for some } \lambda \in \Lambda\} \\ &= \{a \in G : (i, a, \lambda) \varrho (i, 0, \lambda) \text{ for all } i \in I \text{ and for all } \lambda \in \Lambda\}. \end{aligned}$$

Now special properties of ϱ result in special properties of U_ϱ :

- (i) If ϱ is reflexive, then $0 \in U_\varrho \neq \emptyset$ and conversely.
- (ii) If ϱ is reflexive and transitive, then $(U_\varrho, +)$ is a submonoid of G .
- (iii) If ϱ is an equivalence relation (and hence a congruence), then $(U_\varrho, +)$ is a normal subgroup of $(G, +)$.
- (iv) If ϱ is a partial order, then $U_\varrho \cap (-U_\varrho) = \{0\}$, i.e., U_ϱ is a positive cone of ϱ .

Remark 2.5. Let ϱ be a transitive and stable relation on $S = \mathcal{M}(I, G, \Lambda; P)$ and $(i, a, \lambda) \varrho (j, b, \mu)$. Then we get

$$(i, -p_{\nu, j}, \nu) + (i, a, \lambda) + (k, -p_{\mu, k}, \lambda) \varrho (i, -p_{\nu, j}, \nu) + (j, b, \mu) + (k, -p_{\mu, k}, \lambda),$$

$$\text{i.e.,} \quad (i, -p_{\nu, j} + p_{\nu, i} + a + p_{\lambda, k} - p_{\mu, k} - b, \lambda) \varrho (i, 0, \lambda)$$

and hence $-p_{\nu, j} + p_{\nu, i} + a + p_{\lambda, k} - p_{\mu, k} - b \in U_\varrho$ for all $k \in I$ and $\nu \in \Lambda$.

Definition 2.6. Let ϱ be a stable relation on $S = \mathcal{M}(I, G, \Lambda; P)$. Define the relation ϱ_I on I by $i \varrho_I j$ if and only if $(i, a, \lambda) \varrho (j, b, \lambda)$ for some $a, b \in G$ and $\lambda \in \Lambda$ and likewise the relation ϱ_Λ on Λ by $\lambda \varrho_\Lambda \mu$ if and only if $(i, a, \lambda) \varrho (i, b, \mu)$ for some $a, b \in G$ and $i \in I$.

The relation ϱ is called balanced if $i \varrho_I j$ implies $(i, -p_{\lambda, i}, \lambda) \varrho (j, -p_{\lambda, j}, \lambda)$ for all $\lambda \in \Lambda$ and if $\lambda \varrho_\Lambda \mu$ implies $(i, -p_{\lambda, i}, \lambda) \varrho (i, -p_{\mu, i}, \mu)$ for all $i \in I$.

Remark 2.7. Let ϱ be a stable relation on $S = \mathcal{M}(I, G, \Lambda; P)$. Then it is interesting to note that if ϱ is reflexive, then ϱ_I and ϱ_Λ are reflexive. Also, if ϱ is balanced and antisymmetric, then $i \varrho_I j$ and $j \varrho_I i$ imply in particular $(i, 0, o) \varrho (j, 0, o)$ and $(j, 0, o) \varrho (i, 0, o)$; hence $(i, 0, o) = (j, 0, o)$ and ϱ_I is antisymmetric, too. In the similar way it follows that ϱ_Λ is antisymmetric. Again, if ϱ is balanced and transitive, then $i \varrho_I j$ and $j \varrho_I k$ imply $(i, 0, o) \varrho (j, 0, o)$ and $(j, 0, o) \varrho (k, 0, o)$. Hence $(i, 0, o) \varrho (k, 0, o)$, which shows that $i \varrho_I k$, i.e., ϱ_I is transitive. A similar argument shows that ϱ_Λ is transitive. Moreover, from Remark 2.5, it follows that $i \varrho_I j$ implies $-p_{\nu, j} + p_{\nu, i} - p_{\lambda, i} + p_{\lambda, j} \in U_\varrho$ for all $\lambda, \nu \in \Lambda$. Likewise $\lambda \varrho_\Lambda \mu$ implies $-p_{\lambda, i} + p_{\lambda, j} - p_{\mu, j} + p_{\mu, i} \in U_\varrho$ for all $i, j \in I$.

Now, we characterize all balanced order relations on a Rees matrix semiring $\mathcal{M}(I, R, \Lambda; P)$ in terms of the parameters I, R, Λ and P .

Definition 2.8. Let U be a positive cone of an order relation on $(R, +)$ of a skew-ring R satisfying $ac, ca \in U$ for all $a \in U$ and for all $c \in R$, ξ an order relation on I and η an order relation on Λ . The triple (ξ, U, η) is said to be an admissible triple of orders of I, R and Λ respectively if $(i, j) \in \xi$ implies $-p_{\mu,i} + p_{\mu,j} - p_{\lambda,j} + p_{\lambda,i} \in U$ for all $\lambda, \mu \in \Lambda$ and $(\lambda, \mu) \in \eta$ implies $-p_{\mu,i} + p_{\mu,j} - p_{\lambda,j} + p_{\lambda,i} \in U$ for all $i, j \in I$.

First we state the following Theorem.

Theorem 2.9 [5]. *Let $S = \mathcal{M}(I, G, \Lambda; P)$ be a completely simple semigroup and let $\pi(\varrho) = (\varrho_I, U_\varrho, \varrho_\Lambda)$ for any balanced order relation ϱ on S . Then π is an order preserving bijection from the set of all balanced orders on S onto the set of all admissible triples of orders on I, G and Λ , respectively.*

We now extend Theorem 2.9 from completely simple semigroup to completely simple semiring. In fact this is the main theorem in this section.

Theorem 2.10. *Let $S = \mathcal{M}(I, R, \Lambda; P)$ be a completely simple semiring. Then there is an order preserving one-one correspondence between the set of all balanced order relations on S onto the set of all admissible triples of orders of I, R and Λ respectively.*

Proof. Let $\mathcal{B}(S)$ denote the set of all balanced order relations on a completely simple semiring $S = \mathcal{M}(I, R, \Lambda; P)$ and $\mathcal{AT}(S)$ denote the set of all admissible triples of orders of I, R and Λ , respectively. Then $\mathcal{B}(S)$ and $\mathcal{AT}(S)$ are both lattices with respect to set inclusion.

We define $\rho : \mathcal{B}(S) \rightarrow \mathcal{AT}(S)$ by $\rho(\varrho) = (\varrho_I, U_\varrho, \varrho_\Lambda)$ for all $\varrho \in \mathcal{B}(S)$.

Since $(S, +)$ is a completely simple semigroup, we find from Theorem 2.9, that ρ is order preserving and injective.

To complete the proof it suffices to show that ρ is onto. For this let, (ξ, U, η) be an admissible triple of orders of I, R and Λ respectively. We define a relation ϱ on S by $(i, a, \lambda) \varrho (j, b, \mu)$ if and only if $(i, j) \in \xi, (\lambda, \mu) \in \eta$ and $-p_{\nu,i} + p_{\nu,j} + b + p_{\mu,k} - p_{\lambda,k} - a \in U$ for some $\nu \in \Lambda$ and $k \in I$. Then by Theorem 2.9, stated above ϱ is a balanced order relation on $(S, +)$. Moreover, by Theorem 2.9, we have $\varrho_I = \xi, \varrho_\Lambda = \eta$ and $U_\varrho = U$.

We now show that ϱ is stable under multiplication. For this let $(i, a, \lambda), (j, b, \mu), (k, c, \nu) \in S$ and $(i, a, \lambda) \varrho (j, b, \mu)$. Then $(i, j) \in \xi, (\lambda, \mu) \in \eta$ and $-p_{\nu',i} + p_{\nu',j} + b + p_{\mu,k'} - p_{\lambda,k'} - a \in U$ for some $\nu' \in \Lambda$ and $k' \in I$. Now, $(i, j) \in \xi$ implies $(ik, jk) \in \xi$. Similarly, $(\lambda\nu, \mu\nu) \in \eta$.

Again, $-p_{\nu',i} + p_{\nu',j} + b + p_{\mu,k'} - p_{\lambda,k'} - a \in U$ implies

$$(-p_{\nu,i} + p_{\nu,j} + b + p_{\mu,k} - p_{\lambda,k} - a)c \in U,$$

i.e., $bc - ac \in U$ [by (1.4) of Theorem 1.1]. This leads to,

$$-p_{\lambda\nu,ik} + p_{\lambda\nu,jk} + (-p_{\mu\nu,jk} + bc) + p_{\mu\nu,jk} - p_{\lambda\nu,jk} - (-p_{\lambda\nu,ik} + ac) \in U.$$

Hence $(ik, -p_{\lambda\nu,ik} + ac, \lambda\nu) \varrho (jk, -p_{\mu\nu,jk} + bc, \mu\nu)$. This implies

$$(i, a, \lambda)(k, c, \nu) \varrho (j, b, \mu)(k, c, \nu).$$

Similarly, we can show that $(k, c, \nu)(i, a, \lambda) \varrho (k, c, \nu)(j, b, \mu)$ and the proof is completed. \blacksquare

3. NORMAL EXTENSIONS

For the remaining part of this paper, let $E^+(S)$ denote the set of all additive idempotents of the semiring S and by $[E^+(S)]$ we mean the subsemiring of S generated by $E^+(S)$.

Definition 3.1. An ideal K of a completely simple semiring S is said to be a normal ideal of S , and S is said to be a normal extension of K , if (i) K is a completely simple semiring and (ii) $x' + K + x \subseteq K$ for all $x \in S$.

Here it is interesting to mention that the requirement that $x' + K + x \subseteq K$ for all $x \in S$, ensures that K has non-null intersection with every \mathcal{H}^+ -class of S and so is full, i.e., $E^+(S) \subseteq K$.

Notation 3.2. For $S = \mathcal{M}(I, R, \Lambda; P)$, we will write

$$\begin{aligned} \overline{P} &= \text{subskew-ring of } R \text{ generated by the entries of } P, \\ \mathcal{K} &= \text{set of normal ideals of } S, \\ \mathcal{N} &= \text{set of skew-ideals of } R \text{ containing } \overline{P} \end{aligned}$$

where \mathcal{K} and \mathcal{N} are both lattices with respect to set inclusion.

Similar to Rees matrix semigroup, one can easily show the following lemma.

Lemma 3.3. If $S = \mathcal{M}(I, R, \Lambda; P)$, then $[E^+(S)] = \mathcal{M}(I, \overline{P}, \Lambda; P)$.

Definition 3.4. A completely simple semiring S is said to be an essential extension of a normal ideal K if the restriction to K of any non-trivial congruences on S is non-trivial.

Theorem 3.5. *Let $S = \mathcal{M}(I, R, \Lambda; P)$.*

(i) *The mappings*

$$K \longrightarrow K^* = \{a \in R : (o, a, o) \in K\},$$

$$N \longrightarrow N^* = \{(i, a, \lambda) \in S : a \in N\},$$

are mutually inverse lattice isomorphisms of \mathcal{K} and \mathcal{N} .

(ii) *S is an essential extension of $K \in \mathcal{K}$ if and only if R is an essential extension of K^* .*

Proof. (i) By Theorem 2.7 (i) [4], we see that $(K^*, +)$ is a normal subgroup of R containing \bar{P} . To show K^* is a skew-ideal of R , let $a \in K^*$ and $r \in R$. Then $(o, a, o) \in K$. Now, $(o, r, o) \in S$. Since K is an ideal of S , so it follows that $(o, a, o)(o, r, o) \in K$, i.e., $(o, ar, o) \in K$. This implies $ar \in K^*$. Similarly, we can show that $ra \in K^*$. Hence K^* is a skew-ideal of R containing \bar{P} .

Again, if N is a skew-ideal of R then by Theorem 2.7 (i) [4], N^* is a normal subsemigroup of $(S, +)$. To show N^* is a normal ideal of S , let $(i, a, \lambda) \in N^*$ and $(j, r, \mu) \in S$. Then $(i, a, \lambda)(j, r, \mu) = (ij, -p_{\lambda\mu, ij} + ar, \lambda\mu) \in N^*$, since $-p_{\lambda\mu, ij} + ar \in N$. Similarly, we can show that $(j, r, \mu)(i, a, \lambda) \in N^*$. Consequently, N^* is a normal ideal of S .

From Theorem 2.7 (i) [4], it follows that $(K^*)^* = K$ and $(N^*)^* = N$. Hence the theorem.

(ii) This part follows from Theorem 2.7 (ii) [4]. ■

From Theorem 3.5 (i) we can conclude that any normal ideal is completely determined by its intersection with H_e^+ , $e = (o, 0, o)$. Since the concept of a normal ideal is quite independent of any particular representation as a Rees matrix semiring, we have

Corollary 3.6. *Let K be a normal ideal of a completely simple semiring S . Then K is determined by its intersection with any \mathcal{H}^+ -class of S .*

4. THE NORMAL HULL

In this section we define the normal hull of a completely simple semiring. In this regard, we point out that if S is a completely simple semiring and $e \in E^+(S)$, then $e + S + e$ is a subskew-ring of S and conversely for any subskew-ring R of S there exists $f \in E^+(S)$ such that $R = f + S + f$.

Lemma 4.1. *Let S be a completely simple semiring and $a, b \in S$. Then $(a+b)' = (a+b)^0 + b' + (b+a)^0 + a' + (a+b)^0$.*

Proof. This follows from Lemma 3.2 [4]. ■

In the next lemma we define two mappings which will be useful in the remaining part of this paper.

Lemma 4.2. *Let S be a completely simple semiring and $e, f \in E^+(S)$. Then the mapping*

$$\phi_{e,f} : x \rightarrow (f + e)^0 + x + f \quad (x \in H_e^+)$$

is an isomorphism of H_e^+ onto H_f^+ with inverse

$$(4.1) \quad \phi_{e,f}^{-1} : z \rightarrow e + z + (f + e)^0 \quad (z \in H_f^+).$$

Proof. Clearly, $\phi_{e,f}$ is a mapping from H_e^+ to H_f^+ . By Theorem 3.3 [4], we at once have the mapping in (4.1) is the inverse of $\phi_{e,f}$ and $\phi_{e,f}$ is an isomorphism from $(H_e^+, +)$ onto $(H_f^+, +)$.

To show $\phi_{e,f}$ is a homomorphism under multiplication, let $x, y \in H_e^+$. Now,

$$\begin{aligned} & (x\phi_{e,f})(y\phi_{e,f}) \\ &= \left((f + e)^0 + x + f\right)\left((f + e)^0 + y + f\right) \\ &= \left((f + e)^0 + x\right)\left((f + e)^0 + y\right) + \left((f + e)^0 + x\right)f + f\left((f + e)^0 + y\right) + f^2 \\ &= (f + e)^0 + (f + e)^0 e + e(f + e)^0 + xy + f + ef + ef + f + fe + fe + f \\ &= (f + e)^0 + fe + e + ef + e + xy + f + ef + f + fe + f \\ &= (f + e)^0 + xy + f \\ &= (xy)\phi_{e,f}. \end{aligned}$$

Consequently, the mapping $\phi_{e,f}$ is an isomorphism. ■

Notation 4.3. Let $S = \mathcal{M}(I, R, \Lambda; P)$ be a completely simple semiring. Then for all $i \in I$ and $\lambda \in \Lambda$, we define $\theta_{p_{\lambda,i}} : R \rightarrow R$ by $x\theta_{p_{\lambda,i}} = -p_{\lambda,i} + x + p_{\lambda,i}$, for all $x \in R$. Then it is easy to verify that $\theta_{p_{\lambda,i}}$ is an automorphism of R . For any skew-ring R , we will denote by $\mathcal{A}(R)$, the automorphism group of R .

Lemma 4.4. *Let S be a completely simple semiring. Then for any three elements $e, f, g \in E^+(S)$, $\phi_{e,g}\theta_{p_{(g+e)^0, (f+g)^0}}\phi_{g,f} = \phi_{e,f}$. Moreover, if $e\mathcal{L}^+g$ or $g\mathcal{R}^+f$, then $\phi_{e,g}\phi_{g,f} = \phi_{e,f}$.*

Proof. Follows from Lemma 3.5 [4]. ■

Notation 4.5. For any completely simple semiring S , let

$$\Phi(S) = \bigcup \left\{ \mathcal{A}(H_e^+) : e \in E^+(S) \right\}.$$

We define a binary operation \star on $\Phi(S)$ by

$$(4.2) \quad \alpha \star \beta = \phi_{(e+f)^0, e} \alpha \phi_{e, f} \beta \phi_{f, (e+f)^0}$$

for all $\alpha \in \mathcal{A}(H_e^+)$, $\beta \in \mathcal{A}(H_f^+)$. Here we note that $\alpha \star \beta \in \mathcal{A}(H_{(e+f)^0}^+)$ and thus $\Phi(S)$ forms a semigroup.

We now prove that $\Phi(S)$ is a completely simple semigroup with respect to the operation defined in (4.2) by establishing an isomorphism with a Rees matrix semigroup of the following form.

Definition 4.6. For a completely simple semiring $S = \mathcal{M}(I, R, \Lambda; P)$, the completely simple semigroup $\mathcal{M}(I, \mathcal{A}(R), \Lambda; P^*)$ where $p_{\lambda, i}^* = \theta_{p_{\lambda, i}}$ is defined to be the automorphism semigroup of the semiring S and it is denoted by $\mathcal{A}(S)$.

Lemma 4.7. *Let S be a completely simple semiring. Then the mapping*

$$\psi : \alpha \rightarrow ((e+g)^0, \phi_{g, e} \alpha \phi_{e, g}, (g+e)^0) \quad \left(\alpha \in \mathcal{A}(H_e^+) \subseteq \Phi(S); e, g \in E^+(S) \right),$$

is an isomorphism of $\Phi(S)$ onto $\mathcal{A}(S)$.

Proof. The proof follows from Theorem 3.10 [4]. ■

Definition 4.8. For any completely simple semiring S , the (completely simple) semigroup $\Phi(S)$ is called the normal hull of S .

Corollary 4.9. *Let $S = \mathcal{M}(I, R, \Lambda; P)$, $S' = \mathcal{M}(I', R', \Lambda'; P')$ be two isomorphic completely simple semirings. Then $\mathcal{A}(S) \cong \mathcal{A}(S')$.*

Proof. Since the definition of $\Phi(S)$ is independent of the matrix representation of S , $\Phi(S)$ and $\Phi(S')$ are isomorphic. Hence, by Lemma 4.7, $\mathcal{A}(S) \cong \Phi(S) \cong \Phi(S') \cong \mathcal{A}(S')$. ■

Acknowledgement

The author is grateful to the anonymous referee for his valuable suggestions which definitely improved the presentation of this paper.

REFERENCES

- [1] M.K. Sen, S.K. Maity and H.J. Weinert, *Completely simple semirings*, Bull. Cal. Math. Soc **97** (2005) 163–172.
- [2] M.K. Sen, S.K. Maity and K.P. Shum, *On completely regular semirings*, Bull. Cal. Math. Soc **98** (2006) 319–328.
- [3] M.P. Grillet, *Semirings with a completely simple additive semigroup*, J. Austral. Math. Soc.(Series A) **20** (1975) 257–267. doi:10.1017/S1446788700020607
- [4] M. Petrich and N. Reilly, *The normal hull of a completely simple semigroup*, J. Algebra **81** (1983) 232–257. doi:10.1016/0021-8693(83)90218-1
- [5] S.M. Goberstein, *Balanced order relations on completely simple semigroups*, Semigroup Forum **30** (1984) 121–124. doi:10.1007/BF02573443

Received 22 August 2013

Revised 30 October 2013