*Discussiones Mathematicae*
*Differential Inclusions, Control and Optimization 31 (2011) 71–90*

# INTEGRO-DIFFERENTIAL EQUATIONS ON TIME SCALES WITH HENSTOCK-KURZWEIL DELTA INTEGRALS

Aneta Sikorska-Nowak

*Adam Mickiewicz University*
*Faculty of Mathematics and Computer Science*
*Poznań, Poland*

**Abstract**

In this paper we prove existence theorems for integro – differential equations

$$\begin{aligned} & x^{\Delta}(t) = f(t, x(t), \textstyle\int_0^t k(t,s,x(s))\Delta s), \\ & x(0) = x_0 \end{aligned} \qquad t \in I_a = [0,a] \cap T, a \in R_+,$$

where $T$ denotes a time scale (nonempty closed subset of real numbers R), $I_a$ is a time scale interval. Functions $f, k$ are Carathéodory functions with values in a Banach space $E$ and the integral is taken in the sense of Henstock-Kurzweil delta integral, which generalizes the Henstock-Kurzweil integral.

Additionally, functions $f$ and $k$ satisfy some boundary conditions and conditions expressed in terms of measures of noncompactness.

Moreover, we prove an Ambrosetti type lemma on a time scale.

**Keywords:** integro-differential equations, nonlinear Volterra integral equation, time scales, Henstock-Kurzweil delta integral, HL delta integral, Banach space, fixed point, measure of noncompactness, Carathéodory solutions.

**2010 Mathematics Subject Classification:** 34D09, 34D99.

## 1. Introduction

A time scale $T$ is a nonempty closed subset of real numbers R, with the subspace topology inherited from the standard topology of R. Thus, R;Z;N and the Cantor set are the examples of time scales, while Q and $(0;1)$ are not time scales.

Time scales (or a measure chain) was introduced by Hilger in his Ph.D. thesis in 1988, [22]. It was created in order to unify the study of differential and difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies and helps avoid proving results twice – once for differential equations and once again for difference equations. The general idea is to prove a result for a dynamic equation, where the domain of the unknown function is a so-called time scale $T$, which may be an arbitrary closed subset of the reals. This way results not only related to the set of real numbers or set of integers but those pertaining to more general time scales are obtained.

Since the time Hilger formed the definitions of a derivative and integral on a time scale, several authors have extended on various aspects of the theory [1, 2, 4, 8, 9, 14, 17, 20, 23, 24]. Time scales have been shown to be applicable to any field that can be described by means of discrete or continuous models. In recent years there have been many research activities on dynamic equations, in order to unify the results concerning difference equations and differential equations [3, 6, 15, 26].

The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [25]), i.e., when $T =$ R; $T =$ N, $T = q^{N_0} = \{q^t : t \in N_0\}$, where $q > 1$. Dynamic equations on a time scale have an enormous potential for applications such as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in, say, winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population (see [8]). There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics. A recent cover story article in New Scientist [40] discusses several possible applications. Since then several authors have expounded on various aspects of this new theory [9]. The book on the subject of time scale, i.e., measure chain, by Bohner and Peterson [8] summarizes and organizes much of time scale calculus.

In this paper we consider an integro-differential equation. As it is know, ordinary integro-differential equations, an extreme case of integro-differential equations on time scales, find many applications in various mathematical problems: see Corduneanu's book [14] and references therein for details. In addition, the existence of extremal solutions of ordinary integro-differential equations and impulsive integro-differential equations have been studied extensively in [15, 16, 17, 18, 19, 31, 32, 33, 39, 46].

In [43] the authors extended such results to the integro-differential equations on time scales and therefore obtained corresponding criteria which can be employed to study the difference equation of Volterra type [26, 46], q-difference equations of Volterra type, etc.

In [44] the authors proved a new comparison result and developed the monotone iterative technique to show the existence of extremal solutions of the periodic boundary value problems of nonlinear integro-differential equation on time scales.

We extend the results by proving the existence of the Carathéodory type solution of the problem

$$(1)\qquad \begin{aligned} &x^{\Delta}(t) = f(t, x(t), \textstyle\int_0^t k(t, s, x(s))\Delta s), \\ &x(0) = x_0 \end{aligned} \qquad t \in I_a = [0, a] \cap T, a \in R_+,$$

where $f : I_a \times E \times E \to E$, $k : I_a \times I_a \times E \to E$, $T$ denotes a time scale, $0 \in T$, $I_a$ denotes a time scale interval, $(E, \|\cdot\|)$ is a Banach space.

We use a new type of integrals on time scales (the Henstock-Kurzweil delta integral, HL delta integral), which lets us consider the wider class of the function than so far.

The Henstock-Kurzweil delta integral contains the Riemann delta, the Lebesgue delta and the Bochner delta integrals as special cases. These integrals will enable time scale researchers to study more general dynamic equations. A. Petterson and B. Thomson in [35] show that there are highly oscillatory functions that are not delta integrable on a time scale, but are the Henstock-Kurzweil delta integrable.

Let us remark that the existence of the Henstock-Kurzweil integral over $[a, b]$ implies the existence of such integrals over all subintervals of $[a, b]$ but not for all measurable subsets of this interval, so the theory of such integrals on $T$ does not follow from general theory on R.

In [12] M. Cichoń introduced a definition of the Henstock-Kurzweil delta integral ($\Delta$-HK integral) and HL delta integral ($\Delta$-HL integral) on Banach spaces for checking the existence of solutions of differential (or: dynamic) equations in Banach spaces.

Dynamic equations in Banach spaces constitute quite a new research area.

In [13] the authors initiated the study of dynamic equations on Banach spaces and considered the Cauchy dynamic problem

$$\begin{array}{l} x^{\Delta}(t) = f(t, x(t)) \\ x(0) = x_0 \end{array}, \quad t \in T.$$

They offer the existence of the weak solution of this dynamic Cauchy problem on an infinite time scale.

The existence theorems for the Carathéodory type solution, presented in this paper, are new not only for Banach valued functions, but also for real valued functions.

Adopting the Mönch fixed point theorem [34] and the techniques of the theory of the measure of noncompactness [7], we are able to study the existence results for problem (1).

Moreover, we prove an Ambrosetti type lemma on a time scale.

## 2. PRELIMINARIES

Let $(E, \|\cdot\|)$ be a Banach space. Denote, by $C(I_a, E)$, the set of all continuous bounded functions from $I_a$ to $E$ endowed with the topology of almost uniform convergence (i.e., uniform convergence on each closed bounded subsets of $I_a$) and by $C_{rd}(I_a, E)$ denote the set of all rd-continuous bounded functions from $I_a$ to $E$ endowed with the same topology.

By $\mu_{\Delta}$ we denote the Lebesgue measure on $T$. For a precise definition and basic properties of this measure we refer the reader to [10].

This part is divided into three sections.

**I.** To let the reader understand the so-called dynamic equations and follow this paper easily, we present some preliminary definitions and notations of time scales which are very common in the literature (see [1, 8, 9, 22, 23, 24] and references therein).

A time scale $T$ is a nonempty closed subset of real numbers R, with the subspace topology inherited from the standard topology of R.

If $a, b$ are points in $T$, we denote by $I = [a, b] = \{t \in T : a \leq t \leq b\}$ and $I_a = \{t \in T : 0 \leq t \leq a\}$. Other types of intervals are approached similarly. By a subinterval $I_b$ of $I_a$ we mean the time scale subinterval.

**Definition 2.1.** The forward jump operator $\sigma : T \to T$ and the backward jump operator $\rho : T \to T$ are defined by $\sigma(t) = \inf\{s \in T : s > t\}$ and $\rho(t) = \sup\{s \in T : s < t\}$, respectively.

We put $\inf \emptyset = \sup T$ (i.e., $\sigma(M) = M$ if $T$ has a maximum $M$) and $\sup \emptyset = \inf T$ (i.e., $\rho(m) = m$ if $T$ has a minimum $m$).

The jump operators $\sigma$ and $\rho$ allow the classification of points in time scale in the following way: $t$ is called right-dense, right scattered, left-dense, left scattered, dense and isolated if $\sigma(t) = t$, $\sigma(t) > t, \rho(t) = t$, $\rho(t) < t$, $\rho(t) = t = \sigma(t)$ and $\rho(t) < t < \sigma(t)$, respectively.

**Definition 2.2.** We say that $k : T \to E$ is right-dense continuous (rd-continuous) if $k$ is continuous at every right-dense point $t \in T$ and $\lim_{s \to t^-} k(s)$ exists and is finite at every left-dense point $t \in T$.

**Definition 2.3.** Fix $t \in T$. Let $f : J \to E$. Then, we define $\Delta$-derivative $f^{\Delta}(t)$ by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

**Remark 2.4.** The $\Delta$-derivative turns out that

(i) $f^{\Delta} = f'$ is the usual derivative if $T = \mathrm{R}$ and

(ii) $f^{\Delta} = \Delta f$ is the usual forward difference operator if $T = \mathrm{Z}$ and

(iii) $f^{\Delta} = D_q f$ is the $q$-derivative if $T = q^{N_0} = \{q^t : t \in \mathrm{N}_0\}$, $q > 1$.

Hence, the time scale allows us to consider the unification of differential, difference and $q$-difference equations as particular cases (but our results hold also for more exotic time scales which appear in mathematical biology or economics cf. [8, 9, 42], for instance).

**II.** As in classical case ([11], cf. [35] for real valued functions), we need to introduce of vector valued Henstock-Kurzweil $\Delta$-integrals and HL $\Delta$-integrals. Definitions and basic properties of non absolute integrals (HK $\Delta$-integral and HL $\Delta$-integral) were presented in [12].

We will use the notation $\eta(t) := \sigma(t) - t$, where $\eta$ is called the graininess function and $\vartheta(t) := t - \rho(t)$, where $\vartheta$ is called the left-graininess function.

We say that $\delta = (\delta_L, \delta_R)$ is a $\Delta$-gauge for time scale interval $[a, b]$ provided $\delta_L(t) > 0$ on $(a, b]$, $\delta_R(t) > 0$ on $[a, b)$, $\delta_L \geq 0$, $\delta_R \geq 0$ and $\delta_R \geq \eta(t)$ for all $t \in [a, b)$.

We say that a partition $D$ for a time scale interval $[a, b]$ given by $D = \{a = t_0 \leq \xi_1 \leq t_1 \leq ... \leq t_{n-1} \leq \xi_n \leq t_n = b\}$ with $t_i > t_{i-1}$ for $1 \leq i \leq n$ and $t_i, \xi_i \in T$ is $\delta$-fine if $\xi_i - \delta_L(\xi_i) \leq t_{i-1} < t_i \leq \xi_i + \delta_R(\xi_i)$ for $1 \leq i \leq n$.

**Definition 2.5.** A function $f : [a, b] \to E$ is the Henstock-Kurzweil-$\Delta$-integrable on $[a, b]$($\Delta$-HK integrable in short) if there exists a function $F : [a, b] \to E$, defined on the subintervals of $[a, b]$, satisfying the following property: given $\varepsilon > 0$ there exists a positive function $\delta$ on $[a, b]$ such that $D = \{[u, v], \xi\}$ is $\delta$-fine division of a $[a, b]$, we have

$$\left\| \sum_D f(\xi)(v - u) - (F(v) - F(u)) \right\| < \varepsilon.$$

**Definition 2.6.** A function $f : [a, b] \to E$ is the Henstock-Lebesgue-$\Delta$-integrable on $[a, b]$($\Delta$-HL integrable in short) if there exists a function $F : [a, b] \to E$, defined on the subintervals of $[a, b]$, satisfying the following property: given $\varepsilon > 0$ there exists a positive function $\delta$ on $[a, b]$ such that $D = \{[u, v], \xi\}$ is $\delta$-fine division of a $[a, b]$, we have

$$\sum_D \|f(\xi)(v - u) - (F(v) - F(u))\| < \varepsilon.$$

**Remark 2.7.** We note that, by triangle inequality, if $f$ is $\Delta$-HL integrable it is also $\Delta$-HK integrable. In general, the converse is not true. For real-valued functions the two integrals are equivalent.

It is well known that Henstock's Lemma plays an important role in the theory of the Henstock-Kurzweil integral in the real-valued case. On the other hand, in connection with the Henstock-Kurzweil integral for Banach space valued functions, S.S. Cao pointed out in [11] that Henstock's Lemma holds for the case of finite dimension, but it does not always hold for the case of infinite dimension.

In this paper we will use the definition of HL integral which satisfies Henstock's Lemma.

**Theorem 2.8** [35] (Henstock's Lemma)**.** *If $f$ is the Henstock-Kurzweil $\Delta$-integrable on $[a, b]$ with primitive $F$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\delta$-fine division of $[a, b]$ we have*

$$\sum_D |f(\xi)(v-u) - (F(v) - F(u)| < \varepsilon.$$

Theorem 2.8 says that in the definition of the Henstock-Kurzweil delta integral for real valued functions [35], we may put the absolute value sign $|\cdot|$ inside the summation sign $\sum$. We know from [11] that this is no longer true if we replace $|\cdot|$ with $\|\cdot\|$, i.e., Henstock's Lemma is not satisfied by Henstock-Kurzweil integrable Banach valued functions. By the definition of HL integral, an HL integrable function with primitive $F$ satisfies Henstock's Lemma with $|\cdot|$ replaced with $\|\cdot\|$.

**Theorem 2.9** [12]**.** *If $f : [a, b] \to E$ is $\Delta$-HL integrable, then function $F(t) = (\Delta - HL) \int_0^t f(s)\Delta s$ is continuous at each point $t \in T$. Moreover, for every point $t$ of the continuity of $f$ we have $F^{\Delta}(t) = f(t)$.*

**Theorem 2.10.** *Suppose that $f_n : [a, b] \to E, n = 1, 2, \ldots$ is a sequence of $\Delta$-HL integrable functions satisfying the following conditions:*

1. *$f_n(x) \to f(x)$ $\mu_\Delta$ almost everywhere in $[a, b]$, as $n \to \infty$;*
2. *the set of primitives of $f_n, \{F_n(t)\}$, where $F_n(t) = \int_a^t f_n(s)\Delta s$, is uniformly $ACG_*$ in $n$;*
3. *the primitives $F_n$ are equicontinuous on $[a, b]$;*

*then, $f$ is $\Delta$-HL integrable on $[a, b]$ and $\int_a^t f_n \to \int_a^t f\mu_\Delta$ uniformly on $[a, b]$, as $n \to \infty$.*

The proof is similar to that of Theorem 7.6 in [30], see also ([38], Theorem 4).

**Theorem 2.11** [12] Mean Value Theorem**.** *For each $\Delta$-subinterval $[c, d] \subset [a, b]$, if the integral $(\Delta - HK) \int_c^d y(s)\Delta s$ exists, then we have*

$$(\Delta - HK) \int_c^d y(s)\Delta s \in \mu_\Delta([c, d]) \cdot \overline{conv}\ y([c, d]),$$

*where $\overline{conv}\, y([c, d])$ denotes the close convex hull of the set $y([c, d])$.*

**Theorem 2.12** [4] (Gronwall's inequality)**.** *Suppose that* $u, g, h \in C_{rd}(I_a, E)$ *and* $h \geq 0$. *Then,*

$$u(t) \leq g(t) + \int_0^t h(\tau)u(\tau)\Delta\tau, \quad \textit{for each} \quad t \in I_a$$

*implies*

$$u(t) \leq \left(g(t) + \int_0^t g(\tau)h(\tau)\Delta\tau\right) \exp\left(\int_0^t h(\tau)\Delta\tau\right), \quad \textit{for each} \;\; t \in I_a.$$

**III.** The Kuratowski measures of noncompactness is our fundamental tool in this paper.

For any bounded subset $A$ of $E$ we denote by $\alpha(A)$ the Kuratowski measure of noncompactness of $A$, i.e., the infimum of all $\varepsilon > 0$ such that there exists a finite covering of $A$ by sets of diameter smaller than $\varepsilon$.

The properties of the measure of noncompactness $\alpha$ are:

(i) if $A \subset B$ then $\alpha(A) \leq \alpha(B)$;

(ii) $\alpha(A) = \alpha(\bar{A})$, where $\bar{A}$ denotes the closure of $A$;

(iii) $\alpha(A) = 0$ if and only if $A$ is relatively compact;

(iv) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$;

(v) $\alpha(\lambda A) = |\lambda|\alpha(A)$ $(\lambda \in R)$;

(vi) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$;

(vii) $\alpha(convA) = \alpha(A)$, where $conv(A)$ denotes the convex hull of $A$;

(viii) $\alpha(A) < \delta(A)$, where $\delta(A) = \sup\limits_{x,y \in A} \{\|x - y\|\}$.

The lemma below is an adaptation of the corresponding result of Ambrosetti (see [5]).

**Lemma 2.13** [28]. *Let* $H \subset C(I_a, E)$ *be a family of strongly equicontinuous functions. Let* $H(t) = \{h(t) \in E,\ h \in H\}$, *for* $t \in I_a$ *and* $H(I_a) = \bigcup_{t \in I_a} H(t)$. *Then,*

$$\alpha_C(H) = \sup_{t \in I_a} \alpha(H(t)) = \alpha(H(I_a)),$$

*where* $\alpha_C(H)$ *denotes the measure of noncompactness in* $C(I_a, E)$, *and the function* $t \mapsto \alpha(H(t))$ *is continuous.*

We now gather some well-known definitions and results from the literature, which we will use through this paper.

**Definition 2.14.** A function $I_a \times E \times E \to E$ is $L^1$-Carathéodory if the following conditions hold:

1. the map $t \to f(t,x,y)$ is $\mu_\Delta$-measurable on $I_a$ for all $(x,y) \in E^2$;
2. the map $(x,y) \to f(t,x,y)$ is continuous for almost all $t \in I_a$.

**Definition 2.15.** A function $k : I_a \times I_a \times E \to E$ is $L^1$-Carathéodory if the following conditions hold:

1. the map $(t,s) \to k(t,s,y)$ is $\mu_\Delta$-measurable on $I_a \times I_a$ for all $y \in E$;
2. the map $y \to k(t,s,y)$ is continuous for almost all $(t,s) \in I_a^2$.

**Definition 2.16** [21]**.** A family $\mathbb{F}$ of functions $F$ is said to be uniformly absolutely continuous in the restricted sense on $A \subseteq [a,b]$ or in short uniformly $AC_*(A)$ if for every $\varepsilon > 0$ there exists $\eta > 0$ such that for every $F$ in $F$ and for every finite or infinite sequence of non-overlapping intervals $\{[a_i,b_i]\}$ with $a_i, b_i \in A$ and satisfying $\sum_i \mu_\Delta([a_i,b_i]) < \eta$, we have $\sum_i \omega(F,[a_i,b_i]) < \varepsilon$, where $\omega$ denotes the oscillation of $F$ over $[a_i,b_i]$ $(\omega(F,[a_i,b_i]) = \sup\{|F(r)-F(s)| : r,s \in [a_i,b_i]\})$.

A family $\mathbb{F}$ of functions $F$ is said to be uniformly generalized absolutely continuous in the restricted sense on $[a,b]$ or uniformly $ACG_*$ if $[a,b]$ is the union of a sequence of closed sets $A_i$ such that on each $A_i$ the function $F$ is uniformly $AC_*(A_i)$.

## 3. MAIN RESULTS

The proofs of the main theorems are based on the Mönch fixed point theorem.

**Theorem 3.1** [34]**.** *Let $D$ be a closed convex subset of $E$, and let $F$ be a continuous map from $D$ into itself. If for some $x \in D$ the implication*

$$\bar{V} = \overline{conv}(\{x\} \cup F(V)) \Rightarrow V \quad \text{is relatively compact}, \tag{2}$$

*holds for every countable subset $V$ of $D$, then $F$ has a fixed point.*

Now, we will consider the equivalently integral problem

$$x(t) = x_0 + \int_0^t f\left(z, x(z), \int_0^z k(z, s, x(s))\Delta s\right)\Delta z, \tag{3}$$

where $f : I_a \times E \times E \to E$, $k : I_a \times I_a \times E \to E$, $T$ denotes a time scale (nonempty closed subset of real numbers R), $0 \in T$, $I_a$ denotes a time scale interval, $(E, \|\cdot\|)$ is a Banach space and integrals are taken in the sense of HL $\Delta$-integrals.

To obtain the existence result it is necessary to define a notion of a solution.

An $ACG_*$ function $x : I_a \to E$ is said to be a Carathéodory solution to problem (1) if it satisfies the following conditions:

(i) $x(0) = x_0$

(ii) $x^{\Delta}(t) = f\left(t, x(t), \int_0^t k(t, s, x(s))\Delta s\right)$ for $\mu_\Delta$ a.e. $t \in I_a$.

A continuous function $x : I_a \to E$ is said to be a solution to problem (3) if it satisfies $x(t) = x_0 + \int_0^t f\left(z, x(z), \int_0^z k(z, s, x(s))\Delta s\right)\Delta z$, for every $t \in I_a$.

Because we consider a new type of integrals and a new type of solutions is necessary to prove that each solution $x$ to problem (1) is equaivalent to the solutions to problem (3).

Let $x$ be a continuous solution to (1). By definition, $x$ is an $ACG_*$ function and $x(0) = x_0$. Since, for $\mu_\Delta$ a.e $t \in I_a$, $x^{\Delta}(t) = f(t, x(t), \int_0^t k(t, s, x(s))\Delta s)$ and the last is $\Delta$-HL integrable, so it is differentiable $\mu_\Delta$ a.e. Moreover, $\int_0^t f(z, x(z), \int_0^z k(z, s, x(s)\Delta s)\Delta z = \int_0^t x^{\Delta}(s)\Delta s = x(t) - x_0$. Thus (3) is satisfied.

Now assume that $y$ is an $ACG_*$ function and it is clear that $y(0) = x_0$. By the definition of HL $\Delta$-integrals, there exists an $ACG_*$ function $G$ such that $G(0) = x_0$ and $G^{\Delta}(t) = f(t, y(t), \int_0^t k(t, s, y(s))\Delta s) \quad \mu_\Delta$ a.e.

Hence,

$$\begin{aligned} y(t) &= x_0 + \int_0^t f(z, y(z), \int_0^z k(z, s, y(s))\Delta s)\Delta z \\ &= x_0 + \int_0^t G^{\Delta}(s)\Delta s = x_0 + G(t) - G(0) = G(t). \end{aligned}$$

We obtain $y = G$ and then $y^{\Delta}(t) = f(t, y(t), \int_0^t k(t, s, y(s))\Delta s)$.

Let

$$B = \{x \in E : \|x\| \leq \|x_0\| + p, \quad p > 0\},$$

$$\tilde{B} = \{x \in C(I_a, E) : x(0) = x_0,\ \|x\| \leq \|x_0\| + p, \quad p > 0\}.$$

Moreover, let $F(x)(t) = x_0 + \int_0^t f(z, x(z), \int_0^z k(z, s, x(s))\Delta s)\Delta z,\ t \in I_a$, $K = \{F(x) :\ x \in \tilde{B}\}$, $K_1 = \{\int_0^z k(z, s, x(s))\Delta s : z \in [0, t], t \in [0, a], x \in \tilde{B}\}$.

**Theorem 3.2.** *Assume that for each uniformly $ACG_*$ function $x : I_a \to E$ the functions: $k(\cdot, s, x(s)), f\big(\cdot, x(\cdot), \int_0^{(\cdot)} k(\cdot, s, x(s))\Delta s\big)$ are $\Delta$-HL integrable, $f$ and $k$ are $L^1$-Carathéodory functions. Suppose that there exist constants $d_1, d_2, d_3 > 0$ such that*

$$\alpha(f(I, A, C)) \leq d_1 \cdot \alpha(A) + d_2 \cdot \alpha(C), \tag{4}$$

*for each time scale interval $I \subset I_a$ and for each subset $A, C$ of $B$,*

$$\alpha(k(I, I, X)) \leq d_3 \cdot \alpha(X) \tag{5}$$

*for each subset $X$ of $B$ and $I \subset I_a$, where $f(I, A, C) = \{f(t, x_1, x_2) : (t, x_1, x_2) \in I \times A \times A\}, k(I, I, X) = \{k(t, s, x) : (t, s, x) \in I \times I \times A\}$. Moreover, let $K$ and $K_1$ be equicontinuous, equibounded and uniformly $ACG_*$ on $I_a$. Then, there exists a Carathéodory type solution to problem* (1) *on $I_c$, for some $0 < c \leq a$ and $0 < c \cdot d_1 + c^2 \cdot d_2 \cdot d_3 < 1$.*

***Proof.*** Fix an arbitrary $p \geq 0$. Put $B = \{x \in E : \|x\| \leq \|x_0\| + p, p > 0\}$, $\tilde{B} = \{x \in C(I_c, E) : x(0) = x_0,\ \|x\| \leq \|x_0\| + p, p > 0\}$, where $c$ will be given below.

Recall that a set $K$ of continuous functions $F(x) \in K$ defined on a time scale interval $I_a$ is equicontinuous on $I_a$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\big\|F(x)(t) - F(x)(\tau)\big\| < \varepsilon$ for all $x \in \tilde{B}$ whenever $|t - \tau| < \delta$, $t, \tau \in I_a$, for each $F(x) \in K$. Thus, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\big\|\int_\tau^t f(z, x(z), \int_0^z k(z, s, x(s))\Delta s)\Delta z\big\| < \varepsilon$ for all $x \in \tilde{B}$ whenever $|t - \tau| < \delta$ and $t, \tau \in [0, a]$. As a result, there exists a number $c$, $0 < c \leq a$ such that

$$\left\| \int_0^t f\left(z, x(z), \int_0^z k(z, s, x(s))\Delta s\right) \Delta z \right\| \leq p, \quad t \in I_c \ \text{ and } \ x \in \tilde{B}.$$

We now show that the operator $F$ is well defined and maps $\tilde{B}$ into $\tilde{B}$.

$$\|F(x)(t)\| = \left\| x_0 + \int_0^t f\left(z, x(z), \int_0^z k(z,s,x(s))\Delta s\right)\Delta z \right\|$$
$$\leq \|x_0\| + \left\| \int_0^t f\left(z, x(z), \int_0^z k(z,s,x(s))\Delta s\right)\Delta z \right\| \leq \|x_0\| + p.$$

We will show that the operator $F$ is continuous. Let $x_n \to x$ in $\tilde{B}$. Then,

$$\begin{aligned}
&\|F(x_n) - F(x)\| \\
&= \sup_{t\in I_a} \left\| \int_0^t f\Big(z, x_n(z), \int_0^z k(z,s,x_n(s))\Delta s\Big)\Delta z \right. \\
&\qquad \left. - \int_0^t f\Big(z, x(z), \int_0^z k(z,s,x(s))\Delta s\Big)\Delta z \right\| \\
&= \sup_{t\in I_a} \left\| \int_0^t \left[ f\Big(z, x_n(z), \int_0^z k(z,s,x_n(s))\Delta s\Big) \right.\right. \\
&\qquad \left.\left. - f\Big(z, x(z), \int_0^z k(z,s,x(s))\Delta s\Big)\right]\Delta z \right\|.
\end{aligned}$$

Since $k$ is the Carathéodory's function and $x_n \to x$ in $\tilde{B}$, $k(z,s,x_n(s)) \to k(z,s,x(s))\mu_\Delta$ a.e. on $I_a$ and using Theorem 2.10 (see our assumption on $K_1$) we obtain $\int_0^z k(z,s,x_n(s))\Delta s \to \int_0^z k(z,s,x(s))\Delta s$ $\mu_\Delta$ a.e. on $I_a$.

Moreover, because $f$ is the Carathéodory's function, we have

$$f\left(z, x_n(z), \int_0^z k(z,s,x_n(s))\Delta s\right) \to f\left(z, x(z), \int_0^z k(z,s,x(s))\Delta s\right) \ \mu_\Delta$$

a.e. on $I_a$.

Thus, Theorem 2.10 implies $\|F(x_n) - F(x)\| \to 0$.

Suppose that $V \subset \tilde{B}$ satisfies the condition $\bar{V} = \overline{conv}(\{x\} \cup F(V))$. We will prove that $V$ is relatively compact and so (3.1) is satisfied. Since $V \subset \tilde{B}$, $F(V) \subset K$. Then, $V \subset \overline{V} = \overline{conv}(\{x\} \cup F(V))$ is equicontinuous. By Lemma 2.14, $t \mapsto v(t) = \alpha(V(t))$ is continuous on $I_c$.

For fixed $t \in I_c$ we divide the interval $[0,t]$ into $m$ parts in the following way $t_0 = 0$,

$$t_1 = \sup_{s\in I_a}\{s : s \geq t_0, s - t_0 < \delta\}, \quad t_2 = \sup_{s\in I_a}\{s : s > t_1, s - t_1 < \delta\}, \ldots,$$
$$t_n = \sup_{s\in I_a}\{s : s > t_{n-1}, s - t_{n-1} < \delta\}.$$

Since $T$ is closed, $t_i \in I_a$. If some $t_{i+1} = t_i$, then $t_{i+2} = \inf\{t \in T : t > t_{i+1}\}$.
For fixed $z \in [0,t]$ we divide the interval $[0,z]$ into $m$ parts: $z_0 = 0$,

$$z_1 = \sup_{s\in[0,t]} \{s : s \geq z_0, s - z_0 < \delta\}, \quad z_2 = \sup_{s\in[0,t]} \{s : s > z_1, s - z_1 < \delta\}, \ldots,$$

$$z_n = \sup_{s\in[0,t]} \{s : s > z_{n-1}, s - z_{n-1} < \delta\}$$

such that $\mu_\Delta(I_j) = \frac{jz}{m}$, $j = 0, 1, \ldots, m$, $I_j = [z_j,\, z_{j+1}]$.
Let $V([z_j, z_{j+1}]) = \{u(s) : u \in V, z_j \leq s \leq z_{j+1},\ j = 0, 1, \ldots, m-1\}$. By Lemma 2.14 and the continuity of $v$ there exists $s_j \in I_j = [z_j, z_{j+1}]$ such that

$$\alpha(V([z_j, z_{j+1}])) = \sup\{\alpha(V(s)) : z_j \leq s \leq z_{j+1}\} := v(s_j).$$

By Theorem 2.11 and the properties of the $\Delta$-HL integral we have for $x \in V$

$$\begin{aligned}
F(x)(t) &= x_0 + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} f\left(z, x(z), \sum_{j=0}^{m-1} \int_{z_j}^{z_{j+1}} k(z,s,x(s))\Delta s\right)\Delta z \\
&\in x_0 + \sum_{i=0}^{m-1} \mu_\Delta(J_i)\, \overline{conv}\, f\left(J_i, V(J_i), \sum_{j=0}^{m-1} (z_{j+1} - z_j)\, \overline{conv}\, k(I_j, I_j, V(I_j))\right),
\end{aligned}$$

where $J_i = [t_i, t_{i+1}]$, $i = 0, 1, \ldots, m-1$.
Using (4), (5) and properties of the measure of noncompactness we obtain

$$\begin{aligned}
&\alpha(F(V)(t)) \leq \\
&\leq \sum_{i=0}^{m-1} \mu_\Delta(J_i)\alpha\left(f\left(J_i, V(J_i), \sum_{j=0}^{m-1} \mu_\Delta(I_j)\, \overline{conv}\, k(I_j, I_j, V(I_j)\right)\right) \\
&\leq \sum_{i=0}^{m-1} \mu_\Delta(J_i)\alpha\left(f\left(J_i, V(J_i), \sum_{j=0}^{m-1} \mu_\Delta(I_j)\, \overline{conv}\, k(I_j, I_j, V(I_j)\right)\right) \\
&\leq \sum_{i=1}^{m-1} \mu_\Delta(J_i) \cdot d_1 \cdot \alpha(V(J_i)) \\
&\quad + \sum_{i=1}^{m-1} \mu_\Delta(J_i) \cdot d_2 \cdot \alpha\left(\sum_{j=0}^{m-1} \mu_\Delta(I_j) \cdot \overline{conv}\, k(I_j, I_j, V(I_j)\right)
\end{aligned}$$

$$\leq \sum_{i=1}^{m-1} \mu_\Delta(J_i) \cdot d_1 \cdot \alpha(V(I_c)) + \sum_{i=1}^{m-1} \mu_\Delta(J_i) \cdot d_2 \cdot \sum_{j=0}^{m-1} \mu_\Delta(I_j) \cdot \alpha(k(I_j, I_j, V(I_j))$$
$$\leq \alpha(V(I_c)) \cdot d_1 \cdot c + \sum_{i=1}^{m-1} \mu_\Delta(J_i) \cdot d_2 \cdot \sum_{j=0}^{m-1} \mu_\Delta(I_j) \cdot d_3 \cdot \alpha(V(I_j))$$
$$\leq \alpha(V(I_c)) \cdot d_1 \cdot c + \alpha(V(I_c)) \cdot d_2 \cdot d_3 \cdot c^2 = \alpha(V(I_c)) \left(d_1 \cdot c + d_2 \cdot d_3 \cdot c^2\right).$$

Since $\bar{V} = \overline{conv}(\{x\} \cup F(V))$, we obtain $\alpha\left(V(t)\right) = \alpha\left(\overline{conv}\left(\{x\} \cup F(V(t))\right)\right)$ so $\alpha(V(t)) \leq \alpha(V(I_c))\left(d_1 \cdot c + d_2 \cdot d_3 \cdot c^2\right)$, for $t \in I_c$.

Using Lemma 2.14 we have

$$\alpha(V(I_c)) \leq \alpha(V(I_c))(d_1 \cdot c + d_2 \cdot d_3 \cdot c^2).$$

Since $0 < c \cdot d_1 + c^2 \cdot d_2 \cdot d_3 < 1$ we obtain $v(t) = \alpha(V(t)) = 0$ for $t \in I_c$.

Using the Arzelá-Ascoli‘s theorem [44] we deduce that $V$ is relatively compact.

By Theorem 3.1 the operator $F$ has a fixed point. This means that there exists a Carathéodory's solution to problem (1).

**Theorem 3.3.** *Assume that for each uniformly $ACG^*$ function $x : I_a \to E$, the functions $k(\cdot, s, x(s)), f\left(\cdot, x(\cdot), \int_0^{(\cdot)} k(\cdot, s, x(s))ds\right)$ are $\Delta$-HL-integrable and $k, f$ are Carathéodory's functions. Suppose that there exists a constant $d > 0$ and a continuous function $c_1 : I_a \to R_+$ such that*

$$\alpha(f(I, A, C)) \leq d \cdot \alpha(C), \quad \text{for each} \quad A, C \subset B, I \subset I_a, \tag{6}$$

$$\alpha(k(I, I, X)) \leq \sup_{s \in I} c_1(s)\alpha(X), \quad \text{for each} \quad X \subset B, I \subset I_a, \tag{7}$$

*where*

$$f(I, A, C) = \{f(t, x_1, x_2) : (t, x_1, x_2) \in I \times A \times C\},$$
$$k(I, I, X) = \{k(t, s, x) : (t, s, x) \in I \times I \times X\}.$$

*Moreover, let $K$ and $K_1$ be equicontinuous and uniformly $ACG^*$ on $I_a$. Then, there exists Carathéodory's solution to the problem* (1) *on $I_c$, for some $0 < c \leq a$.*

***Proof.*** The first part of the proof is the same as in the proof of the previous theorem. It remains to show the relative compactness of $V$, where $V$ is defined in Theorem 3.2. In this case notice that for $t \in I_c$ and $z_j$ as in Theorem 3.2 we have

$$
\begin{aligned}
&\alpha(V(t)) \leq \\
&\leq \sum_{i=0}^{m-1} \mu_\Delta(J_i) \cdot d \cdot \alpha\left(\sum_{j=0}^{m-1} \mu_\Delta(I_j) \cdot \overline{conv} k(I_j, I_j, V(I_j))\right) \\
&\leq \sum_{i=0}^{m-1} \mu_\Delta(J_i) \cdot d \cdot \sum_{j=0}^{m-1} \mu_\Delta(I_j) \cdot \alpha(k(I_j, I_j, V(I_j))) \\
&\leq \sum_{i=0}^{m-1} \mu_\Delta(J_i) \cdot d \cdot \sum_{j=0}^{m-1} \mu_\Delta(I_j) \cdot \sup_{s \in I_j} c_1(s)\alpha(V(I_j)) \\
&\quad \leq c \cdot d \cdot \sum_{j=0}^{m-1} \mu_\Delta(I_j) \cdot c_1(p_j) v(s_j) \\
&= c \cdot d \left(\sum_{j=0}^{m-1} \mu_\Delta(I_j) \cdot c_1(p_j) v(p_j) + \sum_{j=0}^{m-1} \mu_\Delta(I_j) \left(c_1(p_j)(v(s_j) - v(p_j))\right)\right),
\end{aligned}
$$

for some $p_j \in I_j$. Fix $\varepsilon > 0$. From the continuity of $v$ we may choose $m$ large enough so that $v(s_j) - v(p_j) < \varepsilon$ and so

$$
\begin{aligned}
\alpha(V(t)) &\leq c \cdot d \left(\sum_{j=0}^{m-1} \mu_\Delta(I_j) \cdot c_1(p_j) v(p_j) + \sum_{j=0}^{m-1} \tfrac{z}{m} c_1(p_j) \cdot \varepsilon\right) \\
&\leq c \cdot d \left(\sum_{j-0}^{m-1} \mu_\Delta(I_j) c_1(p_j) v(p_j) + z \cdot \varepsilon \cdot \max_{0 \leq k \leq m-1} c_1(p_k)\right).
\end{aligned}
$$

Since $\varepsilon \to 0$ and $z \cdot \max\limits_{0 \leq k \leq m-1} c_1(p_k)$ is bounded, we have $z \cdot \varepsilon \cdot \max\limits_{0 \leq k \leq m-1} c_1(p_k) \to 0$. Therefore,

$$
v(t) = \alpha(V(t)) \leq c \cdot d \cdot \int_0^t c_1(s) v(s) \Delta s, \quad t \in [0, c].
$$

Using the Gronwall's inequality, we have

$$
v(t) = \alpha(V(t)) = 0, \quad \text{for} \quad t \in [0, c].
$$

By the Arzelá-Ascoli‘s theorem [44] we deduce that $V$ is relatively compact.

By Theorem 3.1 the operator $F$ has a fixed point. This means that there exists a Carathéodory solution to problem (1).

**Remark 3.4.** By a classical solution to (1) we understand a function $x$ in $C_{rd}(I_a, E)$, such that $x(0) = x_0$, and $x(\cdot)$ satisfies (1) for all $t \in I_a$. If we assume a kind of continuity for $f$ and $k$ instead of Carathéodory conditions, we obtain the existence of at least one solution. For such solutions problem (1) is equivalent to problem (3) for each $t \in I_a$.

Similarly to Theorem 3.2 and Theorem 3.3, we can prove the following theorems.

**Theorem 3.5.** *Assume that for each uniformly $ACG_*$ function $x : I_a \to E$, the functions: $k(\cdot, s, x(s)), f\left(\cdot, x(\cdot), \int_0^{(\cdot)} k(\cdot, s, x(s))\Delta s\right)$ are $\Delta$-HL integrable, $f$ and $k$ are rd-continuous functions. Suppose that there exist constants $d_1, d_2, d_3 > 0$ such that*

$$\alpha(f(I, A, C)) \leq d_1 \cdot \alpha(A) + d_2 \cdot \alpha(C), \tag{8}$$

*for each subset $A, C$ of $B$, $I \subset I_a$*

$$\alpha(k(I, I, X)) \leq d_3 \cdot \alpha(X), \tag{9}$$

*for each subset $X$ of $B$, $I \subset I_a$, where*

$$f(I, A, C) = \{f(t, x_1, x_2) : (t, x_1, x_2) \in I \times A \times A\},$$

$$k(I, I, X) = \{k(t, s, x) : (t, s, x) \in I \times I \times A\}.$$

*Moreover, let $K$ and $K_1$ be equicontinuous, equibounded and uniformly $ACG_*$ on $I_a$. Then, there exists a solution to problem (1) on $I_c$, for some $0 < c \leq a$ and $0 < c \cdot d_1 + c^2 \cdot d_2 \cdot d_3 < 1$.*

**Theorem 3.6.** *Assume that for each uniformly $ACG^*$ function $x : I_a \to E$, the functions $k(\cdot, s, x(s)), f\left(\cdot, x(\cdot), \int_0^{(\cdot)} k(\cdot, s, x(s))ds\right)$ are $\Delta$-HL integrable and $k, f$ are rd-continuous functions. Suppose that there exists a constant $d > 0$ and a continuous function $c_1 : I_a \to R_+$ such that*

$$\alpha(f(I, A, C)) \leq d \cdot \alpha(C), \quad \text{for each} \quad A, C \subset B, I \subset I_a, \tag{10}$$

$$\alpha(k(I,I,X)) \le \sup_{s\in I} c_1(s)\alpha(X), \qquad \textit{for each} \qquad X \subset B, I \subset I_a, \tag{11}$$

*where*

$$\begin{aligned} f(I,A,C) &= \{f(t,x_1,x_2) : (t,x_1,x_2) \in I \times A \times C\}\,, k(I,I,X) \\ &= \{k(t,s,x) : (t,s,x) \in I \times I \times X\}\,. \end{aligned}$$

*Moreover, let* $K$ *and* $K_1$ *be equicontinuous and uniformly* $ACG^*$ *on* $I_a$. *Then, there exists a solution to problem* (1) *on* $I_c$, *for some* $0 < c \le a$.

**Remark 3.7.** For discrete time scales the existence of solutions is trivially given without imposing further compactness assumptions on the right-hand side of the equation. If a time scale admits at least one right-dense point, then the continuity assumption is not sufficient for the existence of (rd-continuous) solutions to problem (1).

Nevertheless, we will not distinguish such a discrete case, because some continuity and compactness conditions are necessary to unify the continuous problems and their discretization.

**Remark 3.8.** The conditions in Theorems: 3.2, 3.3, 3.4, 3.5 can be also generalized to the Sadovskii condition [36], the Szufla condition [41] and others and $\alpha$ can be replaced by some axiomatic measure of noncompactness.

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Received 20 April 2010