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TRANSPORTATION FLOW PROBLEMS WITH RADON MEASURE VARIABLES

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Abstract

For a multidimensional control problem $(\mathrm{P})_{\mathrm{K}}$ involving controls $u \in L_\infty$, we construct a dual problem $(\mathrm{D})_{\mathrm{K}}$ in which the variables ν to be paired with u are taken from the measure space $rca\,(\Omega, \mathfrak{B})$ instead of $(L_\infty)^*$. For this purpose, we add to $(\mathrm{P})_{\mathrm{K}}$ a Baire class restriction for the representatives of the controls u. As main results, we prove a strong duality theorem and saddle-point conditions.

Keywords: multidimensional control problems, strong duality, saddle-point conditions, Baire classification.

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1 Introduction

a) The primal problem. We consider the following multidimensional control problem $(\mathrm{P})_{\mathrm{K}}$ (1.1) – (1.4) ("classical deposit problem") introduced by Klötzler [8]:

$$J(x,u) = -\sum_{k=1}^{n} \int_\Omega x_k(t)\, d\alpha_k(t) \longrightarrow \mathrm{Min}\,! \tag{1.1}$$

subject to $(x,u) \in W_p^{1,n}(\Omega) \times L_p^{nm}(\Omega)$, satisfying

$$x_{i;\,t_j}(t) = u_{ij}(t) \quad \text{a.e. on } \Omega, \; {\scriptstyle i=1,\ldots,n;\; j=1,\ldots,m;} \tag{1.2}$$

$$u(t) \in \mathrm{U}(t) = \Big\{ z \in \mathbb{R}^{nm} \,\Big|\, z^{\mathrm{T}} v \leqslant r(t,v) \;\; \forall\, v \in \mathbb{R}^{nm} \Big\} \;\; \forall\, t \in \Omega \tag{1.3}$$

$$x(t) = \varphi(t) \;\; \forall\, t \in \Gamma \text{ where } \Gamma \in \mathrm{Comp}(\Omega),\; \Gamma \neq \emptyset. \tag{1.4}$$

For $m = 2$ we may interpret $(\mathrm{P})_{\mathrm{K}}$ as deposit problem [8, p. 394]: On a region Ω in the plane, n infinitely divisible commodities have to be stored. $x_k(t)$ describes the deposit height of the k^{th} commodity at the position t (fixed in the case of $t \in \Gamma$), $(-\alpha_k)$ the related cost rate, $J(x, u)$ the total deposit cost which is to minimize. The control restrictions may be understood as generalized slope conditions for the resulting deposit hill. From [8] we take the following

Basic assumptions about the data of $(\mathrm{P})_{\mathrm{K}}$**:**

$(\mathrm{V1})_{\mathrm{K}}$: We have $m \geqslant 2$ and $p = \infty$. $\Omega \subset \mathbb{R}^m$ is a compact Lipschitz domain in strong sense, see [11, Definition 3.4.1, p. 72]. (In view of Lemma 2.1, we may assume $m < p < \infty$ instead of $p = \infty$.) Then functions $x \in W_p^{1,n}(\Omega)$, $m < p < \infty$, have continuous representatives, and functions $x \in W_\infty^{1,n}(\Omega)$ are Lipschitz representable [1, Theorem 5.5, p. 185].

$(\mathrm{V2})_{\mathrm{K}}$: $r(\cdot, v)$ is summable on Ω for all $v \in \mathbb{R}^{nm}$; $r(t, \cdot)$ is positively homogeneous of degree one in v (i.e. $r(t, \lambda\, v) = \lambda\, r(t, v)$ for all $\lambda > 0$) and convex; there exist constants $0 < \gamma_1 \leqslant \gamma_2$ with $\gamma_1 |v\,| \leqslant r(t, v) \leqslant \gamma_2 |\,v\,|$ for all $t \in \Omega$ and for all $v \in \mathbb{R}^{nm}$.

$(\mathrm{V3})_{\mathrm{K}}$: α_k are signed regular measures on the σ-algebra $\mathfrak{B}'$ of the Lebesgue sets of Ω satisfying the balance condition $\alpha_k(\Omega) = 0$. (In the following, we only consider the uniquely determined restrictions of α_k on the σ-subalgebra $\mathfrak{B} \subset \mathfrak{B}'$ of the Borel sets of Ω.)

$(\mathrm{V4})_{\mathrm{K}}$: There is $\Gamma = \{t_0\} \subset \partial\Omega$ and $x(t_0) = \mathfrak{o}_n$.

b) Outline and main results of the paper. In [8] and [9], a transportation flow problem $(\mathrm{T})_{\mathrm{K}}$ in which the variables ("flows") come from the space $(L_\infty)^*$ is opposited to $(\mathrm{P})_{\mathrm{K}}$. Both problems are in strong duality. The aim of the present paper is the construction of a strong dual problem for $(\mathrm{P})_{\mathrm{K}}$ with more regular variables, namely Radon measures, in place of $(L_\infty)^*$-functionals (which are representable only by finitely additive set functions, cf. [4, Theorem 16, p. 296]). For this purpose, we restrict the feasible domain of $(\mathrm{P})_{\mathrm{K}}$ under conservation of the minimal value $\inf(\mathrm{P})_{\mathrm{K}}$:

Definition 1.1. For $(\mathrm{P})_{\mathrm{K}}$ and $k \in \mathbb{N}_0$, we consider the *class-qualified problem* $(\mathrm{P})_{\mathrm{K},\mathcal{B}^k}$ (1.1) – (1.5) with

$$x_{i;\,t_j} \text{ admits (at least) one representative from } \mathcal{B}^k(\Omega) \ \forall\, i,\, j. \tag{1.5}$$

Here $\mathcal{B}^k(\Omega)$ denotes the k^{th} Baire function class on Ω (see below), thus we have to distinguish in $(\mathrm{P})_{\mathrm{K},\mathcal{B}^k}$ feasible controls u', u'' taking different values even on a λ^m-null set. The following theorem gives sufficient conditions under which the minimal value of $(\mathrm{P})_{\mathrm{K}}$ is not influenced by addition of the class qualification (1.5) to (1.1) – (1.4).

Theorem 1.2 (Sufficient conditions for $\inf\,(\mathrm{P})_{\mathrm{K}} = \inf\,(\mathrm{P})_{\mathrm{K},\mathcal{B}^k}$). *Let* $(\mathrm{P})_{\mathrm{K}}$ *satisfy assumptions* $(\mathrm{V1})_{\mathrm{K}}$ – $(\mathrm{V4})_{\mathrm{K}}$. *Assume further that the function* $r(t,v)$ *satisfies the condition* $|r(t',v) - r(t'',v)| \leqslant L \cdot |t' - t''| \cdot \tilde{r}(v) \quad \forall\, v \in \mathbb{R}^{nm}$ $\forall\, t',\, t'' \in \Omega$ *with* $L > 0$ *and* $\tilde{r} \in C^0(\mathbb{R}^{nm})$.

Then $(\mathrm{P})_{\mathrm{K}}$ *admits a minimizing sequence* $\{(x^N,\, u^N)\}$ *with representatives of* 0^{th} *Baire class for* $x^N_{i;\,t_j}$, *and the minimal values of* $(\mathrm{P})_{\mathrm{K}}$ *and* $(\mathrm{P})_{\mathrm{K},\mathcal{B}^k}$, $k = 0, 1, \ldots$, *coincide. Furthermore, each* $(x^N,\, u^N)$ *can be determined in such a way that the state equations* (1.2) *are satisfied everywhere on* Ω.

If the assumptions of Theorem 1.2 are satisfied then the problem $(\mathrm{D})_{\mathrm{K}}$ (2.1) – (2.2)

$$G(\nu) = \inf_{\substack{u \in \mathcal{B}^{1,nm}(\Omega) \\ u(t) \in \mathrm{U}(t)\ \ \forall\, t \in \Omega}} \Big[-\sum_{i,j} \int_\Omega u_{ij}(t)\, d\nu_{ij}(t) \Big] \longrightarrow \mathrm{Max}\,! \tag{2.1}$$

subject to $\nu \in \big(rca\,(\Omega, \mathfrak{B})\big)^{nm}$, satisfying the continuity equation

$$\sum_{i,j} \int_\Omega \zeta_{i\,;t_j}(t)\, d\nu_{ij}(t) - \sum_k \int_\Omega \zeta_k(t)\, d\alpha_k(t) = 0 \qquad \forall\, \zeta \in C^{1,n}(\Omega):\ \zeta(t_0) = \mathfrak{o}_n, \tag{2.2}$$

is strongly dual to $(\mathrm{P})_{\mathrm{K}}$ (Theorem 3.4). In analogy to [8, p. 391 ff.], the feasible elements of $(\mathrm{D})_{\mathrm{K}}$ may be understood as time-independent vectorial transportation flows: Assuming that we have to organize the shipment of n infinitely divisible commodities within Ω where $\alpha_k(\mathrm{A})$ is the rate of supply resp. demand of the k^{th} commodity in $\mathrm{A} \in \mathfrak{B}$, the average flow of the k^{th} commodity in A can be described by the vector $(\nu_{k,1}(\mathrm{A}), \ldots, \nu_{k,m}(\mathrm{A}))$.

Theorem 1.3 (Sufficient saddle-point conditions for the problems $(\mathrm{P})_{\mathrm{K},\mathcal{B}^1}$ – $(\mathrm{D})_{\mathrm{K}}$). *Let* $(\mathrm{P})_{\mathrm{K}}$ *satisfy all assumptions of Theorem* 1.2. *Given some feasible element* (x^*, u^*) *of* $(\mathrm{P})_{\mathrm{K},\mathcal{B}^1}$ (*thus the weak derivatives* $x^*_{i;\,t_j}$ *admit representatives of first Baire class*) *and a measure* $\nu^* \in \big(rca\,(\Omega, \mathfrak{B})\big)^{nm}$. *If the following conditions* $(\mathcal{M})^*_0$, $(\mathcal{K})^*_0$ *and* $(\mathcal{D})^*_0$ *are satisfied then* (x^*, u^*) *is a global minimizer of* $(\mathrm{P})_{\mathrm{K},\mathcal{B}^1}$ *and* ν^* *is a global maximizer of* $(\mathrm{D})_{\mathrm{K}}$:

$$(\mathcal{M})_0^*: \qquad \sum_{i,j} \int_\Omega (\, u_{ij}^*(t) - u_{ij}(t)\,)\, d\nu_{ij}^*(t) \geqslant 0$$
$$\forall\, u \in \mathcal{B}^{1,nm}(\Omega):\; u(t) \in \mathrm{U}(t)\ \forall\, t \in \Omega;$$
$$(\mathcal{K})_0^*: \qquad \sum_{i,j} \int_\Omega \zeta_{i;\,t_j}(t)\, d\nu_{ij}^*(t) - \sum_k \int_\Omega \zeta_k(t)\, d\alpha_k(t) = 0$$
$$\forall \zeta \in C^{1,n}(\Omega):\; \zeta(t_0) = \mathfrak{o}_n;$$
$$(\mathcal{D})_0^*: \qquad \sum_k \int_\Omega x_k^*(t)\, d\alpha_k(t) = \sum_{i,j} \int_\Omega u_{ij}^*(t)\, d\nu_{ij}^*(t).$$

The paper is organized as follows: In the rest of this section, we compile some basic notations and definitions. In Section 2, we investigate the relations between the original deposit problem (1.1) – (1.4), its relaxed problem and the class-qualified problem (1.1) – (1.5) and prove Theorem 1.2. Then, in Section 3, we construct the announced dual problem $(\mathrm{D})_\mathrm{K}$ and give the proof of Theorem 1.3. Finally, we prove that a partial converse of Theorem 1.3 is true (Theorem 3.5).

c) Notations. $C^{k,n}(\Omega)$, $L_p^n(\Omega)$ and $W_p^{k,n}(\Omega)$ $(1 \leqslant p \leqslant \infty)$ denote the spaces of n-dimensional vector functions on Ω whose components are k-times continuously differentiable, resp. belong to $L_p(\Omega)$ or to the Sobolev space of $L_p(\Omega)$-functions having weak derivatives up to k^{th} order in $L_p(\Omega)$. Instead of $C^{0,1}(\Omega)$, we write shortly $C^0(\Omega)$. For the classical as well as for the weak partial derivatives of x_i by t_j we use the notation $x_{i;\,t_j}$. The Banach space of Radon measures (signed regular measures) acting on the σ-algebra $\mathfrak{B}$ of the Borel sets of Ω (equipped with the total variation norm) is denoted by $rca\,(\Omega, \mathfrak{B})$. Due to the compactness of Ω, there is an isometric isomorphism between the dual space $(C^0(\Omega))^*$ and $rca\,(\Omega, \mathfrak{B})$ [4, Theorem 3, p. 265] so that each linear, continuous functional on $C^0(\Omega)$ can be represented by an integral w. r. to a Radon measure $\nu \in rca\,(\Omega, \mathfrak{B})$. δ_v denotes the Dirac measure concentrated in v, λ^m the m-dimensional Lebesgue measure and $\mathfrak{o}$ the zero element of the actual space (in particular, $\mathfrak{o}_n$ is the n-dimensional zero vector).

d) Generalized controls. Let $\mathrm{U} = \bigcup_{t \in \Omega} \mathrm{U}(t)$ (U is compact, see Lemma 2.1 below). A family $\mu = \{\mu_t \,|\, t \in \Omega\}$ of probability measures $\mu_t \in rca\,(\Omega, \mathfrak{B}_\mathrm{U})$ acting on the σ-algebra $\mathfrak{B}_\mathrm{U}$ of the Borel sets of U is called a *generalized control* if 1) supp $\mu_t \subseteq \mathrm{U}(t)$ for all $t \in \Omega$ and 2) for any continuous function $f \in C^0(\Omega \times \mathrm{U})$ the function $h_f\colon \Omega \times \mathrm{U} \to \mathrm{I\!R}$ with

$h_f(t) = \int_{\mathrm{U}} f(t,v)\, d\mu_t(v)$ is measurable [5, p. 23]. Two families μ', μ'' can be identified if $\mu'_t \equiv \mu''_t$ for a.e. $t \in \Omega$. The set of all generalized controls is denoted by $\mathfrak{M}_{\mathrm{U}}$. Let us equip $\mathfrak{M}_{\mathrm{U}}$ with the following topology:

$$(3) \quad \{\mu^N\} \to \mu^* \Longleftrightarrow \lim_{N\to\infty} \int_\Omega \int_{\mathrm{U}} f(t,v)\, \mu_t^N(v)\, dt = \int_\Omega \int_{\mathrm{U}} f(t,v)\, d\mu_t^*(v)\, dt$$

for all $f \in C^0(\Omega \times \mathrm{U})$. Due to the compactness of Ω and U, each family $\{\mu_t\}$ is finite in the sense of [5, p. 21 f.], and each function h_f generated by some $\mu \in \mathfrak{M}_{\mathrm{U}}$ is bounded and, consequently, integrable on Ω. The set $\mathfrak{M}_{\mathrm{U}}$ is convex [5, p. 25] and, by [10, Theorem 20, p. 78], sequentially compact in the above introduced topology while the sets $\mathrm{U}(t)$ are nonempty, closed and uniformly bounded (Lemma 2.1) and the set-valued map $\mathrm{U}(t)\colon\, \Omega \to \mathfrak{P}(\mathbb{R}^{nm})$ is upper semicontinuous (see Lemma 2.2).

e) Baire classification. We say that any continuous function ψ defined on the compact set $\Omega \subset \mathbb{R}^m$ is *of 0^{th} Baire class* and write $\psi \in \mathcal{B}^0(\Omega)$. The limit functions of everywhere pointwise convergent sequences $\{\psi^K\}$, $\psi^K \in \mathcal{B}^0(\Omega)$, form the *first Baire class* $\mathcal{B}^1(\Omega)$; the limit functions of everywhere pointwise convergent sequences $\{\psi^K\}$, $\psi^K \in \mathcal{B}^1(\Omega)$, form the *second Baire class* $\mathcal{B}^2(\Omega)$ and so on. Obviously, we have $\mathcal{B}^0(\Omega) \subset \mathcal{B}^1(\Omega) \subset \mathcal{B}^2(\Omega) \subset ...$ If a finite function is contained in any Baire class then it is measurable [3, Theorem 4, p. 404]; conversely, any measurable, essentially bounded function on Ω agrees a.e. with some function of second Baire class [3, Theorem 5, p. 406]. (Consequently, for $k \geqslant 2$ the minimal values of the problems $(\mathrm{P})_{\mathrm{K}}$ and $(\mathrm{P})_{\mathrm{K},\mathcal{B}^k}$ coincide.) Each Baire class is closed under (pointwise) addition und multiplication of finite functions [3, Theorems 6 and 7, p. 397]. For more details, see [3, p. 393 ff.].

f) Theorem 1.5 (Filippov's lemma)**.** *Consider a measure space* $(\Omega, \mathfrak{A}, \lambda)$ *with a σ-finite measure λ and a σ-algebra $\mathfrak{A}$ which is complete w. r. to subsets of λ-null sets. Further, let Y' and Y'' be separable, complete metric spaces, $h(t,v)\colon\, \Omega \times \mathrm{Y}' \to \mathrm{Y}''$ a Carathéodory function and* $\mathrm{S}(t)\colon\, \Omega \to \mathfrak{P}(\mathrm{Y}')$ *a measurable set-valued map* [2, Definition 8.1.1, p. 307] *with nonempty, closed images. Then for every measurable function $z\colon\, \Omega \to \mathrm{Y}''$ satisfying $z(t) \in \{h(t,v)\,|\, v \in \mathrm{S}(t)\}$ for all $t \in \Omega$ there exists a $\mathfrak{A}$-$\mathfrak{B}_{\mathrm{Y}'}$-measurable selection $s\colon\, \Omega \to \mathrm{Y}'$ with*

$$(4) \qquad s(t) \in \mathrm{S}(t)\ \ \forall\, t \in \Omega \ \textit{and}\ z(t) = h(t, s(t))\ \ \textit{for all}\ \ t \in \Omega.$$

[2, Theorem 8.2.10, p. 316, together with Theorem 8.2.9, p. 315].

2 Relations between $(\mathbf{P})_{\mathrm{K}}$ and $(\mathbf{P})_{\mathrm{K},\mathcal{B}^k}$

a) Two auxiliary results.

Lemma 2.1. *Under assumptions* $(\mathrm{V1})_{\mathrm{K}}$ *and* $(\mathrm{V2})_{\mathrm{K}}$*, the sets* $\mathrm{U}(t)$ *are nonempty, convex and compact, satisfying* $\mathrm{K}(\mathfrak{o}_{nm}, \gamma_1) \subseteq \mathrm{U}(t) \subseteq \mathrm{K}(\mathfrak{o}_{nm}, \gamma_2)$*. Consequently, the assumption "*$p = \infty$*" in* $(\mathrm{V1})_{\mathrm{K}}$ *can be replaced by "*$m < p < \infty$*".*

Proof. By [8, Proof of Theorem 1, p. 394], all $\mathrm{U}(t)$ are nonempty, convex and compact. By $(\mathrm{V2})_{\mathrm{K}}$, it holds for arbitrary $z \in \mathrm{K}(\mathfrak{o}_{nm}, \gamma_1)$ and $v \in \mathbb{R}^{nm}$: $z^{\mathrm{T}} v = |\,z\,| \cdot |\,v\,| \cdot \cos\sphericalangle(z, v) \leqslant \gamma_1 \,|\,v\,| \leqslant r(t, v)$, and we see that $\mathrm{K}(\mathfrak{o}_{nm}, \gamma_1) \subseteq \mathrm{U}(t)$. Conversely, if $z \in \mathrm{U}(t)$ then, choosing $v = z\,/|\,z\,|$, we compute $z^{\mathrm{T}} v = |\,z\,| \leqslant r(t,\, z\,/|\,z\,|) \leqslant \gamma_2 \Big|\, z\,/|\,z\,| \Big| = \gamma_2$ what proves the inclusion $\mathrm{U}(t) \subseteq \mathrm{K}(\mathfrak{o}_{nm}, \gamma_2)$. If, consequently, a function $u \in L_p^{nm}(\Omega)$ with $m < p < \infty$ satisfies the control restrictions (1.3) then u is automatically element of $L_\infty^{nm}(\Omega)$, and $(\mathrm{V1})_{\mathrm{K}}$ may be formulated with $m < p < \infty$ instead of $p = \infty$. ■

Lemma 2.2. *If the function* $r(t, v)$ *is continuous in* t *then the set-valued map* $\mathrm{U}(t)\colon \Omega \to \mathfrak{P}(\mathbb{R}^{nm})$ *is upper semicontinuous in the sense of* [2, Definition 1.4.1, p. 38].

Proof. We apply [2, Proposition 1.4.8, p. 42], taking the ball $\mathrm{K}(\mathfrak{o}_{nm},\, \gamma_2)$ endowed with the Euclidean metric as compact image space. Obviuosly,

$$\text{(5)} \quad \mathrm{Graph}\,(\mathrm{U}) = \{(t, z) \in \mathbb{R}^m \times \mathbb{R}^{nm} \,|\, t \in \Omega,\ z^{\mathrm{T}} v \leqslant r(t, v)\ \forall\, v \in \mathbb{R}^{nm}\}.$$

Consider a sequence $\{(t^N, z^N)\} \to (t^*, z^*)$ with $(t^N, z^N) \in \mathrm{Graph}\,(\mathrm{U})$. Then $t^* \in \Omega$ since Ω is closed. From $(t^N, z^N) \in \mathrm{Graph}\,(\mathrm{U})$ it follows $(z^N)^{\mathrm{T}} v \leqslant r(t^N, v)$ for all $v \in \mathbb{R}^{nm}$, and, by continuity of $r(\cdot, v)$, $(z^*)^{\mathrm{T}} v = \lim_{N\to\infty} (z^N)^{\mathrm{T}} v \leqslant \lim_{N\to\infty} r(t^N, v) = r(t^*, v)$. Thus $(t^*, z^*) \in \mathrm{Graph}\,(\mathrm{U})$; $\mathrm{Graph}\,(\mathrm{U})$ is a closed subset of $\Omega \times \mathbb{R}^{nm}$, and the set-valued map $\mathrm{U}(t)$ is upper semicontinuous. ■

b) An approximation theorem. The following theorem generalizes a result of Hüseinov [6] about C^∞-approximations of Lipschitz functions. For its proof, we refer on the author's paper [15] to be published simultaneously.

Theorem 2.3 (Generalized Hüseinov's theorem)**.** *Consider a set-valued map* $\mathrm{S}(t)\colon \Omega \to \mathfrak{P}(\mathbb{R}^{nm})$ *with convex, compact, uniformly bounded images containing the ball* $\mathrm{K}(\mathfrak{o}, \omega)$ *as subset. Assume that* $\mathrm{S}(t)$ *is Lipschitz*

[2, Definition 1.4.5, p. 41]. *Given further a Lipschitz function* $x^* \in W^{1,n}_\infty(\Omega)$ *with* $(x^*_{i;t_j}(t))_{ij} \in \mathrm{S}(t)$ *for a.e.* $t \in \Omega$. *Then* x^* *can be approximated by a sequence of functions* $x^N \in C^{\infty,n}(\Omega)$ *with*

1) $\lim\limits_{N\to\infty} \|x^N - x^*\|_{C^{0,n}(\Omega)} = 0,\ x^N(t_0) = x^*(t_0),$
2) $\lim\limits_{N\to\infty} \|x^N_{i;t_j} - x^*_{i;t_j}\|_{L_1(\Omega)} = 0\ \forall\, i,\, j,$
3) $(x^N_{i;t_j}(t))_{ij} \in \mathrm{S}(t)$ *for all* $t \in \Omega$. [15, Theorem 1.5, p. 2].

c) Relations between $(\mathrm{P})_\mathrm{K}$ **and its relaxed problem.** The standard relaxation of $(\mathrm{P})_\mathrm{K}$ by use of generalized controls (Young measures) leads to the problem $(\overline{\mathrm{P}})_\mathrm{K}$ (6.1) – (6.4)

$$\bar{J}(x,\mu) = -\sum_{k=1}^{n} \int_\Omega x_k(t)\, d\alpha_k(t) \longrightarrow \text{Min!} \tag{6.1}$$

subject to $(x,\mu) \in W^{1,n}_p(\Omega) \times \mathfrak{M}_\mathrm{U}$, satisfying

$$x_{i;t_j}(t) = \int_\mathrm{U} v_{ij}\, d\mu_t(v) \ \text{ a.e. on } \Omega,\ \ \forall\, i,\, j, \tag{6.2}$$

$$\operatorname{supp} \mu_t \subseteq \mathrm{U}(t) = \{\, z \in \mathbb{R}^{nm} \mid z^\mathrm{T} v \leqslant r(t,v) \ \ \forall\, v \in \mathbb{R}^{nm} \} \ \ \forall\, t \in \Omega \tag{6.3}$$

$$x(t) = \varphi(t) \ \ \forall\, t \in \Gamma \ \text{ where } \Gamma \in \mathrm{Comp}(\Omega),\ \Gamma \neq \emptyset. \tag{6.4}$$

Since $(\mathrm{P})_\mathrm{K}$ itself has a linear-convex structure, the problems $(\mathrm{P})_\mathrm{K}$ and $(\overline{\mathrm{P}})_\mathrm{K}$ are equivalent in a sense specified in the following Theorem 2.4. In particular, their minimal values coincide, and there is a one-to-one correspondence between their minimal solutions. Thus in the frame of the present investigation the relaxed problem is of merely technical interest: it allows to evaluate the conditions of the maximum principle from [12] which is designed for relaxed problems. Moreover, the equivalence between $(\mathrm{P})_\mathrm{K}$ and $(\overline{\mathrm{P}})_\mathrm{K}$ leads to a simple existence proof for global minimizers of $(\mathrm{P})_\mathrm{K}$.

Theorem 2.4 (Equivalence of the problems $(\mathrm{P})_\mathrm{K}$ and $(\overline{\mathrm{P}})_\mathrm{K}$)**.** *Let* $(\mathrm{P})_\mathrm{K}$ *satisfy assumptions* $(\mathrm{V1})_\mathrm{K}$ – $(\mathrm{V4})_\mathrm{K}$, *and let the function* $r(t,v)$ *be continuous in* t *for all* $v \in \mathbb{R}^{nm}$. *Then for each feasible element* (x,μ) *of* $(\overline{\mathrm{P}})_\mathrm{K}$ *there exists a generalized control of the form* $\{\sum_{s=1}^{nm+1} \lambda_s(t)\, \delta_{u_s(t)}\}$ *with the following properties:*

1) $u_s \in L^{nm}_\infty(\Omega)$, $u_s(t) \in \mathrm{U}(t)$ *for all* $t \in \Omega$;
2) $\lambda_s(t) \in L_\infty(\Omega)$, $0 \leqslant \lambda_s(t) \leqslant 1$ *and* $\sum_s \lambda_s(t) = 1$ *for all* $t \in \Omega$;
3) $\int_\mathrm{U} v_{ij}\, d\mu_t(v) = \sum_s \lambda_s(t)\, u_{s,ij}(t)$ *for all* $t \in \Omega\ \forall\, i,\, j$;

4) $\bar{J}(x,\mu) = J(x, \sum_s \lambda_s u_s)$,

so that the element $(x, \sum_s \lambda_s u_s)$ *is feasible in* $(\mathrm{P})_{\mathrm{K}}$. *Consequently, the problems* $(\mathrm{P})_{\mathrm{K}}$ *and* $(\overline{\mathrm{P}})_{\mathrm{K}}$ *have the same minimal value.*

Proof. At first, let us define for fixed $t \in \Omega$ the set-valued maps $\mathfrak{M}_{\mathrm{U}}(t)$: $\Omega \to \mathfrak{P}(rca\,(\mathrm{U}, \mathfrak{B}_{\mathrm{U}}))$ and $\overline{\mathrm{Z}}(t)\colon\ \Omega \to \mathfrak{P}(\mathrm{I\!R}^{nm})$ by

$$\mathfrak{M}_{\mathrm{U}}(t) = \{\mu_t \in rca\,(U, \mathfrak{B}_{\mathrm{U}}) |\, \mu_t \geqslant 0,\ \operatorname{supp}\mu_t \subseteq \mathrm{U}(t),\ \mu_t(\mathrm{U}(t)) = 1\}; \tag{7}$$

$$\overline{\mathrm{Z}}(t) = \{\, z \in \mathrm{I\!R}^{nm} \mid z_{ij} = \int_{\mathrm{U}} v_{ij}\, d\mu_t(v),\ \mu_t \in \mathfrak{M}_{\mathrm{U}}(t)\}. \tag{8}$$

Choosing z', $z'' \in \overline{\mathrm{Z}}(t)$ and $\lambda \in [0,1]$, it follows

$$\lambda\, z'_{ij} + (1-\lambda)\, z''_{ij} = \int_{\mathrm{U}} v_{ij}\, [\, \lambda\, d\mu'_t(v) + (1-\lambda)\, d\mu''_t(v)\,] \tag{9}$$

with $\operatorname{supp}\,[\lambda\,\mu'_t + (1-\lambda)\,\mu''_t] \subseteq \operatorname{supp}\mu'_t \cup \operatorname{supp}\mu''_t \subseteq \mathrm{U}(t)$ and, consequently, $\lambda\,\mu'_t + (1-\lambda)\,\mu''_t \in \mathfrak{M}_{\mathrm{U}}(t)$. This proves the convexity of $\overline{\mathrm{Z}}(t)$. Given a sequence $\{z^N\} \to z^*$ with $z^N \in \overline{\mathrm{Z}}(t)$ then there are representations $z^N_{ij} = \int_{\mathrm{U}} v_{ij}\, d\mu^N_t(v)$ with $\mu^N_t \in \mathfrak{M}_{\mathrm{U}}(t)$, and the norm-bounded sequence $\{\mu^N_t\}$ admits some subsequence $\{\mu^{N'}_t\}$ converging to μ^*_t in the sense of (3). It holds

$$z^*_{ij} = \lim_{N'\to\infty} z^{N'}_{ij} = \lim_{N'\to\infty} \int_{\mathrm{U}} v_{ij}\, d\mu^{N'}_t(v) = \int_{\mathrm{U}} v_{ij}\, d\mu^*_t(v), \tag{10}$$

and from [14, Proposition 1.5.1. (iii), p. 47 f.] it follows that μ^*_t is also a probability measure. Thus $\overline{\mathrm{Z}}(t)$ is closed, and from the continuity of the integrand and the uniform boundedness of the sets $\mathrm{U}(t)$ (Lemma 2.1) it follows also compactness. Since the cost functional does not depend on the control variables, the proof can be completed now as in [5, Assertion 8.3, p. 157 ff.], using the version of Filippov's lemma given in Theorem 1.5 above. ■

Theorem 2.5. ($(\mathrm{P})_{\mathrm{K}}$ admits a global minimizer with $\inf\,(\mathrm{P})_{\mathrm{K}} = \inf\,(\overline{\mathrm{P}})_{\mathrm{K}}$)**.** *Let* $(\mathrm{P})_{\mathrm{K}}$ *satisfy assumptions* $(\mathrm{V1})_{\mathrm{K}}$ – $(\mathrm{V4})_{\mathrm{K}}$, *and let* $r(t,v)$ *be continuous in* t *for all* $v \in \mathrm{I\!R}^{nm}$. *Then there exists a global minimizer* (x^*, u^*) *for* $(\mathrm{P})_{\mathrm{K}}$, *and the problems* $(\mathrm{P})_{\mathrm{K}}$ *and* $(\overline{\mathrm{P}})_{\mathrm{K}}$ *have the same minimal value. Furthermore,* (x^*, u^*) *can be determined in such a way that the state equations* (1.2) *are satisfied everywhere on* Ω.

Proof. In view to Theorem 2.4, it suffices to prove that the relaxed problem $(\overline{\mathrm{P}})_{\mathrm{K}}$ admits a global minimizer. Then by [12, Remark after Theorem 2.2, p. 224 f.] we have to check that 1) the basic assumptions (V1) – (V4) from [12] are satisfied (together with the feasibility of the zero solution $(\mathfrak{o}_n, \mathfrak{o}_{nm})$, this follows from our assumptions $(\mathrm{V1})_{\mathrm{K}}$ – $(\mathrm{V4})_{\mathrm{K}}$ and Lemma 2.1) and 2) $\mathrm{U}(t)\colon \Omega \to \mathfrak{P}(\mathbb{R}^{nm})$ is upper semicontinuous in the sense of [2, p. 38, Definition 1.4.1] with nonempty, closed and uniformly bounded images (this is true by Lemmata 2.1 and 2.2). Finally, the assertion about the state equation (1.2) is proved by Theorem 2.4, 3). ∎

d) Comparison of the minimal values of $(\mathrm{P})_{\mathrm{K}}$ **and** $(\mathrm{P})_{\mathrm{K},\mathcal{B}^k}$**.** In Theorem 1.2, sufficient conditions for the coincidence of the minimal values of $(\mathrm{P})_{\mathrm{K}}$ and $(\mathrm{P})_{\mathrm{K},\mathcal{B}^k}$ were formulated. We continue with its proof.

***Proof of Theorem* 1.2.**

Step 1. *We prove first that the set-valued map* $\mathrm{U}(t)$ *is Lipschitz* [2, Definition 1.4.5, p. 41]. Choosing t', $t'' \in \Omega$ and $z \in \mathrm{U}(t')$, we have for arbitrary $v \in \mathbb{R}^{nm}$:

$$z^{\mathrm{T}} v \leqslant r(t', v) = r(t'', v) + \Big(r(t', v) - r(t'', v)\Big). \tag{11}$$

If $v = \mathfrak{o}_{nm}$ then from $(\mathrm{V2})_{\mathrm{K}}$ it follows $r(t, \mathfrak{o}_{nm}) = 0$ for all $t \in \Omega$, and (11) gives $z^{\mathrm{T}} \mathfrak{o}_{nm} \leqslant r(t'', \mathfrak{o}_{nm})$. Let $v \neq \mathfrak{o}_{nm}$, then it holds in consequence of the homogeneity of $r(t, \cdot)$ and of the assumption of the theorem:

$$\begin{aligned} r(t', v) &= r(t'', v) + |v|\Big(r(t', v/|v|) - (t'', v/|v|)\Big) \\ &\leqslant\, r(t'', v) + |v| \cdot L \cdot |t' - t''\,| \cdot \tilde{r}(v/|v|). \end{aligned} \tag{12}$$

Since $z \in \mathrm{K}(\mathfrak{o}_{nm}, \omega) \Longleftrightarrow z^{\mathrm{T}} v \leqslant \omega\,|\,v\,|$ for all $v \in \mathbb{R}^{nm}$, it follows

$$z \in \mathrm{U}(t'') + \mathrm{K}(\mathfrak{o}_{nm}, \omega) \Longleftrightarrow z^{\mathrm{T}} v \leqslant r(t'', v) + \omega\,|\,v\,| \quad \forall\, v \in \mathbb{R}^{nm}. \tag{13}$$

The nonnegative continuous function $\tilde{r}(v)$ takes on its maximum c on the unit sphere of $\mathbb{R}^{nm}$, thus, by (12), z is element of $\mathrm{U}(t'') + \mathrm{K}(\mathfrak{o}, c\,L|t' - t''|)$ what proves the Lipschitz continuity of $\mathrm{U}(t)$.

Step 2. *Application of the generalized Hüseinov's theorem.* In consequence of the assumptions, $r(t, v)$ is continuous in t, and we know then from Theorem 2.5 that $(\mathrm{P})_{\mathrm{K}}$ possesses a global minimizer (x^*, u^*). By Lemma 2.1

and Step 1, we can apply Theorem 2.3 to $x^*_{i;t_j}$ and the set-valued map $\mathrm{U}(t)$. So there exists a sequence of functions $x^N \in C^{\infty,n}(\Omega)$ with the following properties: They converge to x^* uniformly on Ω and share the boundary value with x^* (so that the boundary condition (1.4) is satisfied), their weak derivatives come from the space $C^{\infty,nm}(\Omega)$ and satisfy the inclusions $(x^N_{i;t_j}(t))_{ij} \in \mathrm{U}(t)$ for all $t \in \Omega$. Thus all pairs (x^N, u^N) with $u^N_{ij}(t) = x^N_{i;t_j}(t)$ are feasible in $(\mathrm{P})_\mathrm{K}$, and these elements satisfy the state equations (1.2) everywhere on Ω. From the uniform convergence of $\{x^N\}$ it follows that $J(x^N, u^N) \to J(x^*, u^*)$, and we find some subsequence of $\{(x^N, u^N)\}$ being a minimizing sequence for $(\mathrm{P})_\mathrm{K}$. Since all functions $x^N_{i;t_j}$ are contained in $C^{\infty,nm}(\Omega) \subset \mathcal{B}^{0,nm}(\Omega) \subset \mathcal{B}^{1,nm}(\Omega) \subset \dots$, the proof is complete. ■

Remark. For more general boundary conditions with $\Gamma \subseteq \partial\Omega$ and $\varphi|\Gamma = c \in \mathbb{R}^n$, Theorems 1.2 and 2.5 remain true if there exists a feasible solution at all.

e) The maximum principle for $(\mathrm{P})_\mathrm{K}$**.** By use of Theorem 2.4, the statements [12, Theorem 3.1, p. 225, and Theorem 3.4, p. 231] can be carried over to the unrelaxed deposit problem $(\mathrm{P})_\mathrm{K}$.

Theorem 2.6 (ε-maximum principle for $(\mathrm{P})_\mathrm{K}$)**.** *Let* (x^*, u^*) *be a global minimizer of the problem* $(\mathrm{P})_\mathrm{K}$ *under all assumptions of Theorem* 2.4. *Then for arbitrary* $\varepsilon > 0$ *there exist multipliers* $y^\varepsilon \in L^{nm}_q(\Omega)$ $(p^{-1} + q^{-1} = 1)$ *satisfying the* ε*-maximum condition (in integrated form),* $(\mathcal{M})_\varepsilon$*, and the canonical equation* $(\mathcal{K})_\varepsilon$*:*

$$(\mathcal{M})_\varepsilon: \qquad \varepsilon + \sum_{i,j} \int_\Omega (\, u^*_{ij}(t) - u_{ij}(t)) y^\varepsilon_{ij}(t)\, dt \geqslant 0$$

$$\forall\, u \in L^{nm}_\infty(\Omega):\ u(t) \in \mathrm{U}(t)\ \forall\, t \in \Omega$$

$$(\mathcal{K})_\varepsilon: \qquad \sum_{i,j} \int_\Omega y^\varepsilon_{ij}(t)\, \zeta_{i;t_j}(t)\, dt - \sum_k \int_\Omega \zeta_k t)\, d\alpha_k(t) = 0$$

$$\zeta \in W^{1,n}_p(\Omega):\ \zeta(t_0) = \mathfrak{o}_n.$$

Proof. As mentioned above, the relaxed problem $(\overline{\mathrm{P}})_\mathrm{K}$ satisfies assumptions (V1) – (V4) from [12], and thus we can apply [12, Theorem 3.1, p. 225]. Its proof in [12] is not influenced by the use of the generalized control restrictions supp $\mu_t \subseteq \mathrm{U}(t)$ in the definition of $\mathfrak{M}_\mathrm{U}$. If (x^*, u^*) is a global minimizer of $(\mathrm{P})_\mathrm{K}$ then (x^*, μ^*) with $\mu^*_t = \delta_{u^*(t)}$ forms a global minimizer of $(\overline{\mathrm{P}})_\mathrm{K}$ since both problems have the same minimal value (Theorem 2.4) and $J(x^*, u^*) = \bar{J}(x^*, \mu^*)$. By the above cited theorem, we find for arbitrary

$\varepsilon > 0$ multipliers $y_{ij}^{\varepsilon} \in L_q^{nm}(\Omega)$ which fulfill its ε-maximum condition and the canonical equation together with (x^*, μ^*). In the ε-maximum condition from [12],

$$\varepsilon + \sum_{i,j} \int_{\Omega} \int_{\mathrm{U}} v_{ij} \left[d\delta_{u^*(t)}(v) - d\mu_t(v) \right] y_{ij}^{\varepsilon}(t)\, dt \geqslant 0 \quad \forall\, \mu \in \mathfrak{M}_{\mathrm{U}}, \tag{14}$$

we can substitute each generalized control $\mu \in \mathfrak{M}_{\mathrm{U}}$ by ordinary controls in the sense of Theorem 2.4 and vice versa, so that we arrive at $(\mathcal{M})_{\varepsilon}$ while $(\mathcal{K})_{\varepsilon}$ carries over formally unchanged. ∎

Remark. Theorem 2.6 differs from [8, Theorem 2, p. 395] in the choose of the spaces of the multipliers y^{ε} as well as of the test functions in the canonical equation.

Theorem 2.7 (Maximum principle for $(\mathrm{P})_{\mathrm{K},\mathcal{B}^1}$ with $\varepsilon = 0$)**.** *Let (x^*, u^*) be a global minimizer of the problem $(\mathrm{P})_{\mathrm{K},\mathcal{B}^1}$ (the weak derivatives $x^*_{i;\,t_j}$ have representatives from first Baire class) under all assumptions of Theorem* 2.4. *Then there exist multipliers $\nu \in (rca\,(\Omega, \mathfrak{B}))^{nm}$ satisfying the maximum condition with $\varepsilon = 0$ (in integrated form), $(\mathcal{M})_0$, and the canonical equation $(\mathcal{K})_0$:*

$$(\mathcal{M})_0: \qquad \sum_{i,j} \int_{\Omega} (u^*_{ij}(t) - u_{ij}(t)) d\nu_{ij}(t) \geqslant 0$$
$$\forall\, u \in \mathcal{B}^{1,nm}(\Omega) \colon u(t) \in \mathrm{U}(t)\ \forall\, t \in \Omega$$

$$(\mathcal{K})_0: \qquad \sum_{i,j} \int_{\Omega} \zeta_{i;\,t_j}(t)\, d\nu_{ij}(t) - \sum_k \int_{\Omega} \zeta_k(t)\, d\alpha_k(t) = 0$$
$$\zeta \in C^{1,n}(\Omega) \colon \zeta(t_0) = \mathfrak{o}_n.$$

Proof. By Lemma 2.1, the relaxed problem $(\overline{\mathrm{P}})_{\mathrm{K}}$ satisfies all assumptions of [12, Theorem 3.4, p. 231]. Its proof in [12] is also not influenced by the formal difference in the definition of $\mathfrak{M}_{\mathrm{U}}$. If (x^*, u^*) is a global minimizer of $(\mathrm{P})_{\mathrm{K}}$ having weak derivatives $x^*_{i;\,t_j}$ with representatives from the first Baire class then, as in the proof of Theorem 2.6, (x^*, μ^*) with $\mu^*_t = \delta_{u^*(t)}$ is a global minimizer of $(\overline{\mathrm{P}})_{\mathrm{K}}$. After correcting the error in the choose of the test function space in $(\mathcal{K})'_0$ ($\zeta \in C^{1,n}(\Omega)$ instead of $\zeta \in W^{1,n}_{\infty}(\Omega)$ with $\zeta_{i;\,t_j} \in \mathcal{B}^1(\Omega)$, see [13, Erratum]) and replacing in $(\mathcal{M})'_0$ the generalized controls $\mu \in \mathfrak{M}'_{\mathrm{U}}$ by ordinary controls in the sense of Theorem 2.4. (even generating functions $x_{i;\,t_j}$ from the first Baire class on the whole domain Ω), one has derived from the conditions $(\mathcal{K})_0'$ and $(\mathcal{M})_0'$ of the above cited theorem the demanded conditions $(\mathcal{K})_0$ and $(\mathcal{M})_0$. ■

3 Duality theorems

a) Construction of the dual problem. Two optimization problems, a minimizing problem (P) and a maximizing problem (D), are said to be *weakly dual* in the case that inf (P) $\geqslant$ sup(D), and *strongly dual* if equality holds: inf (P) = sup(D) (cf. Klötzler [7]). Under the assumptions of Theorem 1.2, the minimal values of the problems $(\mathrm{P})_{\mathrm{K}}$, $(\mathrm{P})_{\mathrm{K},\mathcal{B}^0}$ and $(\mathrm{P})_{\mathrm{K},\mathcal{B}^1}$ coincide, and the dual problem can be formulated in relation to $(\mathrm{P})_{\mathrm{K},\mathcal{B}^0}$. *Thus it is possible to use Radon measures as dual variables.*

Definition 3.1. We define the sets X_0, X_1 and Y_0 and a functional $\Phi\colon \mathrm{X}_0 \times \mathrm{Y}_0 \to \mathbb{R}$ by

$$\begin{aligned} \mathrm{X}_0 = \{(x,u) \in W^{1,n}_\infty(\Omega) \times L^{nm}_\infty(\Omega) | x_{i;\,t_j} \in \mathcal{B}^0(\Omega),\ u_{ij} \in \mathcal{B}^1(\Omega), \\ u(t) \in \mathrm{U}(t)\ \ \forall\, t \in \Omega,\ x(t_0) = \mathfrak{o}_n\}; \end{aligned} \tag{15.1}$$

$$\mathrm{X}_1 = \{(x,u) \in W^{1,n}_\infty(\Omega) \times L^{nm}_\infty(\Omega) | x_{i;\,t_j}(t) = u_{ij}(t)\ \text{ a.e. on }\ \Omega\}; \tag{15.2}$$

$$\mathrm{Y}_0 = (rca\,(\Omega, \mathfrak{B}))^{nm}; \tag{15.3}$$

$$\Phi(x,u,\nu) = J(x,u) + \sum_{i,j} \int_\Omega \Big[x_{i;\,t_j}(t) - u_{ij}(t)\Big] d\nu_{ij}(t). \tag{16}$$

Lemma 3.2. *Let* $(\mathrm{P})_{\mathrm{K}}$ *satisfy all assumptions of Theorem* 1.2. *Then the functional* $\Phi(x,\mu,\nu)$ *satisfies the equivalence condition*

$$\inf_{(x,u)\in \mathrm{X}_0 \cap \mathrm{X}_1} J(x,u) = \inf_{(x,u)\in \mathrm{X}_0}\ \sup_{\nu \in \mathrm{Y}_0} \Phi(x,\mu,\nu).$$

Proof. Given a pair $(x,u) \in \mathrm{X}_0$ where $x_{i_0;\,t_{j_0}}(t') - u_{i_0,j_0}(t') > 0$ (without loss of generality) for certain indices i_0, j_0 at a point $t' \in \Omega$. Then we have along the sequence of the measures $\nu^N \in (rca\,(\Omega,\mathfrak{B}))^{nm}$ with $\nu^N_{i_0,j_0} = N \cdot \delta_{t'}$ and $\nu^N_{ij} = \mathfrak{o}$ for $i \neq i_0$ or $j \neq j_0$

$$\lim_{N\to\infty} \Phi(x,u,\nu^N) = J(x,u) + \lim_{N\to\infty} N \cdot \Big[x_{i_0;\,t_{j_0}}(t') - u_{i_0,j_0}(t')\Big] = +\infty.$$

It follows that $\sup_{\nu\in \mathrm{Y}_0} \Phi(x,u,\nu) = J(x,u)$ if $(x,u) \in \mathrm{X}_0$ satisfies (1.2) for all $t \in \Omega$ (consequently, $(x,u) \in \mathrm{X}_1$), and $\sup_{\nu\in \mathrm{Y}_0} \Phi(x,u,\nu) = +\infty$ else. By Theorem 1.2, $(\mathrm{P})_{\mathrm{K},\mathcal{B}^0}$ admits a minimizing sequence $\{(x^N, u^N)\}$ of feasible processes which fulfill the state equations (1.2) everywhere on Ω. Along this sequence, we have $\inf_{(x,u)\in \mathrm{X}_0\cap \mathrm{X}_1} J(x,u) = \lim_{N\to\infty} J(x^N,u^N) = \lim_{N\to\infty} \sup_{\nu\in \mathrm{Y}_0} \Phi(x^N,u^N,\nu) = \inf_{(x,u)\in \mathrm{X}_0} \sup_{\nu\in \mathrm{Y}_0} \Phi(x,u,\nu)$, and the proof is complete. ■

Theorem 3.3 (Weak duality theorem for $(\mathrm{P})_{\mathrm{K}}$)**.** *Let* $(\mathrm{P})_{\mathrm{K}}$ *satisfy all assumptions of Theorem* 1.2. *Then there is weak duality between each of the problems* $(\mathrm{P})_{\mathrm{K}}$, $(\mathrm{P})_{\mathrm{K},\mathcal{B}^0}$ *and* $(\mathrm{P})_{\mathrm{K},\mathcal{B}^1}$ *and the following problem* $(\mathrm{D})'_{\mathrm{K}}$ (17.1) – (17.2):

$$G'(\nu) = \inf_{(x,u)\,\in\,\mathrm{X}_0} \Phi(x,u,\nu) \longrightarrow \mathrm{Max}! \tag{17.1}$$

$$\nu \in \mathrm{Y}_0 = (\, rca\,(\Omega, \mathfrak{B})\,)^{nm}. \tag{17.2}$$

Proof. It holds $\inf(\mathrm{P})_{\mathrm{K}} = \inf(\mathrm{P})_{\mathrm{K},\mathcal{B}^0} = \inf(\mathrm{P})_{\mathrm{K},\mathcal{B}^1}$ (by Theorem 1.2); $\inf(\mathrm{P})_{\mathrm{K},\mathcal{B}^0} = \inf_{(x,u)\in \mathrm{X}_0\cap \mathrm{X}_1} J(x,u)$ (by construction); $\inf_{(x,u)\in \mathrm{X}_0\cap \mathrm{X}_1} J(x,u)$ $= \inf_{(x,u)\in \mathrm{X}_0} \sup_{\nu\in \mathrm{Y}_0} \Phi(x,u,\nu)$ (by Lemma 3.2) and, finally,
$\inf_{(x,u)\in \mathrm{X}_0} \sup_{\nu\in \mathrm{Y}_0} \Phi(x,u,\nu) \geqslant \sup_{\nu\in \mathrm{Y}_0} \inf_{(x,u)\in \mathrm{X}_0} \Phi(x,u,\nu) = \sup(\mathrm{D})'_{\mathrm{K}}$. ■

b) Strong duality. Note that $G'(\nu)$ can be expressed as follows:

$$\begin{aligned} G'(\nu) = &\inf_{\substack{x\in C^{1,n}(\Omega),\, x(t_0)=\mathfrak{o}_n \\ u\,\in\,\mathcal{B}^{1,nm}(\Omega),\ u(t)\in \mathrm{U}(t)\ \forall\, t\in\Omega}} \Big[-\sum_k \int_\Omega x_k(t)\, d\alpha_k(t) \\ &+\sum_{i,j} \int_\Omega x_{i;\,t_j}(t)\, d\nu_{ij}(t) - \sum_{i,j} \int_\Omega u_{ij}(t)\, d\nu_{ij}(t)\Big]. \end{aligned} \tag{18}$$

Then, by restriction of the feasible domain, we receive from $(\mathrm{D})'_{\mathrm{K}}$ the problem $(\mathrm{D})_{\mathrm{K}}$ (2.1) – (2.2) mentioned in the introduction. Obviuosly, it holds

$$\sup(\mathrm{D})'_{\mathrm{K}} \geqslant \sup(\mathrm{D})_{\mathrm{K}};\ G'(\nu) = G(\nu)\ \text{ for all } \nu \text{ feasible in } (\mathrm{D})_{\mathrm{K}}. \tag{19}$$

The feasible set of $(\mathrm{D})_{\mathrm{K}}$ is weak*-closed and convex, the cost functional $G(\cdot)$ is concave in ν, and thus the set of the global maximizers of $(\mathrm{D})_{\mathrm{K}}$ is convex.

Theorem 3.4 (Strong duality theorem for $(\mathrm{P})_{\mathrm{K}}$)**.** *Let* $(\mathrm{P})_{\mathrm{K}}$ *satisfy all assumptions of Theorem* 1.2. *Then the problems* $(\mathrm{D})'_{\mathrm{K}}$ *and* $(\mathrm{D})_{\mathrm{K}}$ *have the same maximal value, and each of the problems* $(\mathrm{D})'_{\mathrm{K}}$ *and* $(\mathrm{D})_{\mathrm{K}}$ *is strongly dual to each of the problems* $(\mathrm{P})_{\mathrm{K}}$, $(\mathrm{P})_{\mathrm{K},\mathcal{B}^0}$, $(\mathrm{P})_{\mathrm{K},\mathcal{B}^1}$ *and* $(\overline{\mathrm{P}})_{\mathrm{K}}$.

Proof. By Theorem 2.5, $(\mathrm{P})_{\mathrm{K}}$ admits a global minimizer (x^*, u^*) (which is eventually not feasible in $(\mathrm{P})_{\mathrm{K},\mathcal{B}^0}$ or $(\mathrm{P})_{\mathrm{K},\mathcal{B}^1}$). Then, by Theorem 2.6, for each $\varepsilon_N = 1/N$, $N \in \mathbb{N}_1$, there exists a multiplier $y^{\varepsilon_N} = y^N$ satisfying the conditions $(\mathcal{M})_\varepsilon$ and $(\mathcal{K})_\varepsilon$ together with (x^*, u^*), and y^N can be interpreted as the density of a λ^m-absolutely continuous measure ν^N. By $(\mathcal{K})_\varepsilon$, each of the measures ν^N is feasible in $(\mathrm{D})'_{\mathrm{K}}$ as well as in $(\mathrm{D})_{\mathrm{K}}$ (since $C^{1,n}(\Omega) \subset W_p^{1,n}(\Omega)$). Then it follows from (18):

$$G'(\nu^N) = \inf_{\substack{x \in C^{1,n}(\Omega),\, x(t_0)=\mathfrak{o}_n \\ u \in \mathcal{B}^{1,nm}(\Omega),\, u(t) \in \mathrm{U}(t)\ \forall\, t \in \Omega}} \Big[J(x^*, u^*) \tag{20}$$

$$- \sum_k \int_\Omega (x_k(t) - x_k^*(t))\, d\alpha_k(t) + \sum_{i,j} \int_\Omega \Big((x_{i;\,t_j}(t)) - x^*_{i;\,t_j}(t) \Big)\, dy^N_{ij}(t)\, dt$$
$$- \sum_{i,j} \int_\Omega \Big(u_{ij}(t) - u^*_{ij}(t) \Big)\, dy^N_{ij}(t)\, dt + \sum_{i,j} \int_\Omega \Big(x^*_{i;\,t_j}(t) - u^*_{ij}(t) \Big)\, dy^N_{ij}(t)\, dt \Big].$$

Together with the conditions $(\mathcal{M})_{\varepsilon^N}$, $(\mathcal{K})_{\varepsilon^N}$ and the feasibility of (x^*, u^*) for $(\mathrm{P})_{\mathrm{K}}$ we conclude that $G'(\nu^N) \geqslant J(x^*, u^*) - 1/N$. Using Theorem 3.3 and (19), we arrive at the inequalities $J(x^*, u^*) = \inf(\mathrm{P})_{\mathrm{K}} \geqslant \sup(\mathrm{D})'_{\mathrm{K}} \geqslant \sup(\mathrm{D})_{\mathrm{K}} \geqslant G(\nu^N) = G'(\nu^N) \geqslant J(x^*, u^*) - 1/N$ for all $N \in \mathbb{N}_1$, and the relation $\inf(\mathrm{P})_{\mathrm{K}} = \sup(\mathrm{D})_{\mathrm{K}}$ is proved. ■

c) Saddle-point conditions. ***Proof of Theorem* 1.3.** Assume that $\nu^* \in (rca\,(\Omega, \mathfrak{B}))^{nm}$ and a feasible pair (x^*, u^*) of $(\mathrm{P})_{\mathrm{K},\mathcal{B}^1}$ satisfy the conditions $(\mathcal{M})_0^*$, $(\mathcal{K})_0^*$ and $(\mathcal{D})_0^*$. By $(\mathcal{K})_0^*$, ν^* is a feasible element of $(\mathrm{D})_{\mathrm{K}}$. From $(\mathcal{M})_0^*$ we deduce

$$G(\nu^*) = \inf_{\substack{u \in \mathcal{B}^{1,nm}(\Omega) \\ u(t) \in \mathrm{U}(t)\ \forall t \in \Omega}} \Big[- \sum_{i,j} \int_\Omega u_{ij}(t)\, d\nu^*_{ij}(t) \Big] = - \sum_{i,j} \int_\Omega u^*_{ij}(t)\, d\nu^*_{ij}(t),$$

from which, together with $(\mathcal{D})_0^*$, it follows that $J(x^*, u^*) = G(\nu^*)$, and (x^*, u^*) and ν^* form a saddle point for the problems $(\mathrm{P})_{\mathrm{K},\mathcal{B}^1} - (\mathrm{D})_{\mathrm{K}}$. ■

Remark. Since the value of the cost functional does not depend on u one can construct from a given global minimizer (x^*, u^*) of $(\mathrm{P})_{\mathrm{K},\mathcal{B}^1}$ non-denumerably many different global minimizers (x^*, u^{**}) of $(\mathrm{P})_{\mathrm{K},\mathcal{B}^1}$ by the setting $u^{**}_{ij}(t) = \chi_{(\Omega \setminus \mathrm{N}_{ij})}(t) \cdot u^*_{ij}(t) + \chi_{\mathrm{N}_{ij}}(t) \cdot u_{ij}(t)$. Here N_{ij} are λ^m-null sets with characteristic functions from the first Baire class while $u \in \mathcal{B}^{1,nm}(\Omega)$ with $u(t) \in \mathrm{U}(t)$ for all $t \in \Omega$ can be chosen arbitrarily. On this fact, it can be founded a partial converse of Theorem 1.3.

Theorem 3.5 (Partial converse of Theorem 1.3)**.** *Let* $(\mathrm{P})_{\mathrm{K}}$ *satisfy all assumptions of Theorem* 1.2. *Assume that* (x^*, u^*) *and* ν^* *are feasible elements of* $(\mathrm{P})_{\mathrm{K},\mathcal{B}^1}$ *resp.* $(\mathrm{D})_{\mathrm{K}}$ *with* $J(x^*, u^*) = G(\nu^*)$. *Let* ν'_{ij} *and* ν''_{ij} *denote the absolutely continuous resp. singular parts of the components of* ν^* *in the Lebesgue decomposition with respect to* λ^m. *Further assume that*

$\lambda^m(\mathrm{supp}\ \nu''^{,+}_{ij}) = 0$, $\lambda^m(\mathrm{supp}\ \nu''^{,-}_{ij}) = 0$ *and* $\mathrm{supp}\ \nu''^{,+}_{ij} \cap \mathrm{supp}\ \nu''^{,-}_{ij} = \emptyset \forall\, i,\, j$.

Then there exists a function $u^{**} \in \mathcal{B}^{1,nm}(\Omega)$ *with the following properties:*

1) $u^*(t) = u^{**}(t)$ *for a.e.* $t \in \Omega$ (u^* *and* u^{**} *belong to the same* L^{nm}_∞*-equivalence class*).
2) $u^{**}(t) \in \mathrm{U}(t)\ \forall\, t \in \Omega$ (*the pair* (x^*, u^{**}) *is feasible in* $(\mathrm{P})_{\mathrm{K},\mathcal{B}^1}$).
3) $J(x^*, u^*) = J(x^*, u^{**})$ (*the triple* (x^*, u^{**}, ν^*) *forms also a saddle point for the problems* $(\mathrm{P})_{\mathrm{K},\mathcal{B}^1}$ – $(\mathrm{D})_{\mathrm{K}}$).
4) *The triple* (x^*, u^{**}, ν^*) *satisfies the saddle-point conditions* $(\mathcal{M})^*_0$, $(\mathcal{K})^*_0$ *and* $(\mathcal{D})^*_0$ *of Theorem* 1.3.

Proof. From the feasibility of ν^* in $(\mathrm{D})_{\mathrm{K}}$ it follows that $(\mathcal{K})^*_0$ is valid. Now we distinguish two cases:

Case 1. u^* *and* ν^* *satisfy* $(\mathcal{M})^*_0$, i.e.

$$-\sum_{i,j} \int_\Omega u^*_{ij}(t)\, d\nu^*_{ij} = \inf_{\substack{u \in \mathcal{B}^{1,nm}(\Omega) \\ u(t) \in \mathrm{U}(t)\ \ \forall\, t \in \Omega}} \Big[-\sum_{i,j} \int_\Omega u_{ij}(t)\, d\nu^*_{ij}(t)\Big] = G(\nu^*). \tag{21}$$

Then from $J(x^*, u^*) = G(\nu^*)$ it results $(\mathcal{D})^*_0$, and the theorem is valid with $u^*(t) = u^{**}(t)$ for all $t \in \Omega$.

Case 2. u^* *and* ν^* *violate* $(\mathcal{M})^*_0$ what means

$$-\sum_{i,j} \int_\Omega u^*_{ij}(t)\, d\nu^*_{ij} > \inf_{u \in \ldots} \Big[-\sum_{i,j} \int_\Omega u_{ij}(t)\, d\nu^*_{ij}(t)\Big] = G(\nu^*). \tag{22}$$

Here and below, the infimum is taken over the same function set as in (21). Using the members $(x^N,\, u^N) \in C^{\infty,n}(\Omega) \times C^{\infty,nm}(\Omega)$ of the minimizing sequence $\{(x^N,\, u^N)\}$ from Theorem 1.2 as test functions in $(\mathcal{K})^*_0$, it follows:

$$J(x^N, u^N) = -\sum_k \int_\Omega x^N_k(t)\, d\alpha_k(t) = -\sum_{i,j} \int_\Omega u^N_{ij}(t)\, d\nu^*_{ij}(t) \Longrightarrow \tag{23}$$

$$J(x^*, u^*) = \lim_{N\to\infty} J(x^N, u^N) = \lim_{N\to\infty} \Big[-\sum_{i,j} \int_\Omega u_{ij}^N(t)\, d\nu_{ij}^*(t) \Big] = G(\nu^*).$$

We subject each of the measures ν_{ij}^* to the Lebesgue decomposition w. r. to the measure λ^m into the absolutely continuous part ν_{ij}' and the singular part ν_{ij}'' [4, Theorem 14, p. 132]. The densities of the absolutely continuous parts are denoted by $y_{ij}' \in L_1(\Omega)$. Since the functions u_{ij}^N are bounded on Ω, from $u_{ij}^N \to^{L_1(\Omega)} u_{ij}^*$ it follows the convergence $u_{ij}^N\, y_{ij}' \to^{L_1(\Omega)} u_{ij}^*\, y_{ij}'$, and we have

$$\begin{aligned} \inf_{u\in\ldots} \Big[-\sum_{i,j} \int_\Omega u_{ij}(t)\, d\nu_{ij}^*(t) \Big] &= \lim_{N\to\infty} \Big[-\sum_{i,j} \int_\Omega u_{ij}^N(t)\, d\nu_{ij}^*(t) \Big] \\ &= -\sum_{i,j} \int_\Omega u_{ij}^*(t)\, y_{ij}'(t)\, dt - \lim_{N\to\infty} \sum_{i,j} \int_\Omega u_{ij}^N(t)\, d\nu_{ij}''(t). \end{aligned} \tag{24}$$

Further, the singular parts are subjected to the Jordan decomposition $\nu_{ij}'' = \nu_{ij}^{\prime\prime,+} - \nu_{ij}^{\prime\prime,-}$ [4, p. 98, Theorem 8]; both parts are still Radon measures [4, Lemma 12, p. 137] whose supports, by assumption, are compact λ^m-null sets. We abbreviate: supp $\nu_{ij}^{\prime\prime,+} = \mathrm{N}_{ij}^+$, supp $\nu_{ij}^{\prime\prime,+} = \mathrm{N}_{ij}^-$, $\mathrm{N}_{ij} = \mathrm{N}_{ij}^+ \cup \mathrm{N}_{ij}^-$ and define the functions

$$u_{ij}^{**}(t) = \chi_{(\Omega\setminus \mathrm{N}_{ij})}(t)\, u_{ij}^*(t) + \chi_{\mathrm{N}_{ij}^+}(t) \inf_N\, u_{ij}^N(t) + \chi_{\mathrm{N}_{ij}^-}(t) \sup_N\, u_{ij}^N(t). \tag{25}$$

All u_{ij}^{**} are contained in the first Baire class since the characteristic functions $\chi_{(\Omega\setminus \mathrm{N}_{ij})}$, $\chi_{\mathrm{N}_{ij}^+}$ and $\chi_{\mathrm{N}_{ij}^-}$ (cf. [12, Lemma 1.4, p. 220]) as well as the pointwise infimum resp. supremum of the sequence $\{u_{ij}^N\}$ of continuous functions have the same property [3, Theorem 10, p. 398]. The values of the functions u_{ij}^* and u_{ij}^{**} differ at most on the null sets N_{ij}. By Theorem 1.2, we have $u^N(t) \in \mathrm{U}(t)$ for all $N \in \mathbb{N}_1$ and for all $t \in \Omega$; then it follows from the closedness of the sets $\mathrm{U}(t)$ (Lemma 2.1) that $\inf_N u_{ij}^N(t) \in \mathrm{U}(t)$ as well as $\sup_N u_{ij}^N(t) \in \mathrm{U}(t)$ for all $t \in \Omega$. Together with $\mathrm{N}_{ij}^+ \cap \mathrm{N}_{ij}^- = \emptyset$ (by assumption) it results that $u^{**}(t) \in \mathrm{U}(t)$ for all $t \in \Omega$. Thus u^{**} fulfills the assertions 1) – 3) of our theorem.

We have still to prove that (x^*, u^{**}, ν^*) satisfies the saddle-point conditions. For this purpose, let us introduce the following abbreviations:

$$\begin{aligned} L &= \lim_{N\to\infty} -\sum_{i,j} \int_\Omega u_{ij}^N(t)\, d\nu_{ij}^*(t); \quad L' = \lim_{N\to\infty} -\sum_{i,j} \int_\Omega u_{ij}^N(t)\, y_{ij}'(t)\, dt; \\ L'' &= \lim_{N\to\infty} -\sum_{i,j} \int_{\mathrm{N}_{ij}} u_{ij}^N(t)\, d\nu_{ij}''(t); \end{aligned}$$

$$J = \inf_{u\in\ldots}\Big[-\sum_{i,j}\int_\Omega u_{ij}(t)\,d\nu^*_{ij}(t)\Big];\quad J' = \inf_{u\in\ldots}\Big[-\sum_{i,j}\int_\Omega u_{ij}(t)\,y'_{ij}(t)\,dt\Big];$$
$$J'' = \inf_{u\in\ldots}\Big[-\sum_{i,j}\int_\Omega u_{ij}(t)\,d\nu''_{ij}(t)\Big].$$

In these notations, it holds obviously

$$L' + L'' = L = J \geqslant J' + J''. \tag{26}$$

Here $J > J' + J''$ leads to a contradiction since one could choose then functions u', $u'' \in \mathcal{B}^{1,nm}(\Omega)$ with $u'(t) \in \mathrm{U}(t)$ and $u''(t) \in \mathrm{U}(t)$ for all $t \in \Omega$ in such a way that

$$J > -\sum_{i,j}\int_\Omega u'_{ij}(t)y'_{ij}(t)\,dt - \sum_{i,j}\int_{\mathrm{N}_{ij}} u''_{ij}(t)\,d\nu''_{ij}(t) \tag{27}$$

but $u_{ij}(t) = \chi_{(\Omega\setminus\mathrm{N}_{ij})}(t)\cdot u'(t) + \chi_{\mathrm{N}_{ij}}(t)\cdot u''(t)$ would be feasible for the construction of J as a function of first Baire class. So we have

$$L' + L'' = J' + J''. \tag{28}$$

As immediate consequence of (28), the single equations $L' = J'$ and $L'' = J''$ result since $L' < J'$ as well as $L'' < J''$ are impossible. $L' = J'$ means

$$\begin{aligned} -\sum_{i,j}\int_\Omega u^*_{ij}(t)\,y'_{ij}(t)\,dt &= -\sum_{i,j}\int_\Omega u^{**}_{ij}(t)\,y'_{ij}(t)\,dt \\ &= \inf_{u\in\ldots}\Big[-\sum_{i,j}\int_\Omega u_{ij}(t)\,y'_{ij}(t)\,dt\Big]. \end{aligned} \tag{29}$$

Further, it holds for all $N \in \mathbb{N}_1$:

(30)
$$\begin{aligned} &-\sum_{i,j}\int_\Omega u^N_{ij}(t)\,d\nu''_{ij}(t) = -\sum_{i,j}\int_{\mathrm{N}^+_{ij}} u^N_{ij}(t)\,d\nu''^{,+}_{ij}(t) + \sum_{i,j}\int_{\mathrm{N}^-_{ij}} u^N_{ij}(t)\,d\nu''^{,-}_{ij}(t) \\ &\geqslant\ -\sum_{i,j}\int_{\mathrm{N}^+_{ij}} \inf_N (u^N_{ij}(t))d\nu''^{,+}_{ij}(t) + \sum_{i,j}\int_{\mathrm{N}^-_{ij}} \sup_N (u^N_{ij}(t)\,)\,d\nu''^{,-}_{ij}(t) \\ &= -\sum_{i,j}\int_\Omega u^{**}_{ij}(t)\,d\nu''_{ij}(t)\ \geqslant\ J''. \end{aligned}$$

The last inequality results from the fact that u^{**} is feasible for the construction of J''. After the limit passage $N \to \infty$ in (30) we arrive at

$$J'' = L'' \geqslant\ -\sum_{i,j}\int_\Omega u^{**}_{ij}(t)\,d\nu''_{ij}(t) \geqslant J''. \tag{31}$$

Together with (28), it results from equations (29) and (31):

$$-\sum_{i,j}\int_\Omega u^{**}_{ij}(t)\,d\nu^*_{ij}(t) = J' + J'' = J = \inf_{u\in\ldots}\Big[-\sum_{i,j}\int_\Omega u_{ij}(t)\,d\nu^*_{ij}(t)\Big], \tag{32}$$

so that u^{**} and ν^* satisfy condition $(\mathcal{M})^*_0$. As in the former case, $(\mathcal{D})^*_0$ is then satisfied also, and the proof is complete. ■

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