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CONTROLLABILITY FOR IMPULSIVE SEMILINEAR FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH A NON-COMPACT EVOLUTION OPERATOR

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Abstract

We study a controllability problem for a system governed by a semilinear functional differential inclusion in a Banach space in the presence of impulse effects and delay. Assuming a regularity of the multivalued non-linearity in terms of the Hausdorff measure of noncompactness we do not require the compactness of the evolution operator generated by the linear part of inclusion. We find existence results for mild solutions of this problem under various growth conditions on the nonlinear part and on the jump functions. As example, we consider the controllability of an impulsive system governed by a wave equation with delayed feedback.

Keywords: evolution differential inclusion, impulsive inclusion, control system, controllability, mild solution, condensing multimap, fixed point.

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1. Introduction

Impulsive differential equations and inclusions form an appropriate model for describing phenomena where systems instantaneously change their state. For this reason they find wide applications in several fields of applied sciences, such as Biology, Economics, and Physics. Concerning the theory of impulsive differential equations and inclusions we refer, for instance, to the monographs [16, 5, 20] and the references therein. Among recent works on the study of impulsive differential equations and inclusions associated to various boundary conditions such as Cauchy initial condition, periodicity conditions and delay conditions, we may point out, e.g., [6, 7, 9, 10, 15], and [23]. Solutions to this type of problems are functions that may be not continuous at some fixed moments. To deal with such functions with values in a Banach space E, we denote by the symbol $\mathcal{C}([a_1, a_2]; E)$ the space of piecewise continuous functions $c : [a_1, a_2] \to E$ with a finite number of discontinuity points $\{t_*\}$ such that $t_* \neq a_2$ and all values

$$c(t_*^+) = \lim_{h \to 0^+} c(t_* + h), \quad c(t_*^-) = \lim_{h \to 0^-} c(t_* + h)$$

are finite. We will consider the space $\mathcal{C}([a_1, a_2]; E)$ as a normed space with the norm:

$$\|c\|_{\mathcal{C}} = \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \|c(t)\|_E \, dt.$$

(Notice that the same construction can be applied also in [6, 7]).

In this paper we deal with a control system governed by a semilinear functional differential inclusion with delay in a Banach space in the presence of impulse effects. More precisely, for a fixed $\tau > 0$ and a given initial function $\psi \in \mathcal{C}([-\tau,0];E)$, we consider the object which is described by the following relations with delay in a separable Banach space E:

$$(1.1)\qquad \begin{cases} y'(t) \in A(t)y(t) + F(t,y_t) + Bu(t) \ \text{ a.e. } t \in [0,b],\ t \neq t_k, \\ k = 1,\dots,N \\ y(t) = \psi(t)\ ,\ t \in [-\tau,0] \\ y(t_k^-) = y(t_k), \quad k = 1,\dots,N; \\ y(t_k^+) = y(t_k) + I_k(y_{t_k})\ ,\ k = 1,\dots,N. \end{cases}$$

Here $\{A(t)\}_{t\in[0,b]}$ is a family of linear (not necessarily bounded) operators in E generating an evolution operator; F is a Carathéodory type multifunction; the function $y_t \in \mathcal{C}([-\tau,0];E)$ is defined by the relation $y_t(\theta) = y(t+\theta)$, $\theta \in [-\tau,0]$; the points $0 = t_0 < t_1 < \cdots < t_N < t_{N+1} = b$ are given and $I_k : \mathcal{C}([-\tau,0];E) \to E$, $k = 1,\dots,N$, are given impulse functions.

Further, we suppose that the control function $u(\cdot)$ is considered in the space $L^2([0,b];U)$, where U is a Banach space of controls and $B : U \to E$ is a bounded linear operator.

We will consider the controllability problem for this system, i.e., we will study conditions under which there exists a trajectory $y(\cdot)$ of the above system reaching a given state at the final time b.

Similar problems are studied in literature, see e.g. [1, 2, 4, 11, 17, 18]. It should be mentioned that in some works a constant linear part is considered and the compactness of the semigroup generated by it is required. Triggiani in [21, 22] has proved that in an infinite-dimensional case this assumption is in contrast to the hypothesis of the controllability of the corresponding linear problem, hypothesis that is quite usual for this type of problems. Using the concept of condensing operator, it is possible to avoid this difficulty. For instance, Obukhovskii and Zecca in [18], assuming the regularity of the nonlinear part of a differential inclusion of type (1.1) in terms of the Hausdorff measure of non compactness, were able to consider the controllability of the considered problem without assuming the compactness of the semigroup generated by the linear part. We prove that it is possible to do the same for a linear part that generates an evolution operator and in the presence of impulse effects and delay.

The paper is organized as follows. In Section 2 we recall some definitions and results from multivalued analysis which will be used later. In Section 3 we present the framework in which the problem is considered. In Section 4 we construct a solution operator, whose fixed points are solutions to problem (1.1) and study some of its properties. In Section 5 we provide various growth conditions on the multimap F and on the jump functions I_k which guarantee the existence of a solution of the controllability problem. At last, in Section 6 we consider, as an example, the problem of controllability of an impulsive system with delayed feedback governed by a wave equation.

2. Preliminaries

Let X, Y be two Hausdorff topological vector spaces.

We denote by $\mathcal{P}(Y)$ the family of all non-empty subsets of Y and put

$$\mathcal{K}(Y) = \{C \in \mathcal{P}(Y) \text{ is compact}\};$$
$$\mathcal{K}v(Y) = \{D \in \mathcal{P}(Y) \text{ is compact and convex}\}.$$

A multivalued map (multimap) $F : X \to \mathcal{P}(Y)$ is said to be:

(i) *upper semicontinuous* (u.s.c.) if $F^{-1}(V) = \{x \in X : F(x) \subset V\}$ is an open subset of X for every open $V \subseteq Y$;

(ii) *closed* if its graph $G_F = \{(x, y) \in X \times Y : y \in F(x)\}$ is a closed subset of $X \times Y$.

Notice the following useful property (see, e.g., [13], Theorem 1.1.7).

Proposition 2.1. *Let $F : X \to \mathcal{K}(Y)$ be a u.s.c. multimap. If $\mathcal{K} \subset X$ is a compact set then its image $F(\mathcal{K}) = \bigcup_{x\in\mathcal{K}} F(x)$ is a compact subset of Y.*

Sometimes we will denote a multimap with non-empty values by the symbol $F : X \multimap Y$.

Let $(\mathcal{A}, \geq)$ be a partially ordered set and $\mathcal{E}$ be a real Banach space. We recall that a map $\beta : \mathcal{P}(\mathcal{E}) \to \mathcal{A}$ is called a *measure of noncompactness* (MNC) in $\mathcal{E}$ if

$$\beta(\overline{co}\Omega) = \beta(\Omega)$$

for every $\Omega \in \mathcal{P}(\mathcal{E})$, where $\overline{co}$ denotes the convex closure of a set (see, e.g. [13] for details). A measure of noncompactness β is called:

(i) *monotone*, if $\Omega_0, \Omega_1 \in \mathcal{P}(\mathcal{E})$, $\Omega_0 \subseteq \Omega_1$ imply $\beta(\Omega_0) \leq \beta(\Omega_1)$;

(ii) *nonsingular*, if $\beta(\{c\} \cup \Omega) = \beta(\Omega)$ for every $c \in \mathcal{E}$, $\Omega \in \mathcal{P}(\mathcal{E})$;

(iii) *real*, if $\mathcal{A} = [0, +\infty]$ with the natural ordering and $\beta(\Omega) < +\infty$ for every bounded Ω;

If $\mathcal{A}$ is a cone in a Banach space, a measure of noncompactness β is called:

(iv) *regular*, if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω;

(v) *algebraically semiadditive*, if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for every $\Omega_0, \Omega_1 \in \mathcal{P}(\mathcal{E})$.

A well known example of a measure of noncompactness satisfying all of the above properties is the Hausdorff MNC

$$\chi_{\mathcal{E}}(\Omega) = \inf \{\varepsilon > 0 : \Omega \, \text{has a finite } \varepsilon\text{-net}\}.$$

For a linear bounded operator $L : \mathcal{E} \to \mathcal{E}$, it is possible to define its χ-norm by

$$\|L\|^{(\chi)} = \chi(LB), \tag{2.1}$$

where $B \subset \mathcal{E}$ is the unit ball. It is easy to see that

$$\|L\|^{(\chi)} \leq \|L\|.$$

A multifunction $\Phi : [a_1, a_2] \to \mathcal{P}(\mathcal{E})$ is said to be:

(i) *integrable*, if it admits a selection $\phi(t) \in \Phi(t)$ for a.e. $t \in [a_1, a_2]$, $\phi \in L^1([a_1, a_2]; \mathcal{E})$;

(ii) *integrably bounded*, if there exists a function $\kappa \in L^1([a_1, a_2]; \mathbb{R}_+)$ such that

$$\|\Phi(t)\| := \sup\{\|y\| \; : \; y \in \Phi(t)\} \; \leq \; \kappa(t) \text{ for a.e. t } \in [a_1, a_2].$$

For an integrable multifunction Φ the *multivalued integral* of Φ is defined in the following way:

$$\int_{a_1}^{t} \Phi(s)\, ds = \left\{ \int_{a_1}^{t} \phi(s)\, ds \; : \; \phi(s) \in \Phi(s), \, \phi \in L^1([a_1, a_2]; \mathcal{E}) \right\}.$$

The following result on the estimate of the multivalued integral holds.

Proposition 2.2 [cf. [13] Theorem 4.2.3]**.** *Let $\Phi : [a_1, a_2] \to \mathcal{P}(\mathcal{E})$ be an integrable, integrably bounded multifunction. If there exists a function $q \in L^1([a_1, a_2]; \mathbb{R}_+)$ such that*

$$\chi_{\mathcal{E}}(\Phi(t)) \le q(t) \quad \textit{for a.e.} \quad t \in [a_1, a_2],$$

then

$$\chi_{\mathcal{E}}\left(\int_{a_1}^{t} \Phi(s)\, ds\right) \le \int_{a_1}^{t} q(s)\, ds$$

for all $t \in [a_1, a_2]$.

We will also use the following MNCs, defined on the Banach space of continuous functions $C([a_1, a_2]; \mathcal{E})$:

(i) *the modulus of fiber noncompactness*

$$\varphi(\Omega) = \sup_{t \in [a_1, a_2]} \chi_{\mathcal{E}}(\Omega(t)),$$

where $\Omega(t) = \{y(t) \,:\, y \in \Omega\}$;

(ii) *the modulus of equicontinuity*

$$\mathrm{mod}_C(\Omega) = \lim_{\delta \to 0} \sup_{y \in \Omega} \max_{|t_1 - t_2| \le \delta} \|y(t_1) - y(t_2)\|.$$

Notice that these MNCs satisfy all the above properties except regularity. Let us also mention that the modulus of fiber noncompactness φ is well defined on the space $\mathcal{C}([a_1, a_2]; \mathcal{E})$ too.

If X is a subset of $\mathcal{E}$, a multimap $\mathcal{F} : X \to \mathcal{K}(\mathcal{E})$ is called *condensing* with respect to a MNC β, or β-condensing, if for every $\Omega \subseteq X$ that is not relatively compact we have

$$\beta(F(\Omega)) \not\ge \beta(\Omega).$$

Let $K \subset \mathcal{E}$ be a convex closed set, $V \subset K$ a nonempty bounded relatively open set, β a monotone nonsingular MNC in $\mathcal{E}$ and $\mathcal{F} : \overline{V} \to \mathcal{K}v(K)$ an u.s.c. β-condensing multimap such that $x \notin \mathcal{F}(x)$ for all $x \in \partial V$, where $\overline{V}$ and ∂V denote the relative closure and the relative boundary of the set V in K. In such a setting the relative topological degree

$$\deg_K(i - \mathcal{F}, \overline{V})$$

of the corresponding multivalued vector field $i - \mathcal{F}$ is well defined and satisfies the standard properties (see e.g. [13]). In particular the condition

$$\deg_K(i - \mathcal{F}, \overline{V}) \neq 0$$

implies that the fixed point set $\mathrm{Fix}\,\mathcal{F} = \{x : x \in \mathcal{F}(x)\}$ is a nonempty compact subset of V.

The application of the topological degree theory yields the following fixed point principles which we will use in the sequel.

Theorem 2.1 ([13], Corollary 3.3.1)**.** *Let $\mathcal{M}$ be a bounded convex closed subset of $\mathcal{E}$ and $\mathcal{F} : \mathcal{M} \to \mathcal{K}v(\mathcal{M})$ an u.s.c. β-condensing multimap. Then* $\mathrm{Fix}\,\mathcal{F}$ *is nonempty and compact.*

Theorem 2.2 (cf. [13], Theorem 3.3.4)**.** *Let $a \in V$ be an interior point and $\mathcal{F} : \overline{V} \to Kv(K)$ an u.s.c. β-condensing multimap satisfying the boundary condition*

$$x - a \notin \lambda\,(\mathcal{F}(x) - a)$$

for all $x \in \partial V$ and $0 < \lambda \leq 1$. Then $Fix\mathcal{F}$ is a nonempty compact set.

3. The setting of the problem

Let $[0, b]$ be a fixed interval of the real line, E a separable Banach space.

Put $\Delta = \{(t, s) \in [0, b] \times [0, b] : 0 \leq s \leq t \leq b\}$, we recall (see, e.g. [14, 19]) that a two parameter family of bounded linear operators $\{T(t, s)\}_{(t,s)\in\Delta}$, $T(t, s) : E \to E$, is called an *evolution system* if the following conditions are satisfied:

1. $T(s, s) = I,\ 0 \leq s \leq b;\quad T(t, r)T(r, s) = T(t, s),\ 0 \leq s \leq r \leq t \leq b;$
2. $T(t, s)$ is strongly continuous on Δ, i.e., for each $x \in E$, the function $(t, s) \in \Delta \to T(t, s)x$ is continuous.

To every evolution system we can assign the corresponding *evolution operator* $T : \Delta \to \mathcal{L}(E)$, where $\mathcal{L}(E)$ is the space of all bounded linear operators in E.

It is known (see, e.g., [14]) that there exists a constant $M = M_\Delta > 0$

such that

$$\|T(t,s)\|_{\mathcal{L}(E)} \leq M, \quad (t,s) \in \Delta. \tag{3.1}$$

We will study problem (1.1) under the following hypotheses:

(A) $\{A(t)\}_{t\in[0,b]}$ is a family of linear not necessarily bounded operators ($A(t) : D(A) \subset E \to E$, $t \in [0,b]$, $D(A)$ a dense subset of E not depending on t), generating an evolution operator $T : \Delta \to \mathcal{L}(E)$; i.e., there exists an evolution system $\{T(t,s)\}_{(t,s)\in\Delta}$ such that, on the region $D(A)$, each operator $T(t,s)$ is strongly differentiable (see, e.g. [14]) relative to t and s, while

$$\frac{\partial T(t,s)}{\partial t} = A(t)T(t,s) \quad \text{and} \quad \frac{\partial T(t,s)}{\partial s} = -T(t,s)A(s)\ , \quad (t,s) \in \Delta.$$

For the multivalued nonlinearity $F : [0,b] \times \mathcal{C}([-\tau,0];E) \to \mathcal{K}v(E)$ we assume:

(F1) the multifunction $F(\cdot,c) : [0,b] \to \mathcal{K}v(E)$ has a measurable selection for every $c \in \mathcal{C}([-\tau,0];E)$, i.e., there exists a measurable function $f : [0,b] \to E$ such that $f(t) \in F(t,c)$ for a.e. $t \in [0,b]$;

(F2) the multimap $F(t,\cdot) : \mathcal{C}([-\tau,0];E) \to \mathcal{K}v(E)$ is u.s.c. for a.e. $t \in [0,b]$;

(F3) for every bounded set $\Omega \subset \mathcal{C}([-\tau,0];E)$ there exists a function $\mu_\Omega \in L^1([0,b];\mathbb{R}_+)$ such that for each $c \in \Omega$:

$$\|F(t,c)\| = \sup\{\|y\| : y \in F(t,c)\} \leq \mu_\Omega(t) \ \text{ for a.e. } t \in [0,b];$$

(F4) there exists a function $m \in L^1([0,b];\mathbb{R}_+)$ such that for every bounded $\Omega \subset \mathcal{C}([-\tau,0];E)$

$$\chi_E(F(t,\Omega)) \leq m(t)\varphi(\Omega) \ \text{ for a.e. } \ t \in [0,b],$$

where χ_E is the Hausdorff MNC in E and φ is the modulus of fiber noncompactness.

Concerning the jump function and the operator B we suppose the following:

(I_k) the jump functions $I_k : \mathcal{C}([-\tau,0];E) \to E$, $k = 1,\dots,N$ are completely continuous, i.e., they are continuous and map bounded sets into relatively compact ones;

(B) $B : U \to E$ is a linear bounded operator with

$$\|B\| \le M_1, \tag{3.2}$$

where M_1 is a positive constant.

Definition 3.1. A piecewise continuous function $y : [-\tau, b] \to E$ is a *mild solution* for the impulsive Cauchy problem (1.1) if

(i) $y(t) = T(t,0)\psi(0) + \sum_{0<t_k<t} T(t,t_k)I_k(y_{t_k}) + \int_0^t T(t,s)f(s)\,ds$
$+ \int_0^t T(t,s)Bu(s)\,ds$, $t \in [0,b]$, where $f \in L^1([0,b];E)$, $f(s) \in F(s, y_s)$ a.e. $s \in [0,b]$, and $u \in L^2([0,b];U)$;

(ii) $y(t) = \psi(t)$, $t \in [-\tau, 0]$;

(iii) $y(t_k^-) = y(t_k)$, $k = 1, \dots, N$;

(iv) $y(t_k^+) = y(t_k) + I_k(y_{t_k})$, $k = 1, \dots, N$.

As mentioned in the introduction, we will consider the controllability problem for the above system, i.e., we will study conditions which guarantee the existence of a mild solution to problem (1.1) satisfying

$$y(b) = x_1, \tag{3.3}$$

where $x_1 \in E$ is a given point. A pair (y, u) consisting of a mild solution $y(\cdot)$ to (1.1) satisfying (3.3) and of the corresponding control $u(\cdot) \in L^2([0,b];E)$ is called a *solution of the controllability problem.*

We assume the standard assumption that the corresponding linear problem without impulses ($F \equiv 0$, $I_k \equiv 0$, $k = 1, \dots, N$) has a solution. More precisely, we suppose that

(W) the controllability operator $W : L^2([0,b];U) \to E$ given by

$$Wu = \int_0^b T(t,s)Bu(s)\,ds$$

has a bounded inverse $W^{-1} : E \to L^2([0,b];U)/\mathrm{Ker}(W)$.

It should be mentioned that we may assume, w.l.o.g., that W^{-1} acts into $L^2([0,b];U)$ (see, e.g., [3]). Let M_2 be a positive constant such that

$$\|W^{-1}\| \leq M_2. \tag{3.4}$$

In the sequel, by the symbol $\mathcal{PC}([0,b];E)$ we denote the space of functions $z:[0,b]\to E$ which are continuous on $[0,b]\setminus\{t_1,\dots,t_N\}$ and such that the left and right limits $z(t_k^-)$ and $z(t_k^+)$, $k=1,\dots,N$ exist and $z(t_k^-)=z(t_k)$. It is easy to see that this space endowed with the norm of uniform convergence is a Banach space and that the space of continuous functions $C([0,b];E)$ is a closed subspace of it. For $z\in\mathcal{PC}([0,b];E)$ we denote by $\tilde z_i$ for $i=0,1,\dots,N$ the function $\tilde z_i\in C([t_i,t_{i+1}];E)$ given by $\tilde z_i(t)=z_i(t)$ for $t\in(t_i,t_{i+1}]$ and $\tilde z_i(t_i)=z(t_i^+)$. Moreover, for a set $D\subset\mathcal{PC}([0,b];E)$, we denote by $\tilde D_i, i=0,1,\dots,N$ the set $\tilde D_i=\{\tilde z_i : z\in D\}$. It is easy to verify the following assertion.

Proposition 3.1. *A set $D\in\mathcal{PC}([0,b];E)$ is relatively compact in $\mathcal{PC}([0,b];E)$ if and only if each set $\tilde D_i$, $i=0,1,\dots,N$ is relatively compact in $C([t_i,t_{i+1}];E)$.*

Now, consider the convex closed subset $\mathcal{D}\in\mathcal{PC}([0,b];E)$ defined by

$$\mathcal{D}=\{z\in\mathcal{PC}([0,b];E), z(0)=\psi(0)\}, \tag{3.5}$$

where $\psi:[-\tau,0]\to E$ is the function from the initial condition of (1.1).

For any $z\in\mathcal{D}$ we define the function $z[\psi]:[-\tau,b]\to E$ as

$$z[\psi]=\begin{cases}\psi(t) & t\in[-\tau,0];\\ z(t) & t\in[0,b].\end{cases}$$

Moreover, for each $\Omega\subset\mathcal{D}$ we denote $\Omega[\psi]=\{z[\psi]: z\in\Omega\}$.

For a given multimap $F:[0,b]\times\mathcal{C}([-\tau,0];E)\to\mathcal{K}v(E)$ satisfying conditions (F1)-(F4), we may consider the multivalued superposition operator $\mathcal{P}_F:\mathcal{D}\multimap L^1([0,b];E)$ defined as

$$\mathcal{P}_F(z)=\left\{f\in L^1([0,b];E): f(s)\in F(s,z[\psi]_s) \text{ a.e. } s\in[0,b]\right\}. \tag{3.6}$$

It is known (see e.g. [13]) that $\mathcal{P}_F$ is well defined. (Notice that the function $s\in[0,b]\to z[\psi]_s\in\mathcal{C}([-\tau,0];E)$ is continuous).

For an abstract operator $S : L^1([0,b];E) \to \mathcal{PC}([0,b],E)$ we consider the following conditions

(S1) $\|Sf - Sg\|_C \le M\|f - g\|_{L^1([0,b];E)}$ for every $f, g \in L^1([0,b];E)$, where $\|\cdot\|_C$ denotes the sup-norm;

(S2) for any compact $K \subset E$ and sequence $\{f_n\}_{n=1}^{\infty}$, $f_n \in L^1([0,b];E)$ such that $\{f_n(t)\}_{n=1}^{\infty} \subset K$ for a.e. $t \in [0,b]$, the weak convergence $f_n \rightharpoonup f_0$ implies the convergence $Sf_n \to Sf_0$.

Applying Proposition 2.1 and Corollary 5.1.2 of [13], we can get the following result.

Proposition 3.2. *Let* $F : [0,b] \times \mathcal{C}([-\tau,0];E) \to \mathcal{K}v(E)$ *satisfy* (F1)–(F4) *and* $S : L^1([0,b];E) \to \mathcal{PC}([0,b],E)$ *obey* (S1), (S2). *Then the composition* $S \circ \mathcal{P}_F : \mathcal{D} \multimap \mathcal{PC}([0,b];E)$ *is an u.s.c. multimap with compact values.*

Now, we consider the generalized Cauchy operator $G : L^1([0,b];E) \to C([0,b];E)$ defined by

$$Gf(t) = \int_0^t T(t,s)f(s)\,ds\,,\ \ t \in [0,b] \tag{3.7}$$

(see [8], Definition 1).

We recall that G has the following property.

Proposition 3.3 (cf. [8], Theorem 2; [13], Lemma 4.2.1)**.** *The generalized Cauchy operator* G *satisfies properties* (S1) *and* (S2).

Finally, let us mention the following result which may be deduced from Theorem 5.1.1 of [13].

Proposition 3.4. *Let* $S : L^1([0,b];E) \to \mathcal{PC}([0,b],E)$ *be an operator satisfying properties* (S1) *and* (S2). *Then for an integrably bounded sequence* $\{f_n\} \subset L^1([0,b];E)$ *such that* $\{f_n(t)\}$ *is relatively compact for a.e.* $t \in [0,b]$, *the sequence* $\{Sf_n\}$ *is relatively compact in* $\mathcal{PC}([0,b],E)$.

4. The solution multioperator

In order to solve controllability problem (1.1, 3.3), we consider the integral multioperator $\Gamma : \mathcal{D} \multimap \mathcal{D}$ defined as:

$$\Gamma(z) = \Big\{ y \in \mathcal{D} : y(t) = T(t,0)\psi(0) + \sum_{0<t_k<t} T(t,t_k) I_k(z[\psi]_{t_k}) + \int_0^t T(t,s)(f(s) + Bu_z(s))\, ds,\ f \in \mathcal{P}_F(z) \Big\}, \tag{4.1}$$

where $u_z(\cdot) \in L^2([0,b];U)$,

$$u_z(t) = = W^{-1}\left(x_1 - T(b,0)\psi(0) - \sum_{k=1}^{N} T(b,t_k) I_k(z[\psi]_{t_k}) - \int_0^b T(b,\eta) f(\eta)\, d\eta \right)(t).$$

It is easy to see that if $y \in \operatorname{Fix}\Gamma$, then $(y[\psi], u_y)$ is a solution to controllability problem (1.1, 3.3).

So our aim is to find a fixed point $y \in \operatorname{Fix}\Gamma$. To do this we will study some properties of the multioperator Γ. First of all, we obtain the following estimate.

Lemma 4.1. *Let $z \in \mathcal{D}$ and $y \in \lambda\Gamma(z)$ for some $0 < \lambda \leq 1$, then for each $t \in [0,b]$ we have*

$$\|y(t)\| \leq M\|\psi(0)\| + M \sum_{0<t_k<t} \|I_k(z[\psi]_{t_k})\| + M \int_0^t \|f(s)\|\, ds + MM_1M_2\sqrt{b} \left(\|x_1\| + M\|\psi(0)\| + M \sum_{k=1}^{N} \|I_k(z[\psi]_{t_k})\| + M \int_0^b \|f(\eta)\|\, d\eta \right)$$

with $f \in \mathcal{P}_F(z)$ and M, M_1, M_2 being constants from estimates (3.1), (3.2), (3.4).

Proof. Let $y \in \lambda\Gamma(z)$, $0 < \lambda \leq 1$, then we have:

$$\|y(t)\| \leq \leq \left\| T(t,0)\psi(0) + \sum_{0<t_k<t} T(t,t_k) I_k(z[\psi]_{t_k}) + \int_0^t T(t,s)(f(s) + Bu_z(s))\, ds \right\|$$

$$
\begin{aligned}
&\le M\|\psi(0)\| + M\sum_{0<t_k<t}\|I_k(z[\psi]_{t_k})\| + M\int_0^t \|f(s)\|\,ds + MM_1\int_0^t \|u_z(s)\|\,ds \\
&= M\|\psi(0)\| + M\sum_{0<t_k<t}\|I_k(z[\psi]_{t_k})\| + M\int_0^t \|f(s)\|\,ds \\
&+ MM_1\int_0^t \bigg\|W^{-1}\bigg(x_1 - T(b,0)\psi(0) - \sum_{k=1}^N T(b,t_k)I_k(z[\psi]_{t_k}) \\
&\quad - \int_0^b T(b,\eta)f(\eta)\,d\eta\bigg)(s)\bigg\|\,ds \\
&\le M\|\psi(0)\| + M\sum_{0<t_k<t}\|I_k(z[\psi]_{t_k})\| + M\int_0^t \|f(s)\|\,ds \\
&+ MM_1\bigg\|W^{-1}\bigg(x_1 - T(b,0)\psi(0) - \sum_{k=1}^N T(b,t_k)I_k(z[\psi]_{t_k}) \\
&\quad - \int_0^b T(b,\eta)f(\eta)\,d\eta\bigg)\bigg\|_{L^1([0,b],U)} \\
&\le M\|\psi(0)\| + M\sum_{0<t_k<t}\|I_k(z[\psi]_{t_k})\| + M\int_0^t \|f(s)\|\,ds \\
&+ MM_1\sqrt{b}\bigg\|W^{-1}\bigg(x_1 - T(b,0)\psi(0) - \sum_{k=1}^N T(b,t_k)I_k(z[\psi]_{t_k}) \\
&\quad - \int_0^b T(b,\eta)f(\eta)\,d\eta\bigg)\bigg\|_{L^2([0,b],U)} \\
&\le M\|\psi(0)\| + M\sum_{0<t_k<t}\|I_k(z[\psi]_{t_k})\| + M\int_0^t \|f(s)\|\,ds \\
&+ MM_1M_2\sqrt{b}\bigg(\|x_1\| + M\|\psi(0)\| + M\sum_{k=1}^N \|I_k(z[\psi]_{t_k})\| + M\int_0^b \|f(\eta)\|\,d\eta\bigg).
\end{aligned}
$$

Proposition 4.1. *The multioperator* Γ*, defined in* (4.1)*, is a u.s.c. operator with convex and compact values.*

Proof. We decompose Γ into the sum of a multioperator $\Gamma_1 : \mathcal{D} \multimap \mathcal{D}$ and two single valued operators Γ_2, $\Gamma_3 : \mathcal{PC}([0,b];E) \to \mathcal{PC}([0,b];E)$ defined in the following way:

$$\Gamma_1(z) = \left\{ y_1 \in \mathcal{D} : y_1(t) = \begin{array}{l} T(t,0)\psi(0) + \int_0^t T(t,s)f(s)\,ds \\ + \int_0^t T(t,s)BW^{-1}\Big(x_1 - T(b,0)\psi(0) \\ - \int_0^b T(t,\eta)f(\eta)\,d\eta\Big)(s)\,ds, \; f \in \mathcal{P}_F(z) \end{array} \right\}$$

$$\Gamma_2(z)(t) = -\int_0^t T(t,s)BW^{-1}\left(\sum_{k=1}^N T(b,t_k)I_k(z[\psi]_{t_k})\right)(s)\,ds$$

$$\Gamma_3(z)(t) = \sum_{0<t_k<t} T(t,t_k)I_k(z[\psi]_{t_k}).$$

First of all, let us observe that from the continuity of the operators B, W^{-1}, and the jump functions $I_k : \mathcal{C}([-\tau,0];E) \to E$, $k = 1,\dots,N$ we conclude that Γ_2 and Γ_3 are continuous operators.

Further, we can consider the multioperator Γ_1 as a composition of the superposition multioperator $\mathcal{P}_F$ with the operator $S = S_1+S_2 : L^1([0,b];E) \to C([0,b];E)$, where:

$$S_1f(t) = T(t,0)\psi(0) + \int_0^t T(t,s)f(s)\,ds$$

and

$$S_2f(t) = \int_0^t T(t,s)BW^{-1}\left(x_1 - T(b,0)\psi(0) - \int_0^b T(t,r)f(\eta)\,d\eta\right)(s)\,ds.$$

In order to apply Proposition 3.2 we prove that the operator S satisfies properties (S1) and (S2). It is so, provided that each of the items S_1 and S_2 satisfies the mentioned properties. From [[8], Theorem 2] it is true for the operator S_1, we prove the same for the operator S_2.

Let $f, g \in L^1([0,b];E)$, then we have the following estimate for each $t \in [0,b]$:

$$\begin{aligned} &\|S_2f(t) - S_2g(t)\|_E \\ &= \left\| \int_0^t T(t,s)BW^{-1}\left[\int_0^b T(b,\eta)(f(\eta)-g(\eta))\,d\eta\right](s)\,ds \right\| \\ &\le MM_1 \int_0^t \left\| W^{-1}\left[\int_0^b T(b,\eta)(f(\eta)-g(\eta))\,d\eta\right](s)\right\| ds \end{aligned}$$

$$
\begin{aligned}
&\leq MM_1 \left\| W^{-1}\left[\int_0^b T(b,\eta)(f(\eta)-g(\eta))\,d\eta\right]\right\|_{L^1([0,b];U)} \\
&\leq MM_1\sqrt{b} \left\| W^{-1}\left[\int_0^b T(b,\eta)(f(\eta)-g(\eta))\,d\eta\right]\right\|_{L^2([0,b];U)} \\
&\leq MM_1M_2\sqrt{b} \left\| \int_0^b T(b,\eta)(f(\eta)-g(\eta))\,d\eta\right\|_{E} \\
&\leq M^2M_1M_2\sqrt{b}\,\|f-g\|_{L^1}.
\end{aligned}
$$

In order to prove $(S2)$ we present operator S_2 in the following way:

$$
S_2 f = G\left(BW^{-1}\left(x_1 - T(b,0)\psi(0) - \zeta G f\right)\right),
$$

where G is defined in (3.7) and $\zeta : C([0,b];E) \to E$, $\zeta(y) = y(b)$ is a linear continuous operator. Then the assertion follows from Proposition 3.3 and the boundedness of the linear operators W^{-1}, B, and ζ.

The conclusion now comes from the fact that the sum of an u.s.c. compact valued multimap Γ_1 with two continuous single-valued maps Γ_2 and Γ_3 is an u.s.c. multimap with compact values (see, e.g., [13]). The convexity of values of Γ follows directly from the convexity of values of the multimap F. ■

Now we want to prove that the multioperator $\Gamma : \mathcal{D} \to \mathcal{K}v(\mathcal{D})$ is condensing. To this aim we need some extra hypotheses.

First of all we observe that from the uniform boundedness of the operators $T(t,s)$, $0 \leq s < t \leq b$ and the operator B it follows that there exist constants $R, N_1 > 0$ such that

$$
\begin{aligned}
&\|T(t,s)\|^{(\chi)} \leq R \leq M, \text{ for any } 0 \leq s \leq t \leq b \\
&\text{and } \|B\|^{(\chi)} \leq N_1 \leq M_1.
\end{aligned} \tag{4.2}
$$

Further, denoting by χ_U the Hausdorff MNC in the space U, we suppose that there exists a function $\delta \in L^1([0,b];\mathbb{R}_+)$ such that for each bounded set $\Xi \subset E$ we have

$$
\chi_U(W^{-1}(\Xi)(t)) \leq \delta(t)\chi_E(\Xi) \text{ a.e. } t \in [0;b]. \tag{4.3}
$$

Finally, let us assume that the following condition holds:

$$\left(R + R^2 N_1 \int_0^b \delta(s)\, ds\right) \int_0^b m(\eta)\, d\eta < 1, \tag{4.4}$$

where $m(\cdot)$ is the function of condition (F4).

Now consider the MNC ν defined on bounded sets $\Omega \subset \mathcal{PC}([0,b];E)$ with values in $(\mathbb{R}^2_+, \geq)$ as:

$$\nu(\Omega) = \left(\varphi(\Omega), \max_{0\leq i\leq N} \mathrm{mod}_C(\tilde{\Omega}_i)\right),$$

where φ and mod_C are the modulus of fiber noncompactness and the modulus of equicontinuity, respectively, defined in Section 2. We observe that the modulus of fiber noncompactness is well defined also on the space $\mathcal{PC}([0,b];E)$. It is easy to see that ν is a monotone, non singular, algebraically semi-additive, and regular MNC.

Proposition 4.2. *Under conditions* (4.3) *and* (4.4) *the multioperator* $\Gamma : \mathcal{D} \to \mathcal{K}v(\mathcal{D})$*, defined in* (4.1)*, is* ν*-condensing.*

Proof. We decompose the multioperator Γ in the same way as in Proposition 4.1, i.e., we consider Γ as the sum of the multioperator $\Gamma_1 : \mathcal{D} \to \mathcal{K}v(\mathcal{D})$ and the two single valued operators Γ_2, $\Gamma_3 : \mathcal{PC}([0,b];E) \to \mathcal{PC}([0,b];E)$. At first, we prove that the operator Γ_1 is ν-condensing. In fact let $\Omega \subset \mathcal{D}$ be a bounded subset such that

$$\nu(\Gamma_1(\Omega)) \geq \nu(\Omega) \tag{4.5}$$

in the sense of semi-order generated by the cone $\mathbb{R}^2_+$. We will show that Ω is relatively compact.

Let us estimate the value $\varphi(\Omega)$. For any $t \in [0,b]$ we have

$$\Gamma_1(\Omega)(t) \subset T(t,0)\psi(0) + G \circ \mathcal{P}_F(\Omega)(t) + S_2 \circ \mathcal{P}_F(\Omega)(t).$$

Then applying (F4) and (4.2) we have

$$\chi_E\left(\{T(t,s)f(s) : f \in \mathcal{P}_F(\Omega)\}\right) \leq Rm(s)\varphi(\Omega[\psi]_s) \leq Rm(s)\varphi(\Omega),$$

where $\Omega[\psi]_s = \{z[\psi]_s : z \in \Omega\}$.

By Proposition 2.2 we have

$$\chi_E(G \circ \mathcal{P}_F(\Omega)(t)) \le R\varphi(\Omega) \int_0^t m(s)\, ds \le R\varphi(\Omega) \int_0^b m(s)\, ds.$$

From the last inequality and estimates (4.2) and (4.3) we obtain

$$\begin{aligned}
&\chi_E\left(\left\{T(t,s)BW^{-1}\left(x_1 - T(b,0)\psi(0) - \int_0^b T(t,\eta)f(\eta)\, d\eta\right)(s),\right.\right.\\
&\qquad \left.\left. f \in \mathcal{P}_F(\Omega)\right\}\right)\\
&\le RN_1\delta(s)\chi_E\left(\left\{\int_0^b T(t,\eta)f(\eta)\, d\eta : f \in \mathcal{P}_F(\Omega)\right\}\right)\\
&\le R^2N_1\varphi(\Omega)\left(\int_0^b m(s)\, ds\right)\delta(s).
\end{aligned}$$

So, again by Proposition 2.2, we have

$$\begin{aligned}
&\chi_E(S_2 \circ \mathcal{P}_F(\Omega)(t)) \le R^2N_1\varphi(\Omega)\left(\int_0^b m(s)\, ds\right)\left(\int_0^t \delta(s)\, ds\right)\\
&\le R^2N_1\varphi(\Omega)\left(\int_0^b m(s)\, ds\right)\left(\int_0^b \delta(s)\, ds\right).
\end{aligned}$$

Therefore, for each $t \in [0,b]$ we have

$$\begin{aligned}
&\chi_E(\Gamma_1(\Omega)(t)) \le \chi_E(G \circ \mathcal{P}_F(\Omega)(t)) + \chi_E(S_2 \circ \mathcal{P}_F(\Omega)(t))\\
&\le \left(R + R^2N_1\int_0^b \delta(s)\, ds\right)\left(\int_0^b m(s)\, ds\right)\varphi(\Omega).
\end{aligned}$$

Hence,

$$\varphi(\Gamma_1(\Omega)) \le q\,\varphi(\Omega), \tag{4.6}$$

where by (4.4)

$$q = \left(R + R^2N_1\int_0^b \delta(s)\, ds\right)\int_0^b m(s)\, ds < 1.$$

Finally, inequalities (4.5) and (4.6) yield

$$\varphi(\Omega) = 0. \tag{4.7}$$

Now we show that $\mathrm{mod}_C(\tilde{\Omega}_i) = 0$ for all $0 \leq i \leq N$, i.e., each set $\tilde{\Omega}_i$ is equicontinuous. We observe that from (4.5) it follows

$$\max_{0\leq j\leq N} \mathrm{mod}_C(\Gamma_1(\tilde{\Omega}_j)) \geq \mathrm{mod}_C(\tilde{\Omega}_i),$$

for each $0 \leq i \leq N$, so it is sufficient to prove that every set $\Gamma_1(\tilde{\Omega}_j)$, $0 \leq j \leq N$ is equicontinuous. This is equivalent to proving it for any sequence $\{y_n\} \subset \Gamma_1(\tilde{\Omega}_j)$. So, given such a sequence there exists a sequence $\{z_n\} \subset \Omega$ and a sequence of selections $\{f_n\}$, $f_n \in \mathcal{P}_F(z_n)$ such that

$$y_n = T(t,0)\psi(0) + (Gf_n)(t) + (S_2 f_n)(t), \ t \in [t_j, t_{j+1}].$$

By condition (F3) we have that the sequence $\{f_n\}$ is integrably bounded, moreover from (4.7) and condition (F4) it follows that

$$\chi_E(\{f_n(t)\}) = 0 \text{ for a.e. } t \in [0,b],$$

i.e., the sequence $\{f_n(t)\}$ is relatively compact for a.e. $t \in [0,b]$. Hence by Propositions 3.3, 3.4, and the fact that the operator S_2 satisfies properties (S1) and (S2), we obtain that the sequence $\{y_n\}$ is relatively compact, implying that $\mathrm{mod}_C(\{y_n\}) = \mathrm{mod}_C(\Gamma_1(\tilde{\Omega}_j)) = 0$. Hence $\mathrm{mod}_C(\tilde{\Omega}_i) = 0$, for each $0 \leq i \leq N$, implying, together with (4.7), by the Arzelà-Ascoli Theorem that every $\tilde{\Omega}_i$ is relatively compact. By using Proposition 3.1 we come to the conclusion that Ω is a relatively compact set.

Now, by the assumptions on I_k, $(k = 1, \dots N)$, B, W^{-1}, and T, we deduce Γ_2 and Γ_3 are compact operators. In fact let $\Omega \subset \mathcal{D}$ be a bounded set, from the compactness of the functions I_k, $(k = 1, \dots N)$ and the continuity of the operators $T(b, t_k) : E \to E$ $(k = 1, \dots N)$ we get that the image set

$$C = \sum_{k=1}^{N} T(b,t_k) I_k(\Omega[\psi]_{t_k})$$

is a compact set, where $\Omega[\psi]_{t_k} = \{z[\psi]_{t_k} : z \in \Omega\}$. Further, from the continuity of the operator $W^{-1} : E \to L^2([0,b];U)$ we obtain the compactness of the set $K = W^{-1}(C)$ in the space $L^2([0,b];U)$. Now, considering the linear continuous operator $\mathcal{B} : L^2([0,b];U) \to L^2([0,b];E)$ defined as $(\mathcal{B}u)(t) = Bu(t)$, $t \in [0, b]$, it follows that the set $Q = \mathcal{B}(K)$ is compact in $L^2([0, b]; E)$. Finally, introducing the operator $\mathcal{T} : L^2([0, b]; E) \to C([0, b]; E)$ defined as

$$(\mathcal{T}q)(t) = -\int_0^t T(t, s)q(s)\, ds,$$

we have that $\Gamma_2(\Omega) = \mathcal{T}(Q)$. Clearly the operator $\mathcal{T}$ is a continuous operator, then we deduce that the set $\Gamma_2(\Omega)$ is a compact set, hence the operator Γ_2 is a compact operator.

Applying Arzelà-Ascoli Theorem, it is easy to see that for each k the family of functions

$$\{T(t, t_k)I_k(z[\psi]_{t_k}) \,:\, z \in \Omega\},$$

is relatively compact, implying that Γ_3 is a compact operator.

Finally, the operator Γ is a ν-condensing multioperator as a sum of a ν-condensing multioperator Γ_1 and two compact operators Γ_2 and Γ_3. Indeed, applying the properties of monotonicity, algebraic semiadditivity, and regularity of the MNC ν, we have for a given bounded subset $\Omega \subset \mathcal{D}$:

$$\nu(\Gamma(\Omega)) \leq \nu\left(\Gamma_1(\Omega) + \Gamma_2(\Omega) + \Gamma_3(\Omega)\right)$$

$$\leq \nu(\Gamma_1(\Omega)) + \nu(\Gamma_2(\Omega)) + \nu(\Gamma_3(\Omega)) = \nu(\Gamma_1(\Omega))$$

and hence the relation

$$\nu(\Gamma(\Omega)) \geq \nu(\Omega)$$

implies

$$\nu(\Gamma_1(\Omega)) \geq \nu(\Omega)$$

yielding the relative compactness of Ω. ■

5. Existence results

The results of the previous section show that the relative topological degree described in Section 2 can be applied to the multioperator Γ defined in (4.1). We can formulate the following general assertion.

Theorem 5.1. *Let $V \subset \mathcal{D}$ be a bounded (relatively) open set such that $z \notin \Gamma(z)$ for all $z \in \partial V$. If $\deg_{\mathcal{D}}(i - \Gamma, \overline{V}) \neq 0$ then controllability problem* (1.1) *has a solution $(y[\psi], u)$ such that $y \in V$.*

Now we will present various conditions under which the fixed point set of Γ is not empty, obtaining in each of these cases an existence result for problem (1.1). In order to do so, we assume hereafter conditions (A), (F1), (F2), (F4), (I_k), (B), (W), (4.3), and (4.4). At the same time, we need to strengthen condition (F3) and to assume some extra hypotheses on the jump functions $I_k,\ k = 1, \dots, N$.

Theorem 5.2. *Suppose that*

(F3′) *there exists a sequence of functions* $\{\omega_n\} \subset L^1([0,b];\mathbb{R}_+),\ n = 1,2\dots$ *such that*

$$\sup_{\|c\|_{\mathcal{C}} \le n} \|F(t,c)\| \le \omega_n(t) \ \textit{for a.e.}\ t \in [0,b],\ n = 1,2,\dots$$

and assume that there exists a sequence $\{H_n\},\ n = 1,2\dots$ *of non negative numbers such that:*

$$\max_{1\le k\le N}\left(\sup_{\|c\|_{\mathcal{C}}\le n} \|I_k(c)\|\right) < H_n. \tag{5.1}$$

If

$$\liminf_{n\to\infty} \frac{1}{n}\int_0^b \omega_n(s)\,ds = 0 \tag{5.2}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} H_n = 0, \tag{5.3}$$

then controllability problem (1.1) *has a solution.*

Proof. We will prove that there exists a number $\mathcal{R} \ge \|\psi(0)\|_{\mathcal{C}}$ such that for a nonempty closed convex set $B_{\mathcal{R}} = \{z \in \mathcal{D} : \|z\| \le \mathcal{R}\}$ we will have $\Gamma(B_{\mathcal{R}}) \subseteq B_{\mathcal{R}}$. By contradiction, there will exist sequences $\{y_n\}$, $\{z_n\} \subset \mathcal{D}$ such that $y_n \in \Gamma(z_n)$, $\|z_n\| \le \frac{n}{2}$, $\|y_n\| > \frac{n}{2}$ for all $n \ge 2\|\psi\|_{\mathcal{C}}$. Then there exists a sequence $\{f_n\} \in \mathcal{P}_F(z_n)$, $n \ge 2\|\psi\|_{\mathcal{C}}$ such that

$$\begin{aligned} y_n(t) = T(t,0)\psi(0) &+ \sum_{0<t_k<t} T(t,t_k) I_k(z_n[\psi]_{t_k}) \\ &+ \int_0^t T(t,s)(f_n(s) + Bu_{z_n}(s))\,ds \end{aligned}$$

with

$$u_{z_n}(t) = \\ = W^{-1}\left(x_1 - T(b,0)\psi(0) - \sum_{k=1}^{N} T(b,t_k)I_k(z_n[\psi]_{t_k}) - \int_0^b T(b,\eta)f_n(\eta)\,d\eta\right)(t).$$

Notice now that for each $t \in [0,b]$ and $n \geq 2\|\psi\|_{\mathcal{C}}$ we have the estimate

$$\|z_n[\psi]_t\|_{\mathcal{C}} \leq \|\psi\|_{\mathcal{C}} + \sup_{0\leq\sigma\leq t} \|z_n(\sigma)\| \leq \|\psi\|_{\mathcal{C}} + \|z_n\| \leq n$$

implying

$$\|I_k(z_n[\psi]_{t_k})\| \leq H_n \text{ for all } n \geq 2\|\psi\|_{\mathcal{C}} \text{ and } k = 1, \dots, N$$

and

$$\|f_n(t)\| \leq \omega_n(t) \text{ a.e. } t \in [0,b],\ n \geq 2\|\psi\|_{\mathcal{C}}.$$

Applying Lemma 4.1, we have

$$\|y_n\| \leq C_1 + C_2\left(NH_n + \int_0^b \|f_n(\eta)\|\,d\eta\right) \leq C_1 + C_2\left(NH_n + \int_0^b \omega_n(\eta)\,d\eta\right),$$

where

$$C_1 = M\|\psi(0)\| + MM_1M_2\sqrt{b}\,(\|x_1\| + M\|\psi(0)\|) \tag{5.4}$$

$$C_2 = M\left(1 + MM_1M_2\sqrt{b}\right). \tag{5.5}$$

But then

$$\frac{1}{2} < \frac{\|y_n\|}{n} \leq \frac{C_1}{n} + \frac{C_2NH_n}{n} + \frac{C_2}{n}\int_0^b \omega_n(\eta)\,d\eta, \quad n \geq 2\|\psi\|_{\mathcal{C}}$$

giving the contradiction to (5.2) and (5.3).

So, if we apply Theorem 2.1 to the restriction $\Gamma : B_{\mathcal{R}} \multimap B_{\mathcal{R}}$, we obtain a fixed point $y \in Fix\Gamma$ and so a solution to problem (1.1). ■

Theorem 5.3. *Suppose that:*

(F3$''$) *there exists a function* $p(\cdot) \in L^1([0,b];\mathbb{R}_+)$ *and a nondecreasing function* $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that for each* $c \in \mathcal{C}([-\tau,0];E)$ *we have*

$$\|F(t,c)\| \leq p(t)\xi(\|c\|_{\mathcal{C}}) \ a.e.\, t \in [0,b]$$

and

$$\|I_k(c)\| \leq \xi(\|c\|_{\mathcal{C}}) \ \ \forall k = 1,\dots,N.$$

Moreover, assume the existence of a constant $L > 0$ *such that*

$$\frac{L}{C_1 + C_2\xi(L + \|\psi\|_{\mathcal{C}})\left(N + \int_0^b p(\eta)\,d\eta\right) + \|\psi(0)\|} > 1, \tag{5.6}$$

where the constants C_1 *and* C_2 *are those given by* (5.4), (5.5).

Then controllability problem (1.1) *has a solution.*

Proof. Denote by $a \in \mathcal{D}$ the function identically equal to $\psi(0)$. Let us demonstrate that there exists an open bounded neighborhood V of a in $\mathcal{D}$ with the property

$$z - a \notin \lambda(\Gamma(z) - a) \tag{5.7}$$

for all $z \in \partial V$ and $0 < \lambda \leq 1$.

Suppose that $z - a \in \lambda(\Gamma(z) - a)$ for some $z \in \mathcal{D}$ and $0 < \lambda \leq 1$, then $z \in \lambda\Gamma(z) + (1-\lambda)a$. Applying the same reasonings as in Lemma 4.1 and in the previous Theorem, we obtain the following estimate

$$\|z(t)\| \leq C_1 + C_2\left(\sum_{k=1}^{N}\|I_k(z[\psi]_{t_k})\| + \int_0^b \|f(\eta)\|\,d\eta\right) + \|\psi(0)\|,$$

where $f \in \mathcal{P}_F(z)$. Applying condition (F3$''$) and using the fact that the function ξ is nondecreasing, we have the estimate

$$\|z\| \leq C_1 + C_2\left(N\xi\left(\|z\| + \|\psi\|_{\mathcal{C}}\right) + \int_0^b p(\eta)\xi\left(\|z[\psi]_\eta\|_{\mathcal{C}}\right)d\eta\right) + \|\psi(0)\|$$

$$\leq C_1 + C_2\xi\left(\|z\| + \|\psi\|_{\mathcal{C}}\right)\left(N + \int_0^b p(\eta)d\eta\right) + \|\psi(0)\|$$

or

$$\frac{\|z\|}{C_1 + C_2\xi\left(\|z\| + \|\psi\|_{\mathcal{C}}\right)\left(N + \displaystyle\int_0^b p(\eta)d\eta\right) + \|\psi(0)\|} \leq 1.$$

So, $\|z\|$ does not equal the constant L appearing in condition (5.6). Now, let us take the relatively open set

$$V = \{z \in \mathcal{D} : \|z\| < L\}.$$

Notice that condition (5.6) implies $a \in V$. We see that condition (5.7) is fulfilled and it remains only to apply Theorem 2.2. ■

Theorem 5.4. *Suppose that*

(F3‴) *there exists a function* $\alpha \in L^1([0,b];\mathbb{R}_+)$ *such that*

$$\|F(t,c)\| \leq \alpha(t)(1 + \|c\|_{\mathcal{C}}) \quad \textit{for a.e. } t \in [0,b]$$

for all $c \in \mathcal{C}([-\tau,0];E)$

and assume the existence of a constant $H > 0$ *such that:*

$$\max_{1\leq k\leq N} \|I_k(c)\| < H$$

for all $c \in \mathcal{C}([-\tau,0];E)$. *Moreover, suppose that*

$$M^2 M_1 M_2 \sqrt{b} \cdot e^{M\int_0^b \alpha(s)\,ds} \int_0^b \alpha(t) e^{-M\int_0^t \alpha(s)\,ds}\,dt < 1. \tag{5.8}$$

Then controllability problem (1.1) *has a solution.*

Proof. We will show that the set of all $z \in \mathcal{D}$ satisfying $z - a \in \lambda(\Gamma(z) - a)$, $0 < \lambda \leq 1$, where $a \in \mathcal{D}$, $a(t) \equiv \psi(0)$ is a priori bounded.

Indeed, let $z - a \in \lambda(\Gamma(z) - a)$ for some $\lambda \in (0,1]$, then $z \in \Gamma(z) + (1-\lambda)a$. Applying Lemma 4.1 and condition (F3‴), we have for each $t \in [0,b]$ the estimate

$$\begin{aligned}\|z(t)\| \leq\ & C_1 + C_2 NH + M^2 M_1 M_2 \sqrt{b} \int_0^b \alpha(s)(1 + \|z[\psi]_s\|_{\mathcal{C}})\,ds \\ & + M \int_0^t \alpha(s)(1 + \|z[\psi]_s\|_{\mathcal{C}})\,ds + \|\psi(0)\|,\end{aligned}$$

where C_1 and C_2 are given by (5.4), (5.5). Further,

$$
\begin{aligned}
&\|z(t)\| \leq \\
&\leq C_1 + C_2NH + M^2M_1M_2\sqrt{b}\|\alpha\|_{L^1} \\
&\quad + M^2M_1M_2\sqrt{b}\int_0^b \alpha(s)\left(\|\psi\|_{\mathcal{C}} + \sup_{0\leq\sigma\leq s}\|z(\sigma)\|\right) ds \\
&\quad + M\|\alpha\|_{L^1} + M\int_0^t \alpha(s)\left(\|\psi\|_{\mathcal{C}} + \sup_{0\leq\sigma\leq s}\|z(\sigma)\|\right) ds + \|\psi(0)\| \\
&\leq C_1 + C_2NH + M^2M_1M_2\sqrt{b}\|\alpha\|_{L^1} + M^2M_1M_2\sqrt{b}\|\alpha\|_{L^1}\|\psi\|_{\mathcal{C}} \\
&\quad + M^2M_1M_2\sqrt{b}\int_0^b \alpha(s)\sup_{0\leq\sigma\leq s}\|z(\sigma)\|\, ds + M\|\alpha\|_{L^1} + M\|\alpha\|_{L^1}\|\psi\|_{\mathcal{C}} \\
&\quad + M\int_0^t \alpha(s)\sup_{0\leq\sigma\leq s}\|z(\sigma)\|\, ds + \|\psi(0)\| \\
&\leq C_1 + C_2NH + C_2\|\alpha\|_{L^1}(1+\|\psi\|_{\mathcal{C}}) + M^2M_1M_2\sqrt{b}\int_0^b \alpha(s)\sup_{0\leq\sigma\leq s}\|z(\sigma)\|\, ds \\
&\quad + M\int_0^t \alpha(s)\sup_{0\leq\sigma\leq s}\|z(\sigma)\|\, ds + \|\psi(0)\|.
\end{aligned}
$$

The last expression is a nondecreasing function in t, so we have the following estimate:

$$
\begin{aligned}
\sup_{0\leq\eta\leq t}\|z(\eta)\| \;\leq\; & K + M^2M_1M_2\sqrt{b}\int_0^b \alpha(s)\sup_{0\leq\sigma\leq s}\|z(\sigma)\|\, ds \\
& + M\int_0^t \alpha(s)\sup_{0\leq\sigma\leq s}\|z(\sigma)\|\, ds,
\end{aligned}
\tag{5.9}
$$

where $K = C_1 + C_2NH + C_2\|\alpha\|_{L^1}(1+\|\psi\|_{\mathcal{C}}) + \|\psi(0)\|$.

Notice that the function $\omega(t) = \sup_{0\leq\eta\leq t}\|z(\eta)\|$ is piecewise continuous, so the function

$$
v(t) = \int_0^t \alpha(s)\omega(s)\, ds
$$

is well defined and nondecreasing, $v(0) = 0$ and we have

$$v'(t) = \alpha(t)\omega(t) \text{ for a.e. } t \in [0, b].$$

Further, applying (5.9) we obtain

$$v'(t) \leq \alpha(t)(K + M^2 M_1 M_2 \sqrt{b} \cdot v(b) + Mv(t)).$$

Multiplying both sides of the above inequality by $e^{-M\int_0^t \alpha(s)\,ds}$, we have

$$v'(t)e^{-M\int_0^t \alpha(s)\,ds} \leq \alpha(t)e^{-M\int_0^t \alpha(s)\,ds}(K + M^2 M_1 M_2 \sqrt{b} \cdot v(b) + Mv(t))$$

implying

$$\left(v(t)e^{-M\int_0^t \alpha(s)\,ds}\right)' \leq \alpha(t)e^{-M\int_0^t \alpha(s)\,ds}(K + M^2 M_1 M_2 \sqrt{b} \cdot v(b)).$$

The integration of both sides of this inequality from 0 to b yields

$$v(b)e^{-M\int_0^b \alpha(s)\,ds} \leq (K + M^2 M_1 M_2 \sqrt{b} \cdot v(b)) \int_0^b \alpha(t)e^{-M\int_0^t \alpha(s)\,ds}\,dt$$

or

$$lv(b) \leq K \int_0^b \alpha(t)e^{-M\int_0^t \alpha(s)\,ds}\,dt,$$

where

$$l = e^{-M\int_0^b \alpha(s)\,ds} - M^2 M_1 M_2 \sqrt{b} \int_0^b \alpha(t)e^{-M\int_0^t \alpha(s)\,ds}\,dt.$$

From condition (5.8) it follows that $l > 0$ and hence

$$v(b) \leq \frac{K \displaystyle\int_0^b \alpha(t)e^{-M\int_0^t \alpha(s)\,ds}\,dt}{l} = K_1 = \text{const}.$$

Since the function v is nondecreasing, we have

$$v(t) \leq K_1 \text{ for all } t \in [0, b].$$

We obtain

$$\|z(t)\| \le K + M^2 M_1 M_2 \sqrt{b} \cdot v(b) + Mv(t) \le K + C_2 K_1,$$

giving the desired a priori boundedness.

Now take an arbitrary $\mathcal{R} > K + C_2 K_1$ and a relatively open set $V = \{z \in \mathcal{D} : \|z\| < \mathcal{R}\}$. Notice that $\|\psi(0)\| \le K$ implying $a \in V$. The application of Theorem 2.2 concludes the proof. ■

6. Example

We consider the motion of a vibrating string fixed at the endpoints $s = 0, 1$ in the presence of a control and impulse effects. We denote by $z(t, s)$ the vertical displacement from the zero position at point $s \in [0, 1]$ and time $t \in [0, b]$ and we assume that the initial displacement and velocity profiles are given as some functions $z_0(t, s)$ and $z_1(t, s)$ on the interval $[-\tau, 0]$. For convenience, let us denote the displacement at fixed time t by $x(t)$, i.e., $x(t) = z(t, \cdot)$ and x will be treated as the function $x : [-\tau, b] \to H_0^1[0, 1] = \{\varsigma \in H^1[0, 1] : \varsigma(0) = \varsigma(1) = 0\}$.

We assume that the control influence upon the motion can be divided into two types of actions: feedback and "absolute". The feedback control is charaterized by an integrable function $f : [0, b] \to L^2[0, 1]$ obeying the feedback relation

$$f(t) \in F(t, x_t) \quad \text{a.e.} \quad t \in [0, b],$$

where the feedback multifunction $F : [0, b] \times \mathcal{C}([-\tau, 0];\ L^2[0, 1]) \to \mathcal{K}v(L^2[0, 1])$ satisfies conditions (F1)–(F4).

The "absolute" control is induced by a bounded linear operator $B : U \to L^2[0, 1]$ from the Hilbert space of controls U.

Further, we allow that at given moments of time $t = t_1, \dots t_N$ the displacement $x(t) = z(t, \cdot)$ and the velocity $x'(t) = \frac{\partial z(t, \cdot)}{\partial t}$ change abruptly. These changes are defined by the given impulse functions

$$I_k : \mathcal{C}([-\tau, 0]; H_0^1[0, 1]) \times \mathcal{C}([-\tau, 0]; L^2[0, 1]) \to H_0^1[0, 1],\ k = 1, \dots, N$$

and

$$\widetilde{I}_k : \mathcal{C}([-\tau, 0]; H_0^1[0, 1]) \times \mathcal{C}([-\tau, 0]; L^2[0, 1]) \to L^2[0, 1],\ k = 1, \dots, N$$

respectively.

Under agreement that we rescal all physical constants to one, this model may be described by the following relations:

$$
(6.1)\quad \begin{cases}
\dfrac{\partial^2 z(t,s)}{\partial t^2} = \dfrac{\partial^2 z(t,s)}{\partial s^2} + f(t) + Bu(t), & t \in [0,b],\ s \in [0,1], \\
 & t \neq t_k,\ k = 1, \ldots, N; \\
f(t) \in F(t, x_t), \quad \text{a.e.} \quad t \in [0,b] & \\
x(t_k^+) = x(t_k) + I_k(x_{t_k}, x'_{t_k}),\ k = 1, \ldots, N; & \\
x(t_k^-) = x(t_k),\ k = 1, \ldots, N; & \\
x'(t_k^+) = x'(t_k) + \widetilde{I}_k(x_{t_k}, x'_{t_k}),\ k = 1, \ldots, N; & \\
x'(t_k^-) = x'(t_k),\ k = 1, \ldots, N; & \\
z(t,s) = z_0(t,s), & t \in [-\tau, 0],\ s \in [0,1]; \\
\dfrac{\partial z(t,s)}{\partial t} = z_1(t,s), & t \in [-\tau, 0],\ s \in [0,1]; \\
z(t,0) = z(t,1) = 0, & t \in [-\tau, b],
\end{cases}
$$

where $u(\cdot) \in L^2([0,b]; U)$.

We are interested in the controllability of the above system, i.e., we want to steer an arbitrary initial displacement and velocity profiles by suitable controls $f(\cdot)$ and $u(\cdot)$ to given profiles, e.g., to the rest position.

We can rewrite our system as a controlled second-order functional differential inclusion with impulses and delay:

$$
(6.2)\quad \begin{cases}
x''(t) \in Ax(t) + F(t, x_t) + Bu(t), & t \in [0,b] \\
x(t) = x_0(t), & t \in [-\tau, 0] \\
x(t_k^-) = x(t_k) & \\
x'(t) = x_1(t), & t \in [-\tau, 0] \\
x(t_k^+) = x(t_k) + I_k(x_{t_k}, x'_{t_k}),\ \ k = 1, ..., N; & \\
x'(t_k^-) = x'(t_k) & \\
x'(t_k^+) = x'(t_k) + \widetilde{I}_k(x_{t_k}, x'_{t_k}),\ \ k = 1, ..., N. &
\end{cases}
$$

Here A denotes the Laplace operator

$$
A = \frac{\partial^2}{\partial s^2}, \quad D(A) = H_0^2[0,1] = \{\varsigma \in H^2[0,1] : \varsigma(0) = \varsigma(1) = 0\}
$$

on the space $E = L^2[0,1]$.

From the fact that $-A$ is a self-adjoint and positive definite operator on E we see that there exists a unique positive definite square root $(-A)^{1/2}$ with domain $D((-A)^{1/2}) = H_0^1[0,1]$. So it is possible to transform the above inclusion to a first order semilinear functional differential inclusion in the following way.

Introduce the Hilbert space $\mathcal{E} = H_0^1[0,1] \times E$ with the inner product

$$\left\langle \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \cdot \begin{pmatrix} q_0 \\ q_1 \end{pmatrix} \right\rangle = \left\langle (-A)^{1/2} p_0 \,|\, (-A)^{1/2} q_0 \right\rangle + \langle p_1 \,|\, q_1 \rangle ,$$

where $\langle\,|\,\rangle$ denotes the inner product in E. Then, we can treat (6.2) as an impulsive control system governed by the following semilinear functional differential inclusion in $\mathcal{E}$:

$$\text{(6.3)} \qquad \begin{cases} y'(t) \in \mathcal{A}y(t) + \mathcal{F}(t, y_t) + \mathcal{B}u(t), & t \in [0,b]; \\ y(t) = \begin{pmatrix} x_0(t) \\ x_1(t) \end{pmatrix}, \ t \in [-\tau, 0]; \\ y(t_k^-) = y(t_k), \ \ k = 1, \dots, N; \\ y(t_k^+) = y(t_k) + \mathcal{I}_k(y_{t_k}), \ \ k = 1, \dots, N, \end{cases}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}, \quad D(\mathcal{A}) = H_0^2[0,1] \times H_0^1[0,1];$$

$$\mathcal{F} : [0,b] \times \mathcal{C}([-\tau, 0]; \mathcal{E}) \to \mathcal{K}v(\mathcal{E}), \qquad \mathcal{F}\left(t, \begin{pmatrix} c^0 \\ c^1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ F(t, c^0) \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} 0 \\ B \end{pmatrix} \in \mathcal{L}(U, \mathcal{E});$$

$$\mathcal{I}_k : \mathcal{C}([-\tau, 0]; \mathcal{E}) \to \mathcal{E}, \ \ \mathcal{I}_k \begin{pmatrix} c^0 \\ c^1 \end{pmatrix} = \begin{pmatrix} I_k(c^0) \\ \widetilde{I}_k(c^1) \end{pmatrix}, \ \ k = 1, \dots, N.$$

It is known (see [12]) that $\mathcal{A}$ generates a group of contractions $e^{\mathcal{A}t}$ on $\mathcal{E}$. Now assume that

(B1) the control operator $B \in \mathcal{L}(U, E)$ is surjective.

Then, the corresponding linear system

$$y'(t) = \mathcal{A}y(t) + \mathcal{B}u(t)$$

is controllable (see [12], Example VI.8.10) and hence there exists the inverse $\mathcal{W}^{-1}$ for the controllability operator $\mathcal{W} : L^2([0,b];U) \to \mathcal{E}$:

$$\mathcal{W} = \int_0^b e^{\mathcal{A}(b-s)}\mathcal{B}u(s)\,ds.$$

Notice that since $e^{\mathcal{A}t}$ are contractions we may take $R = 1$ in estimate (4.2). Then, condition (4.4) takes the form

$$lb\left(1 + N_1\int_0^b \delta(s)ds\right) < 1. \tag{6.4}$$

Theorem 6.1. *Under conditions* (F3′), (5.2), (B), (B1), (4.3) *and* (6.4), *if the jump functions* I_k *and* $\widetilde{I}_k$, $k = 1, \ldots, N$ *satisfy conditions* (I_k), (5.1) *and* (5.3), *then system* (6.1) *is controllable.*

Proof. It is sufficient to observe that conditions of Theorem 5.2 are fulfilled for system (6.1). ■

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