

**A SIMPLE TROLLEY-LIKE MODEL  
IN THE PRESENCE OF A NONLINEAR FRICTION  
AND A BOUNDED FUEL EXPENDITURE <sup>1</sup>**

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**Abstract**

We consider a problem of maximization of the distance traveled by a material point in the presence of a nonlinear friction under a bounded thrust and fuel expenditure. Using the maximum principle we obtain the form of optimal control and establish conditions under which it contains a singular subarc. This problem seems to be the simplest one having a mechanical sense in which singular subarcs appear in a nontrivial way.

**Keywords:** optimal control problem, Pontryagin Maximum Principle, extremals, singular arcs.

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## 1. INTRODUCTION

Consider the following optimal control problem:

$$(1) \quad \begin{cases} \dot{s} = x, & s(0) = 0, & s(T) \rightarrow \max, \\ \dot{x} = u - \varphi(x), & x(0) = 0, & x(t) \text{ is free,} \\ \dot{m} = -u, & m(0) = m_0, & m(T) \geq m_T, \\ 0 \leq u \leq 1. \end{cases}$$

Here  $s(t)$  and  $x(t)$  are one-dimensional position and velocity of a vehicle,  $m(t)$  describes the total mass of vehicle's body and fuel,  $u(t)$  is the rate of fuel expenditure,  $\varphi(x)$  is a twice smooth function describing the "friction" (media resistance) depending on the velocity. We assume that  $\varphi(0) = 0$ ,  $\varphi'(0) \geq 0$ , and  $\varphi''(x) > 0$  for all  $x > 0$ . This object can be considered as a material point moving along a horizontal track and being forced by a nonnegative thrust. Our aim is to maximize the distance passed by the object in a given time  $T$  under a fuel limitation. Here  $m_T \in (0, m_0)$  is the mass of "empty" vehicle without fuel.

This problem can be also considered as a simplification of the Goddard problem [1], where, first, the object moves in a horizontal, not in a vertical direction (which mathematically means that if the speed is zero and the thrust is not applied, the speed keeps zero value on), and second, the change of the mass of the object is not taken into account in the equation for acceleration.

It is well-known that a typical feature of optimal trajectories in the Goddard problem is the presence of singular arcs. However, in the original Goddard problem it is hardly possible to investigate optimality conditions analytically, and so, even qualitative properties of optimal trajectories are obtained by using numerical calculations (see, e.g. [2]–[3]).

Problem (1) includes rather simple equations, which allow one to determine the form of optimal trajectories analytically. At the same time, these trajectories still may contain singular subarcs for some typical forms of the friction function  $\varphi$ . Probably, this problem is the simplest optimal control problem with an underlying mechanical sense in which singular subarcs appear in a nontrivial way.

## 2. PRELIMINARIES

First of all, we note that an optimal process in problem (1) always exists, which immediately follows from the classical Filippov theorem since the problem is linear in the control, and the admissible control set is a convex compactum. Moreover, the following statement takes place.

**Proposition 1.** *The optimal process in problem (1) is unique.*

**Proof.** Suppose, on the contrary, that there exist two optimal controls  $u_1(t)$  and  $u_2(t)$ , and corresponding “velocity” functions  $x_1(t)$  and  $x_2(t)$  such that  $\int_0^T x_1(\tau) d\tau = \int_0^T x_2(\tau) d\tau = s_T$  and  $x_1(0) = x_2(0) = 0$ . If  $u_1 \neq u_2$ , then  $x_1(t) \neq x_2(t)$  on some open interval  $\omega = (t', t'')$ . Consider a new control  $u(t) = (u_1(t) + u_2(t)) / 2$  and a function  $z(t) = (x_1(t) + x_2(t)) / 2$ . Then

$$\dot{z} = u - (\varphi(x_1) + \varphi(x_2)) / 2.$$

Since the function  $\varphi$  is convex, we have  $\dot{z} \leq u - \varphi(z)$ . Moreover, since  $\varphi$  is strictly convex, the last inequality is strict on  $\omega$ .

Define a function  $x(t)$  from the equation

$$\dot{x} = u(t) - \varphi(x), \quad x(0) = 0.$$

Since  $z$  satisfies the equation

$$\dot{z} = v(t) - \varphi(z), \quad z(0) = 0,$$

where  $v(t) \leq u(t)$  and, moreover,  $v(t) < u(t)$  on  $\omega$ , the Chyaplygin comparison theorem implies that  $x(t) \geq z(t)$  for all  $t \geq 0$  and  $x(t) > z(t)$  for  $t > t'$ . Hence,  $\int_0^T x(t) dt > \int_0^T z(t) dt = s_T$ , which contradicts the maximality of  $s_T$ . ■

### 3. MAXIMUM PRINCIPLE FOR PROBLEM (1)

Let  $s(t), x(t), m(t), u(t)$ ,  $t \in [0, T]$  be an optimal process. According to the Pontryagin Maximum Principle (MP), there exist constants  $(\alpha_0, \alpha, \beta_s, \beta_x, \beta_m)$ , not all identically zero, and Lipschitz functions  $\psi_s(t), \psi_x(t), \psi_m(t)$ , that generate the endpoint Lagrange function

$$(2) \quad l = -\alpha_0 s(T) - \alpha(m(T) - m_T) + \beta_s s(0) + \beta_x x(0) + \beta_m(m(0) - m_0)$$

and the Pontryagin function

$$(3) \quad H(s, x, m, u) = \psi_s x + \psi_x (u - \varphi(x)) - \psi_m u,$$

such that the following conditions are satisfied:

- (a) nonnegativity condition:  $\alpha_0 \geq 0, \alpha \geq 0$ ,
- (b) nontriviality condition:  $(\alpha_0, \alpha, \beta_s, \beta_x, \beta_m) \neq (0, 0, 0, 0, 0)$ ,
- (c) complementarity slackness condition:

$$(4) \quad \alpha(m(T) - m_T) = 0,$$

(d) costate (adjoint) equations

$$(5) \quad \begin{cases} -\dot{\psi}_s = H_s = 0, \\ -\dot{\psi}_x = H_x = \psi_s - \psi_x \varphi'(x), \\ -\dot{\psi}_m = H_m = 0, \end{cases}$$

(e) transversality conditions:

$$(6) \quad \begin{cases} \psi_s(0) = \beta_s, & \psi_s(T) = \alpha_0, \\ \psi_x(0) = \beta_x, & \psi_x(T) = 0, \\ \psi_m(0) = \beta_m, & \psi_m(T) = \alpha, \end{cases}$$

(f) the “energy conservation law”:  $H(s, x, m, u) \equiv \text{const}$ ,

(g) and the maximality condition: for almost all  $t$

$$(7) \quad \max_{0 \leq u' \leq 1} H(s(t), x(t), m(t), u') = H(s(t), x(t), m(t), u(t)).$$

According to (5)–(6), in order to simplify further computations we set  $\psi_x \equiv \alpha_0$ ,  $\psi_m \equiv \alpha$ , and write  $\psi(t)$  instead of  $\psi_x(t)$ . Then the maximality condition (7) gives the optimal control in the form

$$(8) \quad u(t) \in \text{Sign}^+(\psi - \alpha),$$

where  $\text{Sign}^+$  is the set-valued function

$$\text{Sign}^+(z) = \begin{cases} \{1\}, & z > 0, \\ [0, 1], & z = 0, \\ \{0\}, & z < 0, \end{cases}$$

and the costate  $\psi(t)$  is determined by the equation

$$(9) \quad \dot{\psi} = -\alpha_0 + \psi \varphi'(x)$$

with the terminal condition  $\psi(T) = 0$ .

Recall that by definition  $\Delta m = m_0 - m_T > 0$ . If  $\Delta m \geq T$ , then the optimal control is obviously  $u \equiv 1$ . So, in further considerations we assume that

$$(10) \quad 0 < \Delta m < T.$$

## 4. ANALYSIS OF THE MAXIMUM PRINCIPLE

Consider first the abnormal case  $\alpha_0 = 0$ . Then equation (11) for  $\psi(t)$  restricts to a homogeneous one, and condition  $\psi(T) = 0$  yields  $\psi(t) \equiv 0$ . Hence  $\beta_x = 0$  from (6), and the nontriviality condition gives  $\alpha > 0$ . Then (8) yields  $u(t) \equiv 0$ , and from equations (1) we have  $m(t) = \text{const} = m_0$ , which contradicts complementarity slackness condition (4). Hence the normal case  $\alpha_0 > 0$  is realised and we may take  $\alpha_0 = 1$ . Thus, equation (9) reads

$$(11) \quad \dot{\psi} = -1 + \psi \varphi'(x).$$

**Proposition 2.**  $\psi(t) > 0$  for all  $t < T$ .

**Proof.** According to (11),  $\dot{\psi}$  is continuous and  $\dot{\psi}(T) = -1$ . Then  $\psi(t) > 0$  in a left neighborhood of  $T$ . Suppose there exists  $t' < T$  such that  $\psi(t') = 0$  and  $\psi(t) > 0$  on  $(t', T)$ . From (11) we again have  $\dot{\psi}(t') = -1$ , whence  $\psi(t) < 0$  in a right neighborhood of  $t'$ , which contradicts the previous inequality. ■

**Proposition 3.** Any trajectory satisfying the maximum principle is globally optimal in problem (1).

**Proof.** Since  $\psi(t) \geq 0$  for all  $t$ , the Pontryagin function (3) is concave w.r.t. the pair  $(x, u)$ . Moreover, the endpoint constraints are linear, the control set  $[0, 1]$  is convex, and the multiplier at the cost  $\alpha_0 = 1$ . It is well known that in this case the maximum principle guarantees the global optimality. ■

**Proposition 4.**  $\alpha > 0$ .

**Proof.** Suppose that  $\alpha = 0$ . Then from (8) we have  $u \equiv 1$  for a.a.  $t$ . Hence  $\Delta m = T$ , which contradicts (10). ■

From the last proposition and (8) it follows that there exists  $t_2 < T$  such that  $u = 0$  for a.a.  $t \in (t_2, T)$ . Moreover, since  $\alpha > 0$ , condition (4) gives  $m(T) = m_T$ , and hence

$$(12) \quad \int_0^T u \, dt = \Delta m > 0.$$

**Remark 5.** If the friction is linear:  $\varphi(x) = \gamma x$  ( $\gamma > 0$ ), the analysis of MP is quite simple. Then (11) determines  $\psi(t) = (1 - e^{\gamma(t-T)})/\gamma$  which is positive on  $[0, T]$  and decreases monotonically from  $\psi(0) > 0$  to  $\psi(T) = 0$ . In this case condition (8) gives that the optimal control always has a bang-bang form  $u = (1, 0)$  on  $((0, \Delta m), (\Delta m, T))$ , i.e.  $u = 1$  on  $(0, \Delta m)$  and  $u = 0$  on  $(\Delta m, T)$ . Such a case is not interesting, and this is why we assume that  $\varphi(x)$  is strictly convex.

Now, define the set  $M = \{t : \psi(t) = \alpha\}$ . Obviously,  $M$  is closed. Moreover, it is not empty (otherwise  $\psi < \alpha$  on  $(0, T)$ , hence  $u \equiv 0$ , which contradicts (12)).

**Proposition 6.** *The set  $M$  is connected.*

**Proof.** Suppose the opposite. Then there exists an interval  $\omega = (t', t'')$  such that  $\psi(t') = \psi(t'') = \alpha$ , and either i)  $\psi(t) < \alpha$  on  $\omega$ , or ii)  $\psi(t) > \alpha$  on  $\omega$ .

Consider the case i). Since  $\dot{\psi}(t') \leq 0$  and  $\dot{\psi}(t'') \geq 0$ , from (11) it follows that  $\varphi'(x(t')) \leq \varphi'(x(t''))$  and hence  $x(t') \leq x(t'')$  by the strict monotonicity of  $\varphi'$ . But  $u \equiv 0$  on  $\omega$ , so  $\dot{x} = -\varphi(x)$  by (1), whence  $x(t)$  cannot increase along  $\omega$ . Hence,  $x(t) = \text{const}$  on  $\omega$ , and since  $\varphi(x) > 0$  for  $x > 0$ , we get  $x(t) \equiv 0$  on  $\omega$ . Since  $\varphi'(0) = b \geq 0$ , equation (11) reads  $\dot{\psi} = -1 + b\psi$ . Its solution is either increasing or decreasing function, which cannot have equal values at  $t'$  and  $t''$ , a contradiction. Case ii) is analysed similarly. ■

Thus,  $M$  is a segment  $[t_1, t_2]$  with possible  $t_1 = t_2$ .

**Proposition 7.**  $M \subset (0, T)$ , i.e.,  $t = 0$  and  $t = T$  do not belong to  $M$ .

**Proof.** Since  $\psi(T) = 0 < \alpha$ , the right end  $T \notin M$ . So, we just need to show that  $0 \notin M$ . Taking into account that  $M$  is a segment  $[t_1, t_2]$ , suppose first that  $M = \{0\}$ . Then  $\psi < \alpha$  on  $(0, T)$ , so  $u \equiv 0$  for a.a.  $t < T$ , which contradicts (12). Now suppose  $M = [0, t_2]$ , where  $0 < t_2 < T$ . Then along  $[0, t_2]$  we have  $x = \text{const} = x(0) = 0$ ,  $u = \varphi(x) = 0$ , hence  $u \equiv 0$  along the whole  $[0, T]$ , which again contradicts (12). ■

**Proposition 8.**  $\psi(t) > \alpha$  on  $(0, t_1)$ .

**Proof.** Suppose this is wrong. Then, since  $\psi(t) \neq \alpha$  on  $(0, t_1)$ , we have  $\psi < \alpha$ , which yields  $u \equiv 0$  on  $(0, t_1)$ , whence also  $x \equiv 0$  and  $\varphi(x) \equiv 0$ . Since  $\psi(t) \leq \alpha$  for all  $t$ , and  $\psi(t_1) = \alpha$ , we have  $\dot{\psi}(t_1) = 0$ , whence (11) yields  $\varphi'(0) = 1/\alpha$ . Thus, on  $(0, t_1)$  we get  $\dot{\psi} = -1 + \psi/\alpha < 0$  since  $\psi < \alpha$ . Hence,  $\psi < \alpha$  and decreases on  $(0, t_1)$ , so  $\psi(t_1) < \alpha$ , which contradicts the relation  $t_1 \in M$ . ■

The next two propositions hold for any function  $\psi(t)$ , not only for the "true" costate function from MP. (We will need such an extension below, in the numerical algorithm.)

**Proposition 9.** *Let  $\psi(t)$  satisfy (11) on an interval  $(0, t_1)$  where  $u = 1$ , and moreover,  $\psi(t_1) = \alpha > 0$  and  $\dot{\psi}(t_1) \leq 0$ . Then  $\psi(t)$  strictly decreases on  $(0, t_1)$ .*

**Proof.** First, we show that  $\psi(t) > \alpha$  in a left neighborhood of  $t_1$ . If  $\dot{\psi}(t_1) < 0$ , this is obvious. If  $\dot{\psi}(t_1) = 0$ , then

$$\ddot{\psi}(t_1) = \psi \varphi''(x)(1 - \varphi(x)) > 0,$$

since  $\varphi(x(t)) < 1$  for all  $t \geq 0$ , and the required property again holds true.

Now, if  $\psi$  does not strictly decrease on  $(0, t_1)$ , one can show that there exist  $t' < t'' < t_1$  such that  $\psi(t') = \psi(t'') = c > \alpha$  and  $\psi(t) \geq c$  on  $(t', t'')$ . Then  $\dot{\psi}(t') \geq 0$  and  $\dot{\psi}(t'') \leq 0$ , which in view of (11) means that  $\psi(t')\varphi'(x(t')) \geq 1$  and  $\psi(t'')\varphi'(x(t'')) \leq 1$ , and hence,  $\varphi'(x(t')) \leq \varphi'(x(t''))$ . But since  $x(t') < x(t'')$  (because  $u = 1$ ) and  $\varphi'(x)$  strictly increases in  $x$ , we obtain a contradiction. ■

**Proposition 10.** *Let  $\psi(t)$  satisfy (11) on an interval  $[t_2, T]$  where  $u = 0$ , and let  $\psi(t_2) > 0$  with  $\dot{\psi}(t_2) \leq 0$ . Then  $\psi(t)$  strictly decreases on  $[t_2, T]$ .*

**Proof.** Since  $u = 0$ , we have  $\dot{x} = -\varphi(x)$ . In view of (11),  $\ddot{\psi} = \dot{\psi}\varphi' - \psi\varphi''\varphi$  is a continuous function. Since  $\dot{\psi}(t_2) \leq 0$ , we have  $\ddot{\psi}(t_2) < 0$ , so  $\ddot{\psi} < 0$  in a right neighborhood of  $t_2$ . Hence, in this neighborhood,  $\dot{\psi} < 0$ , so  $\psi < \psi(t_2)$ , and then, since  $x$  decreases, it follows from (11) that these inequalities hold on the whole interval  $(t_2, T]$ . ■

Thus, when it comes to the true costate function  $\psi$  (i.e. satisfying MP), it has the following form. First,  $\psi$  decreases on  $(0, t_1)$  from  $\psi(0)$  to  $\psi(t_1) = \alpha > 0$ . Then  $\psi(t) = \alpha$  on  $[t_1, t_2]$ . Finally,  $\psi(t) < \alpha$  on  $(t_2, T)$ , decreasing from  $\psi(t_2) = \alpha$  to  $\psi(T) = 0$ .

To describe the control function, we use for convenience the notation  $u = (u_1, u_2, \dots)$  on  $(\Delta_1, \Delta_2, \dots)$ , where  $\Delta_1, \Delta_2, \dots$  are some intervals, if  $u(t) = u_1$  on  $\Delta_1$ ,  $u(t) = u_2$  on  $\Delta_2$ , etc.

The following two different cases are possible:

- (i)  $\psi$  crosses the value  $\alpha$  only at time  $t_1$ . Here,  $u = (1, 0)$  on  $((0, t_1), (t_1, T))$  is a bang-bang control.
- (ii)  $\psi$  stays at the value  $\alpha$  along an interval  $[t_1, t_2]$  with  $t_1 < t_2$ . Here the control is bang-singular-bang, and, as usual, the maximality condition doesn't allow to find the singular control on  $[t_1, t_2]$  directly. However, differentiating the equality  $\psi(t) \equiv \alpha$ , we obtain  $\dot{\psi} = -1 + \alpha\varphi'(x) \equiv 0$ . Since  $\varphi'$  strictly increases in  $x$ , we get  $x(t) = \text{const}$ , whence  $\dot{x} = u - \varphi(x) = 0$ , and so,

$$(13) \quad u_{\text{sing}}(t) = \varphi(x(t)) \quad \text{on } (t_1, t_2).$$

Thus, here we get  $u = (1, \varphi(x(t_1)), 0)$  on  $((0, t_1), (t_1, t_2), (t_2, T))$  with a singular subarc  $(t_1, t_2)$ .

Note that, for a given starting point  $t_1$  of singular subarc, the corresponding end point is uniquely determined as

$$(14) \quad t_2 = t_1 + \frac{m_0 - t_1 - m_T}{\varphi(x(t_1))}.$$

This can be easily obtained from equations (1). Indeed, since  $u \equiv 1$  on  $(0, t_1)$ , we have  $m(t_1) = m_0 - t_1$ . Since  $u \equiv 0$  on  $[t_2, T]$ , we get  $m(t_2) = m_T$ . On  $(t_1, t_2)$  we have  $u = \varphi(x(t_1))$ , which leads to  $m(t_1) - m(t_2) = \varphi(x(t_1))(t_2 - t_1)$ . The last relation implies (14).

Let us find conditions under which each of the above cases is realized.

##### 5. NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMAL CONTROL TO BE OF THE BANG-BANG FORM

Let  $\hat{x}(t)$  be the trajectory corresponding to the bang-bang control  $\hat{u}(t) = 1$  for  $0 \leq t \leq \gamma = \Delta m$  and  $\hat{u}(t) = 0$  for  $\gamma < t < T$ . Thus,  $\hat{x}(t)$  satisfies the relations  $\dot{\hat{x}}(t) = 1 - \varphi(\hat{x}(t))$  on  $[0, \gamma]$  with  $\hat{x}(0) = 0$  and  $\dot{\hat{x}}(t) = -\varphi(\hat{x}(t))$  on  $[\gamma, T]$ .

The optimality of this trajectory is equivalent to the existence of a function  $\psi(t)$  satisfying (11) such that  $\psi(t) > \psi(\gamma) = \alpha$  for  $t \in [0, \gamma)$  and  $\psi(t) < \alpha$  for  $t \in (\gamma, T]$  with  $\psi(T) = 0$ .

According to Propositions 9 and 10, this is equivalent to that  $\psi(t)$  satisfies (11) with

$$\psi(\gamma) = \alpha > 0, \quad \dot{\psi}(\gamma) \leq 0, \quad \text{and} \quad \psi(T) = 0.$$

The second condition means that  $\psi(\gamma) \varphi'(\hat{x}(\gamma)) \leq 1$ , i.e.,

$$\alpha = \psi(\gamma) \leq \alpha_{\max} = 1/\varphi'(\hat{x}(\gamma)).$$

This is an upper bound for  $\alpha$ . Taking any  $\alpha \leq \alpha_{\max}$ , we obtain a unique  $\psi = \psi(\alpha, t)$  as a solution to (11) with the initial value  $\psi(\gamma) = \alpha$ , and then we can determine  $T = T(\alpha)$  such that  $\psi(\alpha, T) = 0$ . Let us establish the dependence of  $T$  on  $\alpha$ .

**Proposition 11.**  $\frac{\partial \psi(\alpha, t)}{\partial \alpha} > 0$  for all  $t$ .

**Proof.** Note that (11) is a linear (nonhomogeneous) equation. If  $\alpha$  obtains an increment  $\bar{\alpha}$ , then the corresponding increment  $\bar{\psi}(t) = \psi(\alpha + \bar{\alpha}, t) - \psi(\alpha, t)$  satisfies the linear homogeneous equation

$$\dot{\bar{\psi}}(t) = \bar{\psi}(t) \varphi'(\hat{x}), \quad \bar{\psi}(\gamma) = \bar{\alpha}.$$

If  $\bar{\alpha} > 0$ , then obviously  $\bar{\psi}(t) > 0$  for all  $t$ . ■



Hence, if  $\alpha \leq \alpha_{\max}$ , then, in particular,  $\forall t \geq \gamma$  we have  $\psi(\alpha, t) \leq \psi(\alpha_{\max}, t)$  and therefore,  $T(\alpha) \leq T(\alpha_{\max}) = T_{\max}$ . If  $\alpha = \psi(\gamma)$  decreases from  $\alpha_{\max}$  to  $+0$ , then  $\dot{\psi}(\gamma) = -1 + \alpha \varphi'(\hat{x}(\gamma))$  decreases from 0 to  $-1+0$ , and the corresponding  $T(\alpha)$  decreases from  $T_{\max}$  to  $\gamma+0$ .

Thus, for any  $T \in (\gamma, T_{\max}]$  there is a unique  $\alpha \leq \alpha_{\max}$ , such that the function  $\psi(\alpha, t)$  fulfils the MP for the trajectory  $\hat{x}(t)$  on  $[0, T]$ . If  $T \leq \gamma$ , then, as was already noted above,  $\hat{x}(t)$  with  $\hat{u} \equiv 1$  is obviously optimal, and if  $T > T_{\max}$ , then the corresponding  $\alpha \leq \alpha_{\max}$  does not exist, and so the bang-bang  $\hat{x}(t)$  is not optimal.

Thus, we proved the following

**Theorem 12.** *The optimal control in problem (1) is of bang-bang form, which is described above, if and only if  $T \leq T_{\max}$ .*

Examples of application of Theorem 12 are presented in section 8 below.

## 6. GEOMETRICAL ARGUMENTS FOR THE EXISTENCE OF SINGULAR ARC

In this section we obtain a sufficient condition for the presence of singular arc along an optimal trajectory by using geometrical considerations. Since  $s(t) = \int_0^t x(\tau) d\tau$ , our problem (1) consists in finding such a bang-singular switching time moment  $t_1 \in [0, \gamma + \Delta m]$  that the square under the graph of corresponding  $x(t)$  is maximal.

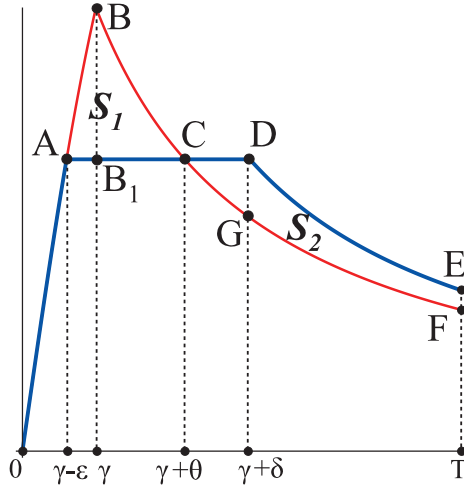


Figure 1.

Take a small  $\varepsilon > 0$  and compare squares under the graphs of bang-bang trajectory  $\hat{x}(t)$  with  $\hat{u} = (1, 0)$  on  $((0, \gamma), (\gamma, T))$ , corresponding to the line  $OBF$  in Figure 1, and bang-singular-bang trajectory  $x(t)$  with  $u = (1, \varphi(\hat{x}(\gamma - \varepsilon), 0))$  on  $((0, \gamma - \varepsilon), (\gamma - \varepsilon, \gamma + \delta(\varepsilon)), (\gamma + \delta(\varepsilon), T))$ , corresponding to the line  $OADE$ .

From geometrical considerations it is easily seen that, if we obtain  $S_2(\varepsilon) > S_1$ , where  $S_1$  is the square of  $ABC$  and  $S_2(\varepsilon)$  is the square of  $CDEF$ , then the bang-bang trajectory is not optimal.

To simplify further formulas, define  $x_\gamma = \hat{x}(\gamma)$ ,  $\varphi_\gamma = \varphi(x_\gamma)$  and  $p_\gamma = 1 - \varphi_\gamma$ . One can show that, up to terms of order  $\varepsilon^2$ ,

$$(15) \quad \begin{aligned} S_1 &= \frac{p_\gamma}{2\varphi_\gamma} \varepsilon^2, & \delta(\varepsilon) &= \frac{p_\gamma}{\varphi_\gamma} \left( \varepsilon + \frac{\varphi'_\gamma}{\varphi_\gamma} \varepsilon^2 \right), \\ |DG| &= x(\gamma + \delta) - \hat{x}(\gamma + \delta) = \Delta x = \frac{p_\gamma \varphi'_\gamma}{2\varphi_\gamma} \varepsilon^2. \end{aligned}$$

The last relation gives that  $S_{GCD} \leq \frac{1}{2}|CD| \cdot |DG| = o(\varepsilon^2)$ . Hence,

$$S_2(\varepsilon) = S_{DGFE} + o(\varepsilon^2) = \int_{\gamma+\delta}^T (\hat{x}(\tau) - x(\tau)) d\tau + o(\varepsilon^2).$$

On the interval on  $[\gamma + \delta, T]$ , we have

$$\begin{cases} \dot{\hat{x}} = -\varphi(\hat{x}), \\ \hat{x}(\gamma + \delta) = x(\gamma - \varepsilon) - \Delta x, \end{cases} \quad \text{and} \quad \begin{cases} \dot{x} = -\varphi(x), \\ x(\gamma + \delta) = x(\gamma - \varepsilon), \end{cases}$$

whence  $\bar{x}(t) = x(t) - \hat{x}(t)$  satisfies (up to terms of order  $\varepsilon^2$ )

$$\dot{\bar{x}} = -\varphi'(\hat{x}(t)) \bar{x}, \quad \bar{x}(\gamma + \delta) = \Delta x,$$

and now our aim is to find  $S_2(\varepsilon) = \int_{\gamma+\delta}^T \bar{x}(\tau) d\tau$ . We will use the following well-known property of linear ODE systems.

**Lemma 13.** *Let  $\bar{x}(t)$  be defined by the equation  $\dot{\bar{x}} = A(t)\bar{x}$ , where  $A(t)$  is an integrable function. Then the linear functional*

$$\bar{J} = l \bar{x}(T) + \int_{t_0}^T c(t) \bar{x}(t) dt,$$

where  $c(t)$  is an integrable function and  $l$  is a real number, can be represented in the form  $\bar{J} = -\bar{\psi}(t_0)\bar{x}(t_0)$ , where the function  $\bar{\psi}(t)$  is determined by the initial value problem (IVP)

$$\dot{\bar{\psi}} = -\bar{\psi}A(t) + c(t), \quad \bar{\psi}(T) = -l.$$

**Proof.** We can write  $\bar{J} = l\bar{x}(T) + \int_{t_0}^T (c\bar{x} + \bar{\psi}(\dot{\bar{x}} - A\bar{x})) dt$ . Integrating  $\bar{\psi}\dot{\bar{x}}$  by parts we get  $\int_{t_0}^T \bar{\psi}\dot{\bar{x}} dt = \bar{\psi}(T)\bar{x}(T) - \bar{\psi}(t_0)\bar{x}(t_0) - \int_{t_0}^T \dot{\bar{\psi}}\bar{x} dt$ , so we obtain

$$\bar{J} = (l + \bar{\psi}(T)) \bar{x}(T) - \bar{\psi}(t_0)\bar{x}(t_0) + \int_{t_0}^T \left( c - (\dot{\bar{\psi}} + \bar{\psi}A) \right) \bar{x} dt = -\bar{\psi}(t_0)\bar{x}(t_0).$$

■

(Note that Lemma 13 is valid also in the case where  $\bar{x}$  is a vector function.)

In our case  $A(t) = -\varphi'(\hat{x}(t))$ ,  $l = 0$ ,  $c(t) \equiv 1$ ,  $\bar{\psi}(t) = -\psi(t)$ , so  $S_2(\varepsilon) = \psi(\gamma) \frac{p_\gamma \varphi'_\gamma}{2\varphi_\gamma} \varepsilon^2$ , and condition  $S_2(\varepsilon) > S_1$  in view of (15) is equivalent to

$$\psi(\gamma + \delta(\varepsilon)) \varphi'(x(\gamma + \delta(\varepsilon))) > 1.$$

The existence of such  $\varepsilon > 0$  that the last inequality takes place is guaranteed if  $\psi(\gamma) \varphi'(x_\gamma) > 1$ , which allows us to formulate a sufficient condition for the existence of singular arc.

**Theorem 14.** *Let  $\hat{x}(t)$  be a bang-bang trajectory (may be not optimal one) with a switching time  $\gamma = \Delta m$ , and let  $\psi(t)$  be determined according to (11) on  $[\gamma, T]$ . If*

$$\psi(\gamma) \varphi'(\hat{x}(\gamma)) > 1 \quad (\text{i.e. } \alpha > \alpha_{\max}),$$

*then the optimal trajectory in problem (1) contains a singular subarc.*

This theorem, obtained by geometrical arguments, is in accordance with the above Theorem 12.

## 7. ALGORITHM TO FIND A SINGULAR SUBINTERVAL

Suppose that, for a given  $T$  the optimal control in (1) is of bang-singular-bang form. The below algorithm allows one to find the start and end times of the singular subarc.

Given  $\Delta m = m_0 - m_T$ , a function  $\varphi(x)$ , and an accuracy  $\varepsilon > 0$ , define  $\gamma = \Delta m$  and a function  $\hat{x}(t)$  on  $[0, \gamma]$  from the equation  $\dot{x} = 1 - \varphi(x)$ ,  $x(0) = 0$ . Taking  $k = 0$  and  $t_1^0 \in (0, \hat{t})$ , go to Step 1 of the following algorithm.

1) Compute  $x_1^k = \hat{x}(t_1^k)$  and set  $t_2^k = t_1^k + (m_0 - t_1^k - m_T) / \varphi(x_1^k)$ .

2) If  $t_2^k \geq T$ , increase  $t_1^k$  and go to Step 1. Else, go to step 3.

3) Find  $\psi^k(T)$  from the following IVP on  $[t_2^k, T]$ :

$$(16) \quad \begin{cases} \dot{\psi}^k(t) = -1 + \psi^k(t) \varphi'(x(t)), & \psi^k(t_2^k) = \alpha^k = 1/\varphi'(x_1^k), \\ \dot{x}(t) = -\varphi(x(t)), & x(t_2^k) = x_1^k. \end{cases}$$

4) (a) If  $|\psi^k(T)| \leq \varepsilon$ , finish the computations with the resulting  $t_1^* = t_1^k$ .

(b) If  $\psi^k(T) > \varepsilon$ , take some  $t_1^{k+1} > t_1^k$  and go to Step 1.

(c) If  $\psi^k(T) < -\varepsilon$ , take some  $t_1^{k+1} < t_1^k$  and go to Step 1.

Note that the increase of  $t_1^k$  corresponds to the decrease of  $\alpha^k$ . As a result, we obtain that the singular subarc of optimal trajectory begins at  $t_1^*$  and ends at  $t_2^* = t_1^* + \alpha^*(m_0 - m_T - t_1^*)$ , where  $\alpha^* = 1/\varphi(\hat{x}(t_1^*))$ .

The described algorithm uses the properties of functions  $\psi^k(t)$  defined by (16), which are given in Propositions 9–11.

## 8. NUMERICAL EXPERIMENTS

In this section we consider the case of simple  $\varphi(x) = x^2/2$  with  $m_0 = 1$  and  $m_T = 0.1$ . Thus, we have  $\gamma = \Delta m = 0.9$ . Choose the accuracy  $\varepsilon = 10^{-6}$ .

1. Find  $\alpha_{\max}$  and the “threshold value”  $T_{\max} = T(\alpha_{\max})$  as was shown in Section 5. Easy computations give  $\alpha_{\max} = 1.087745$  and  $T_{\max} = 3.414422$ .
2. Consider  $T = 3 < T_{\max}$ . According to Theorem 12, the optimal control is of bang-bang form:  $u = 1$  on  $(0, \gamma)$  and  $u = 0$  on  $(\gamma, T)$ .
3. Consider  $T = 6 > T_{\max}$ . According to Theorem 12, the optimal control is of bang-singular-bang form. Applying the algorithm in Section 7, we obtain that the singular subarc starts at  $t_1^* = 0.600071$  and ends at  $t_2^* = 2.469399$ .

All computations were performed on a computer with Pentium(R) Dual-Core CPU 2.2 GHz, 2 GB RAM under 32-bit Windows 7 operating system.

## 9. CONCLUSION

For a trolley-like problem with a nonlinear friction, the form of optimal control is obtained analytically from the Pontryagin Maximum Principle. Necessary and sufficient conditions for a bang-bang form of this control are given (Theorem 12) together with a geometrical form of sufficient conditions for the existence of singular subarc (Theorem 14). An iterative algorithm to find the bounds of singular arc is suggested. Theorems 12 and 14 allow also to determine a threshold value  $T_{\max}$  such that, for all  $0 < T \leq T_{\max}$  the optimal trajectory is of bang-bang form, and for all  $T > T_{\max}$  it contains a singular subarc.

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