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CONTROLLABILITY ON INFINITE TIME HORIZON FOR FIRST AND SECOND ORDER FUNCTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES

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Abstract

In this paper, we shall establish sufficient conditions for the controllability on semi-infinite intervals for first and second order functional differential inclusions in Banach spaces. We shall rely on a fixed point theorem due to Ma, which is an extension on locally convex topological spaces, of Schaefer's theorem. Moreover, by using the fixed point index arguments the implicit case is treated.

Keywords: controllability, mild solution, evolution, fixed point.

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1. Introduction

In this paper, we shall establish sufficient conditions for the controllability on semi-infinite intervals of functional differential inclusions of first and second order in Banach spaces. More precisely, in Section 3 we study the controllability of functional differential inclusions of the form

$$y' - Ay \in F(t, y_t) + (Bu)(t), \quad t \in J = [0, \infty), \tag{1}$$

$$y_0 = \phi, \tag{2}$$

where $F : J \times C(J_0, E) \longrightarrow 2^E$ (here $J_0 = [-r, 0]$) is a bounded, closed, convex valued multivalued map, $\phi \in C(J_0, E)$, A is the infinitesimal generator of a strongly continuous semigroup $T(t), t \geq 0$ and E a real Banach space with the norm $|\cdot|$. Also the control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions with U as a Banach space. Finally, B is a bounded linear operator from U to E. For any continuous function y defined on the interval $[-r, \infty)$ and any $t \in J$, we denote by y_t the element of $C(J_0, E)$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in J_0.$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t.

In Section 4, we investigate the controllability of functional integrodifferential inclusions

$$y' - Ay \in \int_0^t K(t, s)F(s, y_s)ds + (Bu)(t), \quad t \in J = [0, \infty), \tag{3}$$

$$y_0 = \phi, \tag{4}$$

where F, ϕ, A, B are as in the problem (1) – (2) and $K : D \longrightarrow \mathbb{R}$, $D = \{(t, s) \in J \times J : t \geq s\}$.

In Section 5, we study the controllability of second order functional differential inclusions of the form

$$y'' - Ay \in F(t, y_t) + (Bu)(t), \quad t \in J = [0, \infty), \tag{5}$$

$$y_0 = \phi, \ y'(0) = y_1 \tag{6}$$

where F, ϕ, B are as in the problem (1) – (2), $y_1 \in E$ and A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$.

Controllability results of nonlinear functional differential systems and nonlinear integrodifferential systems, on compact intervals, in Banach spaces, by using the Schauder fixed point theorem, were studied by Balachandran, Balasubramaniam and Dauer in [1], [2]. On the other hand, controllability results on functional differential and integrodifferential inclusions, on compact intervals in Banach spaces, were studied by the authors in [3] by using a fixed point theorem for condensing maps due to Martelli [17].

In this paper, we define a new notion, *the infinite controllability*, and study the controllability of systems (1) – (2), (3) – (4) and (5) – (6) based on a fixed point theorem due to Ma [16], which is an extension on locally convex topological spaces, of Schaefer's theorem.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

J_m is the compact real interval $[0, m]$ ($m \in \mathbb{N}$).
$C(J, E)$ is the linear metric Fréchet space of continuous functions from J into E with the metric (see Corduneanu [5], Dugundji and Granas [7])

$$d(y, z) = \sum_{m=0}^{\infty} \frac{2^{-m}\|y - z\|_m}{1 + \|y - z\|_m} \quad \text{for each } y, z \in C(J, E),$$

where

$$\|y\|_m := \sup\{|y(t)| : t \in J_m\}.$$

A measurable function $y : J \longrightarrow E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. For properties of the Bochner integral we refer to Yosida [20].

$L^1(J, E)$ denotes the linear space of equivalence classes of all measurable functions $y : J \longrightarrow E$.

V_p denotes the neighbourhood of 0 in $C(J, E)$ defined by

$$V_p := \{y \in C(J, E) : \|y\|_m \leq p \ \text{ for each } \ m \in \mathbb{N}\}.$$

The convergence in $C(J, E)$ is the uniform convergence on compact intervals, i.e. $y_j \longrightarrow y$ in $C(J, E)$ if and only if for each $m \in \mathbb{N}, \|y_j - y\|_m \longrightarrow 0$ in $C(J_m, E)$ as $j \longrightarrow \infty$.

$M \subseteq C(J,E)$ is a bounded set if and only if there exists a positive function $\varphi \in C(J,\mathbb{R})$ such that

$$|y(t)| \leq \varphi(t) \quad \text{for all} \quad t \in J \quad \text{and all} \quad y \in M.$$

A set $M \subseteq C(J,E)$ is compact if and only if for each $m \in \mathbb{N}$, M is a compact set in the Banach space $(C(J_m,E), \|\cdot\|_m)$.

Let $(X, \|\cdot\|)$ be a Banach space. A multivalued map $G : X \longrightarrow 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets, if $G(D) = \cup_{x\in D} G(x)$ is bounded in X for any bounded set D of X (i.e. $\sup_{x\in D}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called upper semicontinuous (u.s.c.) on X, if for each $x_* \in X$, the set $G(x_*)$ is a nonempty, closed subset of X, and if for each open set $\overline{V}$ of X containing $G(x_*)$, there exists an open neighbourhood V of x_* such that $G(V) \subseteq \overline{V}$.

G is said to be completely continuous, if $G(D)$ is relatively compact, for every bounded subset $D \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$).

G has a fixed point if there is $x \in X$ such that $x \in G(x)$.
In the following, $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $G : J \longrightarrow BCC(E)$ is said to be measurable, if for each $x \in E$, the function $Y : J \longrightarrow \mathbb{R}$ defined by

$$Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\}$$

is measurable. For more details on multivalued maps see the books of Deimling [6], Górniewicz [11] and Hu and Papageorgiou [14].

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is a strongly continuous cosine family if

(i) $C(0) = I$ (I is the identity operator in E),
(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s,t \in \mathbb{R}$,
(iii) the map $t \longmapsto C(t)y$ is strongly continuous for each $y \in E$.

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated with the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y = \int_0^t C(s)y ds, \quad y \in E, \ t \in \mathbb{R}.$$

The infinitesimal generator $A : E \longrightarrow E$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ay = \frac{d^2}{dt^2} C(0)y.$$

For more details on strongly continuous cosine and sine families, we refer the reader to Goldstein [10], Heikkila and Lakshmikantham [13] and to Fattorini [8], [9] and Travis and Webb [18], [19].

The considerations of this paper are based on the following fixed point result.

Lemma 2.1 [16]. *Let X be a locally convex space and $N : X \longrightarrow 2^X$ be a compact convex valued, u.s.c. multivalued map such that for every closed neighbourhood V_p of 0, $N(V_p)$ is a relatively compact set for each $p \in \mathbb{N}$. If the set*

$$\Omega := \{y \in X : \lambda y \in N(y) \ \ \text{for some} \ \ \lambda > 1\}$$

is bounded, then N has a fixed point.

3. First order functional differential inclusions

Definition 3.1. A function $y \in C([-r, \infty), E)$ is called a mild solution to (1) – (2) if there exists a function $v \in L^1(J, E)$ such that $v(t) \in F(t, y_t)$ a.e. on J, $y_0 = \phi$, and

$$y(t) = T(t)\phi(0) + \int_0^t T(t-s)(Bu)(s)\, ds + \int_0^t T(t-s)v(s) ds.$$

Definition 3.2. The system (1) – (2) is said to be infinite controllable on the interval $[-r, \infty)$, if for every initial function $\phi \in C([-r, 0], E)$, for every $y_1 \in E$ and for every $m > 0$ there exists a control $u \in L^2(J_m, U)$, such that the mild solution $y(t)$ of (1) – (2) satisfies $y(m) = y_1$.

Let us list the following hypotheses:

(H1) A is the infinitesimal generator of a compact semigroup $T(t), t \geq 0$ and there exists $M \geq 1$ such that $|T(t)| \leq M, t \geq 0$.

(H2) $F : J \times C(J_0, E) \longrightarrow BCC(E); (t, u) \longmapsto F(t, u)$ is measurable with respect to t for each $u \in C(J_0, E)$, u.s.c. with respect to u for each $t \in J$ and for each fixed $u \in C(J_0, E)$ the set

$$S_{F,u} = \Big\{ g \in L^1(J, E) : \ g(t) \in F(t, u) \text{ for a.e. } \ t \in J \Big\}$$

is nonempty;

(H3) for every $m > 0$ the linear operator $W : L^2(J_m, U) \to E$, defined by

$$Wu = \int_0^m T(m - s)Bu(s)\, ds,$$

has an invertible operator W^{-1} which takes values in $L^2(J_m, U) \backslash kerW$ and there exist positive constants M_1 and M_2 such that $\|B\| \leq M_1$ and $\|W^{-1}\| \leq M_2$.

(H4) $\|F(t, u)\| := \sup\{|v| : v \in F(t, y)\} \leq p(t)\psi(\|u\|)$ for almost all $t \in J$ and all $u \in E$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \longrightarrow (0, \infty)$ is continuous and increasing with

$$\int_{c_m}^{\infty} \frac{du}{\psi(u)} = \infty;$$

where $c_m = M(\|\phi\| + M_0)$ and

$$M_0 = mM_1M_2\Big[|y_1| + M\|\phi\| + M \int_0^m p(s)\psi(\|y\|)\, ds\Big].$$

Remark 3.3. Examples with $W : L^2(J, U) \to E$ such that W^{-1} exists and is bounded are discussed in [4].

The following lemma is crucial in the proof of our main theorems.

Lemma 3.4 [15]**.** *Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying $(H2)$ and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$, then the operator*

$$\Gamma \circ S_F : C(I,X) \longrightarrow BCC(C(I,X)),\ y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Now, we are able to state and prove our main theorem.

Theorem 3.5. *Assume that hypotheses $(H1)$ – $(H4)$ are satisfied. Then the problem (1) – (2) is infinite controllable on $[-r,\infty)$.*

Proof. Let $C([-r,\infty)$ be the Fréshet space of continuous functions from $[-r,\infty)$ into E endowed with the seminorms

$$\|y\|_{r,m} := \sup\{|y(t)| : t \in [-r,m]\}, \quad \text{for } y \in C([-r,\infty).$$

Using hypothesis (H3) for an arbitrary function $y(\cdot)$ define the control

$$u_y^m(t) = W^{-1}\left[y_1 - T(m)\phi(0) - \int_0^m T(m-s)g(s)ds\right](t),$$

where

$$g \in S_{F,y} = \Big\{g \in L^1(J,E) : g(t) \in F(t,y_t) \text{ for a.e. } t \in J\Big\}.$$

We shall now show that when using this control, the operator $N : C([-r,\infty),E) \longrightarrow 2^{C([-r,\infty),E)}$ defined by:

$$N(y) := \left\{ h \in C([-r,\infty),E) : h(t) = \begin{cases} \phi(t), & \text{if } t \in J_0 \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s)(Bu_y^m)(s)ds & \\ \quad + \displaystyle\int_0^t T(t-s)g(s)ds, & \text{if } t \in J \end{cases} \right\}$$

has a fixed point. This fixed point is then the mild solution to the system (1) – (2). Clearly, $y_1 \in N(y)(m)$.

We shall show that $N(V_q)$ is relatively compact for each neighbourhood V_q of $0 \in C([-r,\infty),E)$ with $q \in \mathbb{N}$ and the multivalued map N has bounded, closed and convex values and it is u.s.c. The proof will be given in several steps.

Step 1. $N(y)$ is convex for each $y \in C([-r,\infty),E)$.

This step is obvious. However, for completness, we give the proof. If $h_1,\ h_2$ belong to $N(y)$, then there exist $g_1, g_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$h_i(t) = T(t)\phi(0) + \int_0^t T(t-s)(Bu_y^m)(s)\,ds + \int_0^t T(t-s)g_i(s)\,ds,\ \ i=1,2.$$

Let $0 \le \alpha \le 1$. Then for each $t \in J$ we have

$$\begin{aligned}(\alpha h_1 + (1-\alpha)h_2)(t) &= T(t)\phi(0) + \int_0^t T(t-s)(Bu_y^m)(s)\,ds \\ &+ \int_0^t T(t-s)[\alpha g_1(s) + (1-\alpha)g_2(s)]ds.\end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), then

$$\alpha h_1 + (1-\alpha)h_2 \in N(y).$$

Step 2. $N(V_q)$ is bounded in $C([-r,\infty),E)$ for each $q \in \mathbb{N}$.

Indeed, it is enough to show that for each $m \in \mathbb{N}$ there exists a positive constant ℓ_m such that for each $h \in N(y), y \in V_q$ one has $\|h\|_{r,m} \le \tilde{\ell}_m$. If $h \in N(y)$, then there exists $g \in S_{F,y}$ such that for each $t \in J_m$ we have

$$h(t) = T(t)\phi(0) + \int_0^t T(t-s)(Bu_y^m)(s)\,ds + \int_0^t T(t-s)g(s)ds.$$

By (H1), (H3) and (H4) we have for each $t \in J_m$

$$\begin{aligned}\|h(t)\|_m \le\ & |T(t)\phi(0)| + \left\|\int_0^t T(t-s)(Bu_y^m)(s)\,ds\right\| + \left\|\int_0^t T(t-s)g(s)ds\right\| \\ \le\ & \|\phi\| + mMM_1M_2\left[|y_1| + M\|\phi\| + M\sup_{y\in[0,q]}\psi(y)\Big(\int_0^m p(s)ds\Big)\right] \\ & + M\sup_{y\in[0,q]}\psi(y)\Big(\int_0^m p(s)ds\Big) := \ell_m.\end{aligned}$$

Then for each $h \in N(V_q)$ we have

$$\|h\|_{r,m} \leq \tilde{\ell}_m = \max\{\|\phi\|, \ell_m\}.$$

Step 3. For each $q \in \mathbb{N}$, $N(V_q)$ is equicontinuous for $V_q \in C([-r,\infty),E)$.

Let $t_1, t_2 \in J_m,\ 0 < t_1 < t_2$ and V_q be a neighbourhood of 0 in $C([-r,\infty),E)$ for $q \in \mathbb{N}$.

For each $y \in V_q$ and $h \in N(y)$, there exists $g \in S_{F,y}$ such that

$$h(t) = T(t)\phi(0) + \int_0^t T(t-s)(Bu_y^m)(s)ds + \int_0^t T(t-s)g(s)ds, \quad t \in J.$$

Thus

$$\begin{aligned}
&|h(t_2) - h(t_1)| \leq |[T(t_2) - T(t_1)]\phi(0)| \\
&+ \left\| \int_0^{t_2} [T(t_2 - s) - T(t_1 - s)]\, BW^{-1} \Big[y_1 - T(m)\phi(0) \right. \\
&\quad \left. - \int_0^m T(m-s)g(s)ds \Big](\eta)d\eta \right\| \\
&+ \left\| \int_{t_1}^{t_2} T(t_1 - s)BW^{-1} \left[y_1 - T(m)\phi(0) - \int_0^m T(m-s)g(s)ds \right](\eta)d\eta \right\| \\
&+ \left\| \int_0^{t_2} [T(t_2 - s) - T(t_1 - s)]\, g(s)ds \right\| + \left\| \int_{t_1}^{t_2} T(t_1 - s)g(s)ds \right\| \\
\leq\ & |T(t_2) - T(t_1)|\|\phi\| \\
&+ \int_0^{t_2} \|T(t_2 - s) - T(t_1 - s)\|\, M_1 M_2 \Big[|y_1| + M\|\phi\| \\
&\quad + M \int_0^m p(s)\psi(\|y(s)\|)ds \Big](\eta)d\eta \\
&+ \int_{t_1}^{t_2} \|T(t_1 - s)\| M_1 M_2 \left[|y_1| + M\|\phi\| + M \int_0^m p(s)\psi(\|y(s)\|)ds \right](\eta)d\eta \\
&+ \left\| \int_0^{t_2} [T(t_2 - s) - T(t_1 - s)] \int_0^s g(\tau)d\tau ds \right\| + M \sup_{y \in [0,q]} \psi(y) \Big(\int_{t_1}^{t_2} p(s)ds \Big).
\end{aligned}$$

As $t_2 \longrightarrow t_1$ the right-hand side of the above inequality tends to zero.

The equicontinuity for the cases $t_1 < t_2 \leq 0$ and $t_1 \leq 0 \leq t_2$ follows from the uniform continuity of ϕ on the interval J_0 and from the relation

$$|h(t_2) - h(t_1)| = |h(t_2) - \phi(t_1)| \leq |h(t_2) - h(0)| + |\phi(0) - \phi(t_1)|$$

respectively.

As a consequence of Step 2, Step 3, together with the fact that $T(t)$ is compact and the definition of the metric of the Fréchet space $C([-r,\infty),E)$, we can conclude that $N(V_q)$ is relatively compact in $C([-r,\infty),E)$.

Step 4. N has a closed graph.

Let $y_n \longrightarrow y_*$, $h_n \in N(y_n)$, and $h_n \longrightarrow h_*$. We shall prove that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that

$$h_n(t) = T(t)\phi(0) + \int_0^t T(t-s)(Bu_{y_n}^m)(s)ds + \int_0^t T(t-s)g_n(s)ds, \quad t \in J,$$

where

$$u_{y_n}^m(t) = W^{-1}\Big[y_1 - T(m)\phi(0) - \int_0^m T(m-s)g_n(s)ds\Big](t).$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that

$$\begin{aligned} h_*(t) = T(t)\phi(0) &+ \int_0^t T(t-s)(Bu_{y_*}^m)(s)ds \\ &+ \int_0^t T(t-s)g_*(s)ds, \quad t \in J, \end{aligned} \tag{7}$$

where

$$u_{y_*}^m(t) = W^{-1}\Big[y_1 - T(m)\phi(0) - \int_0^m T(m-s)g_*(s)ds\Big](t).$$

Set

$$\overline{u}_y^m(t) = W^{-1}\Big[y_1 - T(m)\phi(0)\Big](t).$$

The idea is then to use the facts that

(i) $h_n \longrightarrow h_*$;

(ii)

$$h_n(t) - T(t)\phi(0) - \int_0^t T(t-s)(B\overline{u}^m_{y_n})(s)ds \in \Gamma(S_{F,y_n}).$$

where

$$\Gamma : L^1(J,E) \longrightarrow C(J,E)$$

$$g \longmapsto \Gamma(g)(t) = \int_0^t T(t-s)\Big[BW^{-1}\Big(\int_0^m T(m-\sigma)g(\sigma)d\sigma\Big)(s) + g(s)\Big]ds.$$

If $\Gamma \circ S_F$ was a closed graph operator, we would be done. But we do not know whether $\Gamma \circ S_F$ is a closed graph operator. So, we cut the functions $y_n, h_n(t) - T(t)\phi(0) - \int_0^t T(t-s)(B\overline{u}^m_{y_n})(s)ds, g_n$ and we consider them defined on the interval $[k, k+1]$ for aN(y) $k \in \mathbb{N} \cup \{0\}$. Then, using Lemma 3.4, in this case we are able to affirm that (7) is true on the compact interval $[k, k+1]$, i.e.

$$h_*(t)\Big|_{[k,k+1]} = T(t)\phi(0) - \int_0^t T(t-s)(B\overline{u}^m_{y_*})(s)ds + \int_0^t T(t-s)g^k_*(s)ds$$

for a suitable L^1-selection g^k_* of $F(t, y_*(t))$ on the interval $[k, k+1]$.

At this point we can paste the functions g^k_* obtaining the selection g_* defined by

$$g_*(t) = g^k_*(t) \quad \text{for} \quad t \in [k, k+1).$$

We obtain then that g_* is an L^1-selection and (7) is satisfied. We give now the details. Since f, W^{-1} are continuous, then $\overline{u}^m_{y_n}(t) \longrightarrow \overline{u}^m_{y_*}(t)$ for $t \in J$.

Clearly, we have that

$$\Big\|\Big(h_n - T(t)\phi(0) - \int_0^t T(t-s)(B\overline{u}^m_{y_n})(s)ds\Big)$$

$$-\Big(h_* - T(t)\phi(0) - \int_0^t T(t-s)(B\overline{u}^m_{y_*})(s)ds\Big)\Big\|_m \to 0, \quad \text{as} \quad n \longrightarrow \infty.$$

Now, we consider for all $k \in \mathbb{N} \cup \{0\}$, the mapping

$$S^k_F : C([k, k+1], E) \longrightarrow L^1([k, k+1], E)$$

$$u \longmapsto S^k_{F,u} := \{h \in L^1([k, k+1], E) : h(t) \in F(t, u) \text{ for a.e. } t \in [k, k+1]\}.$$

Also, we consider the linear continuous operators

$$\Gamma_k : L^1([k,k+1],E) \longrightarrow C([k,k+1],E)$$

$$g \longmapsto \Gamma(g)(t) = \int_0^t T(t-s)\Big[BW^{-1}\Big(\int_0^m T(m-\sigma)g(\sigma)d\sigma\Big)(s) + g(s)\Big]ds.$$

Clearly, Γ is linear and continuous. Indeed, one has

$$\|\Gamma g\|_\infty \le M(mMM_1M_2+1)\|g\|_{L^1}.$$

From Lemma 3.4, it follows that $\Gamma_k \circ S_F^k$ is a closed graph operator for all $k \in \mathbb{N} \cup \{0\}$. Moreover, we have that

$$\Big(h_n(t) - T(t)\phi(0) - \int_0^t T(t-s)(B\overline{u}^m_{y_n})(s)ds\Big)\Big|_{[k,k+1]} \in \Gamma_k(S^k_{F,y_n}).$$

Since $y_n \longrightarrow y_*$, it follows from Lemma 3.4 that

$$\Big(h_*(t) - T(t)\phi(0) - \int_0^t T(t-s)(B\overline{u}^m_{y_*})(s)ds\Big)\Big|_{[k,k+1]} = \int_0^t T(t-s)g^k_*(s)ds$$

for some $g^k_* \in S^k_{F,y_*}$. So the function g_* defined on J by

$$g_*(t) = g^k_*(t) \quad \text{for} \quad t \in [k,k+1)$$

is in S_{F,y_*} since $g_*(t) \in F(t,y_{*t})$ for a.e. $t \in J$.

Therefore $N(V_q)$ is relatively compact for each neighbourhood V_q of $0 \in C([-r,\infty),E)$ with $q \in \mathbb{N}$ and the multivalued map N has bounded, closed and convex values and it is u.s.c.

Step 5. The set

$$\Omega := \{y \in C([-r,\infty),E) : \lambda y \in N(y) \quad \text{for some} \quad \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$\begin{aligned} y(t) \;=\; & \lambda^{-1}T(t)\phi(0) \\ & + \lambda^{-1}\int_0^t T(t-s)BW^{-1}\Big[y_1 - T(m)\phi(0) - \int_0^m T(m-s)g(s)\,ds\Big](\eta)d\eta \\ & + \lambda^{-1}\int_0^t T(t-s)g(s)ds, \quad t \in J. \end{aligned}$$

This implies by (H1), (H3) – (H4) that for each $t \in J_m$ we have

$$\begin{aligned}
\|y(t)\|_m &\leq M\|\phi\| + mMM_1M_2\left[|y_1| + M\|\phi\| + M\int_0^m p(s)\psi(\|y_s\|)ds\right] \\
&\quad + M\left\|\int_0^t g(s)ds\right\| \\
&\leq M\|\phi\| + mMM_1M_2\left[|y_1| + M\|\phi\| + M\int_0^m p(s)\psi(\|y_s\|)ds\right] \\
&\quad + M\int_0^t p(s)\psi(\|y_s\|)ds.
\end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq m.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J_m$, by the previous inequality we have for $t \in J_m$

$$\begin{aligned}
\mu(t) &\leq M\|\phi\| + mMM_1M_2\left[|y_1| + M\|\phi\| + M\int_0^m p(s)\psi(\|y_s\|)ds\right] \\
&\quad + M\int_0^{t^*} p(s)\psi(\|y_s\|)ds \\
&\leq M\|\phi\| + mMM_1M_2\left[|y_1| + M\|\phi\| + M\int_0^m p(s)\psi(\mu(s))ds\right] \\
&\quad + M\int_0^t p(s)\psi(\mu(s))ds.
\end{aligned}$$

If $t^* \in J_0$ then $\mu(t) = \|\phi\|$ and the previous inequality obviously holds.

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$v(0) = M(\|\phi\| + M_0), \ \mu(t) \leq v(t), \ t \in J_m$$

and

$$v'(t) = Mp(t)\psi(\mu(t)), \ t \in J_m.$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq Mp(t)\psi(v(t)), \ t \in J_m.$$

This implies for each $t \in J_m$ that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq M \int_0^t p(s)ds \leq M \int_0^m p(s)ds < +\infty.$$

From (H4) we have that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} = \infty$$

thus there exists a constant $L = L(m, p, \psi)$ such that $v(t) \leq L,\ t \in J_m$, and hence $\mu(t) \leq L,\ t \in J_m$. Since for every $t \in J_m$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\|_{r,m} := \sup\{|y(t)| : -r \leq t \leq m\} \leq L,$$

where L depends only on m and on the functions p and ψ. This shows that Ω is bounded.

Set $X := C([-r, \infty), E)$. As a consequence of Lemma 2.1 we deduce that N has a fixed point and thus the system (1) – (2) is infinite controllable on $[-r, \infty)$. ■

4. First order integrodifferential inclusions

Now, we shall study the controllability of the problem (3) – (4).

Definition 4.1. A function $y \in C([-r, \infty), E)$ is called a mild solution to (3) – (4) if there exists a function $v \in L^1(J, E)$ such that $v(t) \in F(t, y_t)$ a.e. on J, $y_0 = \phi$, and

$$y(t) = T(t)\phi(0) + \int_0^t T(t-s)(Bu)(s)\,ds + \int_0^t T(t-s) \int_0^s K(s,\tau)v(s)d\tau ds.$$

Definition 4.2. The system (3) – (4) is said to be infinite controllable on the interval $[-r, \infty)$, if for every initial function $\phi \in C([-r, 0], E)$, for every $y_1 \in E$ and every $m > 0$ there exists a control $u \in L^2(J_m, U)$, such that the mild solution $y(t)$ to (3) – (4) satisfies $y(m) = y_1$.

We need the following assumptions:

(H5) for each $t \in J_m$, $K(t, s)$ is measurable on $[0, t]$ and

$$K(t) = \text{ess sup}\{|K(t,s)|,\quad 0 \leq s \leq t\},$$

is bounded on J_m;

(H6) the map $t \longmapsto K_t$ is continuous from J to $L^\infty(J_m, \mathbb{R})$; here $K_t(s) = K(t,s)$;

(H7) $\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \leq p(t)\psi(\|u\|)$ for almost all $t \in J$ and all $u \in C(J_0, E)$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \longrightarrow (0,\infty)$ is continuous and increasing with

$$\int_{c'_m}^{\infty} \frac{d\tau}{\psi(\tau)} = +\infty;$$

where $c'_m = M(\|\phi\| + M'_0)$, and

$$M'_0 = mM_1M_2\Big[|y_1| + M\|\phi\| + mM \sup_{t\in J_m} K(t) \int_0^m p(s)\psi(\|y\|)ds\Big].$$

Theorem 4.3. *Assume that hypotheses* (H1) – (H3), (H5) – (H7) *are satisfied. Then the problem* (3) – (4) *is infinite controllable on* $[-r,\infty)$.

Proof. Using hypothesis (H3) for an arbitrary function $y(\cdot)$ define the control

$$u_y^m(t) = W^{-1}\left[y_1 - T(m)\phi(0) - \int_0^m T(m-s)\int_0^s K(s,\tau)g(\tau)\,d\tau ds\right](t),$$

where

$$g \in S_{F,y} = \Big\{g \in L^1(J,E) : g(t) \in F(t,y_t) \ \text{ for a.e. } \ t \in J\Big\}.$$

We shall now show that when using this control, the multivalued map, $N : C([-r,\infty),E) \longrightarrow 2^{C([-r,\infty),E)}$ defined by:

$$N(y) := \left\{ h \in C([-r,\infty),E) : h(t) = \left\{ \begin{array}{ll} \phi(t), & \text{if } t \in J_0 \\ T(t)\phi(0) + \displaystyle\int_0^t T(t-s)(Bu_y^m)(s)ds & \\ + \displaystyle\int_0^t T(t-s)\int_0^s K(s,u)g(u)duds, & \\ & \text{if } t \in J \end{array} \right\} \right\}$$

has a fixed point. This fixed point is then the mild solution to the system (3) – (4). Clearly $y_1 \in N(y)(m)$.

As in Theorem 3.5 we can show that $N(V_q)$ is relatively compact for each neighbourhood V_q of $0 \in C([-r,\infty),E)$ with $q \in \mathbb{N}$ and the multivalued map N has bounded, closed and convex values and it is u.s.c.. We repeat only the Step 5, i.e. we show that the set

$$\Omega := \{y \in C([-r,\infty),E) : \lambda y \in N(y) \ \text{ for some } \ \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$\begin{aligned} y(t) &= \lambda^{-1} T(t)\phi(0) \\ &+ \lambda^{-1} \int_0^t T(t-s) B W^{-1} \Big[y_1 - T(m)\phi(0) \\ &\quad - \int_0^m T(m-s) \int_0^s K(s,\tau) g(\tau)\, d\tau ds \Big](\eta) d\eta \\ &+ \lambda^{-1} \int_0^t T(t-s) \int_0^s K(s,\tau) g(\tau) d\tau ds, \ t \in J. \end{aligned}$$

This implies by (H1), (H3), (H5) – (H7) that for each $t \in J_m$ we have

$$\begin{aligned} &\|y(t)\|_m \\ &\le M\|\phi\| + mMM_1M_2 \Big[|y_1| + M\|\phi\| + mM \sup\nolimits_{t\in J_m} K(t) \int_0^m p(s)\psi(\|y_s\|) ds \Big] \\ &\quad + M \left\| \int_0^t \int_0^s K(s,\tau) g(\tau) d\tau ds \right\| \\ &\le \ M\|\phi\| + mMM_1M_2 \Big[|y_1| + M\|\phi\| + mM \sup\nolimits_{t\in J_m} K(t) \int_0^m p(s)\psi(\|y_s\|) ds \Big] \\ &\quad + Mm \sup_{t \in J_m} K(t) \int_0^t p(s)\psi(\|y_s\|) ds. \end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \le s \le t\}, \quad 0 \le t \le m.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0,m]$, by the previous inequality we have for $t \in [0,m]$

$$\mu(t) \le M\|\phi\| + mMM_1M_2\Big[|y_1| + M\|\phi\| + mM \sup_{t\in J_m} K(t)\int_0^m p(s)\psi(\|y_s\|)ds\Big]$$

$$+ Mm \sup_{t\in J_m} K(t) \int_0^{t^*} p(s)\psi(\|y_s\|)ds$$

$$\le M\|\phi\| + mMM_1M_2\Big[|y_1| + M\|\phi\| + mM \sup_{t\in J_m} K(t)\int_0^m p(s)\psi(\|y_s\|)ds\Big]$$

$$+ Mm \sup_{t\in J_m} K(t) \int_0^{t} p(s)\psi(\mu(s))ds.$$

If $t^* \in J_0$, then $\mu(t) = \|\phi\|$ and the previous inequality holds, since $M \ge 1$.

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$v(0) = M[\|\phi\| + M_0'],\ \mu(t) \le v(t),\ t \in J_m$$

and

$$v'(t) = mM \sup_{t\in J_m} K(t)p(t)\psi(\mu(t)),\ t \in J_m.$$

Using the nondecreasing character of ψ we get

$$v'(t) \le Mm \sup_{t\in J_m} K(t)p(t)\psi(v(t)),\ t \in J_m.$$

This implies for each $t \in J_m$ that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \le Mm \sup_{t\in J_m} K(t) \int_0^t p(s)ds \le Mm \sup_{t\in J_m} K(t) \int_0^m p(s)ds < +\infty.$$

From (H7) we have that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} = \infty$$

thus there exists a constant $L = L(m, p, \psi)$ such that $v(t) \le L,\ t \in J_m$, and hence $\mu(t) \le L,\ t \in J_m$. Since for every $t \in J_m$, $\|y_t\| \le \mu(t)$, we have

$$\|y\|_{r,m} := \sup\{|y(t)| : -r \le t \le m\} \le L,$$

where L depends only on m and on the functions p and ψ. This shows that Ω is bounded.

Set $X := C([-r,\infty), E)$. As a consequence of Lemma 2.1 we deduce that N has a fixed point and thus the system (3) – (4) is infinite controllable on $[-r,\infty)$. ∎

5. Second order functional differential inclusions

The controllability of the system (5) – (6) is considered in this Section.

Definition 5.1. A function $y \in C([-r,\infty), E)$ is called a mild solution to (5) – (6) if there exists a function $v \in L^1(J, E)$ such that $v(t) \in F(t, y_t)$ a.e., on J, $y_0 = \phi$, and

$$y(t) = C(t)\phi(0) + S(t)y_1 + \int_0^t S(t-s)v(s)ds + \int_0^t S(t-s)Bu(s)ds.$$

Definition 5.2. The system (5) – (6) is said to be infinite controllable on the interval $[-r,\infty)$, if for every $y_0, y_1, x_1 \in E$, and every $m > 0$ there exists a control $u \in L^2(J_m, U)$, such that the mild solution $y(t)$ to (5) – (6) satisfies $y(m) = x_1$.

For the proof of the main result in this Section we need furthermore the following assumptions:

(H8) A is the infinitesimal generator of a given strongly continuous and bounded cosine family $\{C(t) : t \in J\}$. Assume that $C(t), t > 0$ is compact and there exists $M > 0$ such that $M = \sup\{|C(t)|; t \in J\}$;

(H9) for each $m > 0$ the linear operator $W : L^2(J_m, U) \to E$, defined by

$$Wu = \int_0^m S(m-s)Bu(s)\,ds,$$

has an invertible operator W^{-1} which takes values in $L^2(J_m, U)\backslash kerW$ and there exist positive constants M_1 and M_2 such that $\|B\| \le M_1$ and $\|W^{-1}\| \le M_2$.

(H10) $\|F(t,u)\| := \sup\{|v| \in F(t,y)\} \le p(t)\psi(\|u\|)$ for almost all $t \in J$ and all $u \in E$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \longrightarrow (0,\infty)$ is continuous and increasing with

$$\int_{c_m^*}^{\infty} \frac{du}{\psi(u)} = +\infty$$

where $c_m^* = M(\|\phi\| + m|y_1| + M_0^*)$, and

$$M_0^* = mM_1M_2\left[|x_1| + M\|\phi\| + mM|y_1| + mM\int_0^m p(s)\psi(\|y\|)ds\right].$$

Now, we are able to state and prove our main theorem.

Theorem 5.3. *Assume that hypotheses* (H2) *and* (H8) – (H10) *are satisfied. Then the problem* (5) – (6) *is infinite controllable on* $[-r,\infty)$.

Proof. Using hypothesis (H9) for an arbirtary function $y(\cdot)$ define the control

$$u_y^m(t) = W^{-1}\left[x_1 - C(m)\phi(0) - S(m)y_1 - \int_0^m S(m-s)g(s)ds\right](t)$$

where

$$g \in S_{F,y} = \Big\{g \in L^1(J,E) : g(t) \in F(t,y_t) \ \text{ for a.e. } \ t \in J\Big\}.$$

We shall now show that, when using this control, the operator $N : C([-r,\infty),E) \longrightarrow 2^{C([-r,\infty),E)}$ defined by:

$$N(y) := \left\{ h \in C([-r,\infty),E) : h(t) = \begin{cases} \phi(t), & \text{if } t \in J_0 \\ C(t)\phi(0) + S(t)y_1 & \\ \quad + \displaystyle\int_0^t S(t-s)(Bu_y)(s) & \\ \quad + \displaystyle\int_0^t S(t-s)g(s)ds, & \text{if } t \in J \end{cases} \right\}$$

where $g \in S_{F,y}$, has a fixed point. This fixed point is then the mild solution to the system (5) – (6). Clearly, $x_1 \in N(y)(m)$.

Similarly, as in the proof of Theorem 3.5 one can show that $N(V_q)$ is relatively compact for each neighbourhood V_q of $0 \in C([-r,\infty),E)$ with $q \in \mathbb{N}$ and the multivalued map N has bounded, closed and convex values and it is u.s.c. from which the result follows.

6. The implicit case

According to §74 of [11] we would like to consider the implicit functional differential inclusions (for details and references see [11]).

Let $F : J \times C(J_0, E) \times E \to 2^E$ be a multivalued map and A, as in Section 1, be an infinitesimal generator (values of F are not necessarily convex).

We would like to study the following differential inclusion

$$(y' - Ay) \in F(t, y_t, y' - Ay). \tag{6.1}$$

To do it we shall consider the map associated with (6.1)

$$G : J \times C(J_0, E) \to 2^E \tag{6.2}$$

defined as follows:

$$F(t, x) = \mathrm{Fix}(F(t, x \cdot)) = \{y \in E \mid y \in F(t, x, y)\}.$$

Evidently, (6.1) is equivalent to following one:

$$(y' - Ay) \in G(t, x_t). \tag{6.3}$$

So, it is enough to solve the problem (6.3).

Usually, (under natural assumptions on F) the map G is u.s.c. but not in general with convex values.. Therefore, the following assumption on F is necessary:

$$\forall t \in J \ \forall x \in C(J_0, E) \ \ \dim \mathrm{Fix}(F(t, x, \,\cdot\,)) = 0, \tag{6.4}$$

where dim stands for the topological dimension.

By using the fixed point index arguments (comp. [11]) we can prove:

Theorem 6.1. *If F satisfies all assumptions of Section* 3 *and* (6.4)*, then G possess a lower semicontinuous selector η with compact values.*

The proof is strictly analogous to the proof of (74.7) in [11].

Finally, the problem (6.1) is reduced to the following one:

$$(y' - Ay) \in \eta(t, x_t), \tag{6.5}$$

where η is an l.s.c. map with compact values.

It is well known that (6.5) is solvable under typical assumptions.

Note that even for singlevalued $F = f$ the map η is in general multivalued. The only case, when η is singlevalued is when $\mathrm{Fix}(F(t, x, \cdot))$ is a singleton, i.e., for example if $F = f$ satisfies the Lipschitz condition with respect to the last variable.

Finally, we recommend [12] for considering problem (1) on a thin domain contained in E. Let us remark also that using the method presented in this section the second order inclusions can be considered (comp. [11]). Note that implicit problems can be formulated on a thin domain, i.e., on a closed subset of E but it is an open problem how to formulate the second order case on thin domains (comp. [12]).

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