

ON RELATIONS AMONG THE GENERALIZED SECOND-ORDER DIRECTIONAL DERIVATIVES

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Abstract

In the paper, we deal with the relations among several generalized second-order directional derivatives. The results partially solve the problem which of the second-order optimality conditions is more useful.

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1. Introduction and preliminaries

Second-order nonsmooth analysis by virtue of generalized second-order directional derivatives has been extensively studied. Applications of second-order directional derivatives are presented for optimality conditions related to nonsmooth optimization problems in, e.g., [3], [7], [12], [18], [20] and of optimal control problems in [14].

Various second-order directional derivatives have been introduced and studied. It seems to be useful to give relations among them. In [1, 2], equivalence relations between second-order differentiability of a convex function and the $*$ -weak Gâteaux differentiability of its subgradients in the Banach space are given. Relations between the Chaney second-order directional derivative and a type of the Ben-Tal-Zowe directional derivative are investigated for a locally Lipschitz function in [11].

X.Q. Yang established connections among several upper and lower generalized second-order directional derivatives and gave applications of his results in [19, 20].

The main purpose of this paper is to establish connections among the generalized lower second-order directional derivative in the sense of Michel-Penot [12], the second-order directional derivatives given in [16], the Schwartz second-order directional derivative (see for example [10]), and the second-order directional derivatives defined in [7]. We apply these relations in characterizing the convexity property and to optimization theory.

Throughout this paper, we assume that X is a normed space and X^* denotes its topological dual. The unit sphere in X is denoted by S_X . We reserve a symbol $\langle x^*, x \rangle$ for the value of a functional $x^* \in X^*$ on an element $x \in X$.

We use a symbol (a, b) for an open segment with endpoints a, b . Moreover, we suppose that $a \neq b$. If X is just \mathbf{R} , then we assume that $a < b$.

The domain of a function f is denoted by $D(f)$.

We recall at first several facts.

Definition 1.1. Let $f : X \rightarrow \mathbf{R}$ be a function, $x, v \in X$. The lower Dini-directional derivative of f at x in direction v is defined as

$$D_+f(x, v) = \liminf_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t}.$$

Definition 1.2 [4]. Let $f : X \rightarrow \mathbf{R}$ be Lipschitz near x , and let $v \in X$. The Clarke upper generalized directional derivative of f at x in the direction v is defined by

$$f^\circ(x, v) = \limsup_{y \rightarrow x, t \downarrow 0} \frac{f(y + tv) - f(y)}{t},$$

and the Clarke generalized gradient of f at x is defined by

$$\partial_c f(x) = \{x^* \in X^* : \langle x^*, v \rangle \leq f^\circ(x, v) \ \forall v \in X\}.$$

Theorem 1.1 (the Lebourg theorem of a mean value [13]). *Let $f : X \rightarrow \mathbf{R}$ be Lipschitz on an open set U , $x, y \in U$. Then there exists a point $u \in (x, y)$ with the property*

$$f(y) - f(x) \in \langle \partial_c f(u), y - x \rangle.$$

Proposition 1.1 [5]. *If $f_i : X \rightarrow \mathbf{R}$ is Lipschitz near $x \in X$ and $s_i \in \mathbf{R}$ for every $i = 1, 2, \dots, n$, then*

$$\partial_c \left(\sum_{i=1}^n s_i f_i(x) \right) \subset \sum_{i=1}^n s_i \partial_c f_i(x).$$

Proposition 1.2 [5]. *Let $f_1 : X \rightarrow \mathbf{R}, f_2 : X \rightarrow \mathbf{R}$ be Lipschitz near $x \in X, f_2(x) \neq 0$. Then the function $\frac{f_1}{f_2}$ is Lipschitz near x too, and we have*

$$\partial_c \left(\frac{f_1}{f_2} \right) (x) \subset \frac{f_2(x) \partial_c f_1(x) - f_1(x) \partial_c f_2(x)}{f_2^2(x)}.$$

Definition 1.3 [7]. Let $f : X \rightarrow \mathbf{R}$ be a function, $x, u, v \in X$. The upper and lower generalized second-order directional derivatives are defined, respectively, by

$$f^\infty(x)(u, v) = \limsup_{y \rightarrow x, s, t \downarrow 0} \frac{f(y + su + tv) - f(y + tu) - f(y + sv) + f(y)}{st},$$

$$f_\infty(x)(u, v) = \liminf_{y \rightarrow x, s, t \downarrow 0} \frac{f(y + su + tv) - f(y + tu) - f(y + sv) + f(y)}{st}.$$

Definition 1.4 [15]. Let $f : X \rightarrow \mathbf{R}$ be a $C^{1,1}$ function, i.e., f is Gâteaux differentiable with locally Lipschitz derivative. Then the generalized lower second-order directional derivative of f at x in the sense of Michel-Penot is defined by

$$f^\infty(x)(u, v) = \inf_{z \in X} \liminf_{s \downarrow 0} \frac{\langle \nabla f(x + sz + su), v \rangle - \langle \nabla f(x + sz), v \rangle}{s},$$

where $\nabla f(x)$ denotes the Gâteaux derivative of f at x .

Proposition 1.3 [5]. *Let $f : X \rightarrow \mathbf{R}$ be Lipschitz near x , and let f be continuously differentiable at x . Then $\partial_c f(x) = \{\nabla f(x)\}$.*

Definition 1.5. Let $f : X \rightarrow \mathbf{R}$ be a function, $x \in X, h \in X$. The Schwarz generalized second-order directional derivative is defined by

$$f_S''(x)(h, h) = \limsup_{t \downarrow 0} \frac{f(x + th) + f(x - th) - 2f(x)}{t^2}.$$

Definition 1.6 [16]. Let $f : X \rightarrow \mathbf{R}$ be Lipschitz near x , $(u, v) \in X^2$. The generalized second-order directional derivative $f''(x)(u, v)$ of f at x in direction (u, v) , the right generalized second-order directional derivative $f''_+(x)(u, v)$ of f at x in direction (u, v) , the left generalized second-order directional derivative $f''_-(x)(u, v)$ of f at x in direction (u, v) , and the generalized strict second-order directional derivative $f^{**}(x)(u, v)$ of f at x in direction (u, v) are defined respectively as follows:

$$\begin{aligned} f''(x)(u, v) &= \liminf_{t \rightarrow 0} \left\{ \frac{1}{t} \langle q - p, v \rangle : q \in \partial_c f(x + tu), p \in \partial_c f(x) \right\}, \\ f''_+(x)(u, v) &= \liminf_{t \downarrow 0} \left\{ \frac{1}{t} \langle q - p, v \rangle : q \in \partial_c f(x + tu), p \in \partial_c f(x) \right\}, \\ f''_-(x)(u, v) &= \liminf_{t \uparrow 0} \left\{ \frac{1}{t} \langle q - p, v \rangle : q \in \partial_c f(x + tu), p \in \partial_c f(x) \right\}, \\ f^{**}(x)(u, v) &= \liminf_{y \rightarrow x, t \downarrow 0} \left\{ \frac{1}{t} \langle q - p, v \rangle : q \in \partial_c f(y + tu), p \in \partial_c f(y) \right\}. \end{aligned}$$

Proposition 1.4 [16]. Let $f : X \rightarrow \mathbf{R}$ be Lipschitz near x , $(u, v) \in X^2$, then

1. $f''(x)(u, v) = \min\{f''_+(x)(u, v), f''_-(x)(u, v)\}$.
2. $f''_-(x)(u, v) = f''_+(x)(-u, -v)$.

The first assertion holds also for the generalized second-order directional derivative and for the left generalized second-order directional derivative.

Analogously as in the smooth analysis, it is very useful to give the characterization of convexity by the generalized second-order directional derivatives for various classes of functions.

Theorem 1.2 [10]. Assume that U is an open subset of X , and suppose that $f : U \rightarrow \mathbf{R}$ is continuous. Then f is convex if and only if for each $x \in X$ and for every $h \in S_X$,

$$f''_S(x)(h, h) \geq 0.$$

Theorem 1.3 [16]. Let $f : X \rightarrow \mathbf{R}$ be a locally Lipschitz function on an open convex set $U \subset X$. Then the following conditions are equivalent

- (a) f is convex on U .

- (b) $f^{**}(x)(h, h) \geq 0$ for every $x \in U$, $h \in S_X$.
- (c) $f_+''(x)(h, h) \geq 0$ for every $x \in U$, $h \in S_X$.
- (d) $f''(x)(h, h) \geq 0$ for every $x \in U$, $h \in S_X$.

Following Proposition 3.1 and Corollary 4.2 in [12], we obtain

Theorem 1.4. *Let $U \subset X$ be an open convex set, and let $f : U \rightarrow \mathbf{R}$ be a $C^{1,1}$ function. Then f is convex if and only if for every $x \in U$ and for every $u \in S_X$,*

$$f^\infty(x)(u, u) \geq 0.$$

Much attention has been focused on nonsmooth sufficient second-order conditions for strict local minimum.

Definition 1.7 [6]. A function $f : X \rightarrow \mathbf{R}$ is called twice uniformly locally Lipschitzian at x if there exist neighbourhoods X_0 of x and U of zero such that $f^\infty(X_0)(U, U)$ is bounded in \mathbf{R} .

Proposition 1.5 [6]. *A function f is twice uniformly locally Lipschitzian at x if and only if $f \in C^{1,1}$.*

In [8], it is considered the following kind of directional derivative for a function $f : X \rightarrow \mathbf{R}$, $x, u, v \in X$, defined by

$$f_{+u}^\circ(x, u) = \limsup_{s, \lambda \downarrow 0} \frac{f(x + \lambda a + su) - f(x + \lambda a)}{s}.$$

Theorem 1.5 [8]. *Let $f : \mathbf{R}^n \rightarrow \mathbf{R}$ be Lipschitz near \hat{x} and twice locally Lipschitzian. If $f_{+u}^\circ(\hat{x}, u) \geq 0$ for all $u \in X$, then a sufficient condition for x to be a strict local minimum is that $f_\infty(\hat{x})(u, u) > 0$ for every $u \in X, u \neq 0$.*

Theorem 1.6 [16]. *Let $f : \mathbf{R}^n \rightarrow \mathbf{R}$ be a $C^{1,1}$ function, $\hat{x} \in \mathbf{R}^n$, $\nabla f(\hat{x}) = 0$, and $f_+''(\hat{x})(h, h) > 0$ whenever $h \in S_{\mathbf{R}^n}$. Then f attains a strict local minimum at \hat{x} .*

2. Second-order characterizations of a convex function

With respect to Proposition 1.3, it is straightforward to verify the following relation [16].

Proposition 2.1. *Let $f : X \rightarrow \mathbf{R}$ be a $C^{1,1}$ function, $x \in X$, and let $(u, v) \in X^2$. Then*

$$f''_+(x)(u, v) \geq f^{\diamond\diamond}(x)(u, v) \geq f^{**}(x)(u, v).$$

Corollary 2.1. *Theorem 1.4 now follows immediately from Theorem 1.3 and Proposition 2.1.*

Now we give a relation among the Schwartz generalized second-order directional derivative $f''_S(x)(h, h)$ and the generalized second-order directional derivative $f''(x)(h, h)$ for a locally Lipschitz function. We note that for a function of one variable, $f''_S(x) = f''_S(x)(1, 1)$ and $f''(x) = f''(x)(1, 1)$.

Proposition 1.1 and Proposition 1.2 imply the following lemma.

Lemma 2.1. *Let $\varphi : (a, b) \rightarrow \mathbf{R}$ be Lipschitz near $x \in (a, b), x \neq 0$. Then*

$$\begin{aligned} \partial_c \left(\frac{\varphi(x) + \varphi(-x)}{x} \right) &\subset \frac{x[\partial_c \varphi(x) - \partial_c \varphi(-x)] - [\varphi(x) + \varphi(-x)]}{x^2} \\ &= \frac{\partial_c \varphi(x) - \partial_c \varphi(-x)}{x} - \frac{\varphi(x) + \varphi(-x)}{x^2}. \end{aligned}$$

Lemma 2.2. *Let φ be a function of one real variable which is Lipschitz near 0, $\varphi(0) = 0$, $0 \in \partial_c \varphi(0)$, and*

$$\liminf_{x \downarrow 0} \frac{\varphi(x) + \varphi(-x)}{x^2} = -\infty.$$

Then $\varphi''(0) = -\infty$.

Proof. Consider an arbitrary $K < 0$. By the hypothesis, for every $\delta > 0$ there exists $x, 0 < |x| < \delta$ with the property

$$(1) \quad \frac{\varphi(x)}{x^2} < K.$$

Suppose that $x > 0$. Thanks to the Lebourg theorem of a mean value, we can find $0 < \lambda(x) < x$ satisfying

$$(2) \quad \partial_c \varphi(\lambda(x)) = \frac{\varphi(x)}{x}.$$

It follows from (1) and (2) that

$$\frac{\partial_c \varphi(\lambda(x))}{\lambda(x)} < \frac{\partial_c \varphi(\lambda(x))}{x} < K.$$

The same result we obtain also in the case $x < 0$. Since $0 \in \partial_c \varphi(0)$ and K was arbitrary, $\varphi''(0) = -\infty$. ■

We note that the Lebourg theorem is true also under weaker assumptions (compare with its proof in [5]). We give a version for a function of one real variable, which we will use in the sequel.

Lemma 2.3. *Let $f : [a, b] \rightarrow \mathbf{R}$ be a continuous function, which is locally Lipschitz on (a, b) . Then there exists a point $c \in (a, b)$ with the property*

$$f(b) - f(a) \in \partial_c f(c)(b - a).$$

Proposition 2.2. *Let $f : (a, b) \rightarrow \mathbf{R}$ be Lipschitz near $x \in (a, b)$. Then*

$$f''(x) \leq f_S''(x).$$

Proof. Fix $K \in \partial_c f(x)$ and consider a function φ of one real variable given near 0 by

$$\varphi(y) = f(x + y) - f(x) - Ky.$$

Since

$$\begin{aligned} \varphi(0) &= 0, & f''_-(x) &= \varphi''_-(0), & f_S''(x) &= \varphi_S''(0), \\ f''_+(x) &= \varphi''_+(0), & f''(x) &= \varphi''(0), & 0 &\in \partial_c \varphi(0), \end{aligned}$$

it suffices to show that $\varphi''(0) \leq \varphi_S''(0)$.

According to Lemma 2.2, this inequality is true for $\varphi_S''(0) = +\infty$, and $\varphi_S''(0) = -\infty$. So we can suppose that

$$\limsup_{t \downarrow 0} \frac{\varphi(t) + \varphi(-t)}{t^2} \in \mathbf{R}.$$

Then one has

$$(3) \quad \lim_{t \rightarrow 0} \frac{\varphi(t) + \varphi(-t)}{t} = 0.$$

Let us consider a function $g : D(\varphi) \rightarrow \mathbf{R}$ given by

$$g(x) = \begin{cases} \frac{\varphi(x) + \varphi(-x)}{x}, & \text{if } x \in D(\varphi) \setminus \{0\}, \\ 0, & \text{if } x = 0. \end{cases}$$

By Proposition 1.2 and (3), g is a continuous function on $D(\varphi)$, which is locally Lipschitz on $D(\varphi) \setminus \{0\}$. Applying Lemma 2.3, for every $x > 0$, there exists $\lambda(x) \in (0, x)$ satisfying

$$g(x) - g(0) \in \partial_c g(\lambda(x))x.$$

Using Lemma 2.1,

$$(4) \quad \frac{\varphi(x) + \varphi(-x)}{x^2} \in \frac{\partial_c \varphi(\lambda(x)) - \partial_c \varphi(-\lambda(x))}{\lambda(x)} - \frac{\varphi(\lambda(x)) + \varphi(-\lambda(x))}{\lambda(x)^2}.$$

This yields

$$\begin{aligned} & \liminf_{x \downarrow 0} \left(\frac{\varphi(x) + \varphi(-x)}{x^2} + \frac{\varphi(\lambda(x)) + \varphi(-\lambda(x))}{\lambda(x)^2} \right) \\ (5) \quad & \geq \liminf_{x \downarrow 0} \frac{\partial_c \varphi(\lambda(x)) - \partial_c \varphi(-\lambda(x))}{\lambda(x)} \\ & \geq \liminf_{x \downarrow 0} \frac{\partial_c \varphi(\lambda(x))}{\lambda(x)} + \liminf_{x \downarrow 0} \frac{\partial_c \varphi(-\lambda(x))}{-\lambda(x)} \\ & \geq \varphi''_+(0) + \varphi''_-(0) \geq 2\varphi''(0). \end{aligned}$$

Furthermore,

$$\begin{aligned} & \liminf_{x \downarrow 0} \left(\frac{\varphi(x) + \varphi(-x)}{x^2} + \frac{\varphi(\lambda(x)) + \varphi(-\lambda(x))}{\lambda(x)^2} \right) \\ (6) \quad & \leq \limsup_{x \downarrow 0} \frac{\varphi(x) + \varphi(-x)}{x^2} + \limsup_{x \downarrow 0} \frac{\varphi(\lambda(x)) + \varphi(-\lambda(x))}{\lambda(x)^2} \\ & = 2\varphi''_S(0). \end{aligned}$$

Inequalities (5), (6) give $\varphi''(0) \leq \varphi''_S(0)$, which was to be proved. ■

Example 2.1. For a strict inequality $f''(x) < f_S''(x)$ in Proposition 2.2, consider a function

$$f(x) = \begin{cases} x, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Then $f''(0) = 0$, and $f_S''(0) = +\infty$.

Lemma 2.4 [5]. *Suppose that $f : X \rightarrow \mathbf{R}$ is locally Lipschitz on an open set U , $[x, y] \subset U$. Denote $x_t = x + t(y - x)$. Then a function $g : (0, 1) \rightarrow \mathbf{R} : g(t) = f(x_t)$ is locally Lipschitz, and*

$$\partial_c g(t) \subset \langle \partial_c f(x_t), y - x \rangle.$$

Theorem 2.1. *Let $U \subset X$ be an open set, $f : U \rightarrow \mathbf{R}$ a locally Lipschitz function, $x \in U$, $h \in X$. Then*

$$f''(x)(h, h) \leq f_S''(x)(h, h).$$

Proof. Take an arbitrary open interval $(\alpha, \beta) \subset \mathbf{R}$ such that $0 \in (\alpha, \beta)$ and that for every $t \in (\alpha, \beta)$ we have $x + th \in U$. Let us consider a function $g : (\alpha, \beta) \rightarrow \mathbf{R} : g(t) = f(x + th)$. Thanks to Lemma 2.4,

$$\partial_c g(t) \subset \langle \partial_c f(x + th), h \rangle$$

whenever $t \in (\alpha, \beta)$. Hence

$$(7) \quad g''(0) \geq f''(x)(h, h).$$

Since $g_S''(0) = f_S''(x)(h, h)$, by inequality (7) and Proposition 2.2, we reach the conclusion. ■

We conclude this section with application of Theorem 2.1 in characterizing the convexity property.

Theorem 2.2. *Let $f : X \rightarrow \mathbf{R}$ be a locally Lipschitz function on an open convex set $U \subset X$. Then the following conditions are equivalent*

- (a) f is convex on U ,
- (b) $f^{**}(x)(h, h) \geq 0$ for every $x \in U, h \in S_X$,
- (c) $f_+''(x)(h, h) \geq 0$ for every $x \in U, h \in S_X$,
- (d) $f''(x)(h, h) \geq 0$ for every $x \in U, h \in S_X$,

(e) $f_S''(x)(h, h) \geq 0$ for every $x \in U, h \in S_X$.

Moreover, if f is a $C^{1,1}$ function, then the previous conditions are equivalent with

(f) $f^{\diamond\diamond}(x)(u, u) \geq 0$ for every $x \in U, h \in S_X$.

Proof. Considering the well known fact that the subdifferential of a continuous convex function is a monotone operator [17], we obtain $(a) \Rightarrow (b)$. Implications $(b) \Rightarrow (f)$, $(f) \Rightarrow (c)$, and $(d) \Rightarrow (e)$ follow from Proposition 2.1 and Theorem 2.1, respectively. Symmetry of S_X and Proposition 1.4 give $(c) \Rightarrow (d)$. Finally, $(e) \Rightarrow (a)$ holds thanks to Theorem 1.2. ■

Corollary 2.2. *Theorem 1.3 and Theorem 1.4 follow immediately from the above relations among the generalized second-order directional derivatives and Theorem 1.2.*

3. Second-order sufficient optimality condition

In this section, we establish connections among the lower second-order directional derivative $f_\infty(x)(u, v)$ on the one hand, and the generalized second-order directional derivative $f_+''(x)(u, v)$ and the generalized strict second-order directional derivative $f^{\diamond\diamond}(x)(u, v)$, on the other hand, for $C^{1,1}$ functions.

The following lemma has appeared in [8].

Lemma 3.1. *Let $f : X \rightarrow \mathbf{R}$ be a continuous function. Let $x, u, v \in X$ and suppose that $D_+(f(\cdot), u)$ is finite near x . Then*

$$f_\infty(x)(u, v) = \liminf_{y \rightarrow x, s \downarrow 0} \frac{1}{s} (D_+f(y + sv, u) - D_+f(y, u)).$$

Proposition 3.1. *Let $f : X \rightarrow \mathbf{R}$ be Lipschitz near $x, u, v \in X$. Then*

$$f^{**}(x)(u, v) \leq f_\infty(x)(u, v).$$

Furthermore, if f is a continuously Gâteaux differentiable function near x , then

$$f^{**}(x)(u, v) = f_\infty(x)(u, v) \leq f_+''(x)(u, v).$$

Proof. We note that it holds

$$(8) \quad f^{**}(x)(u, v) = \liminf_{y \rightarrow x, t \downarrow 0} \frac{f_{\circ}(y + tu, v) - f^{\circ}(y, v)}{t}$$

and

$$(9) \quad f''_{+}(x)(u, v) = \liminf_{t \downarrow 0} \frac{f_{\circ}(x + tu, v) - f^{\circ}(x, v)}{t}.$$

One can see immediately that

$$(10) \quad f_{\circ}(y + tu, v) - f^{\circ}(y, v) \leq D_{+}f(y + tu, v) - D_{+}f(y, v)$$

for every y sufficiently near x and every sufficiently small $t > 0$. By (8) and Lemma 3.1,

$$(11) \quad f^{**}(x)(u, v) \leq f_{\infty}(x)(u, v).$$

For continuously Gâteaux differentiable functions,

$$f_{\circ}(y + tu, v) = D_{+}f(y + tu, v) = f^{\circ}(y + tu, v)$$

for every y sufficiently near x and every sufficiently small $t > 0$.

This implies that the inequality (11) becomes an equality and using (9), we obtain

$$\begin{aligned} f_{\infty}(x)(u, v) &= \liminf_{y \rightarrow x, t \downarrow 0} \frac{1}{t} (D_{+}f(y + tu, v) - D_{+}f(y, v)) \\ &= \liminf_{y \rightarrow x, t \downarrow 0} \frac{1}{t} (f_{\circ}(y + tu, v) - f^{\circ}(y, v)) \\ &\leq \liminf_{t \downarrow 0} \frac{1}{t} (f_{\circ}(x + tu, v) - f^{\circ}(x, v)) = f''_{+}(x)(u, v). \quad \blacksquare \end{aligned}$$

Example 3.1. We give an example of a function for which a strict inequality $f_{\infty}(x)(u, v) < f''_{+}(x)(u, v)$ holds.

Define $f : \mathbf{R}^2 \rightarrow \mathbf{R}$, partially in polar coordinates, as

1. $f(x, y) = (y - x^{\frac{3}{2}})^2$, if $[x, y] \in \left\{ [x, y] \in \mathbf{R}^2, x \geq 0, x^{\frac{3}{2}} \leq y \leq x^{\frac{3}{2}} + 1 \right\}$,
2. $f(x, y) = 1$, if $[x, y] \in \left\{ [x, y] \in \mathbf{R}^2, x \geq 0, x^{\frac{3}{2}} + 1 \leq y \right\}$,

3. $f(r \sin \varphi, r \cos \varphi) = \hat{f}(r, \varphi) = \left(r - \frac{\sin^2 \varphi}{\cos^3 \varphi}\right)^2$, if $\varphi \in (0, \frac{\pi}{2})$ and $r > \frac{\sin^2 \varphi}{\cos^3 \varphi}$,
4. $f(r \sin \varphi, r \cos \varphi) = \hat{f}(r, \varphi) = r^2$, if $\varphi \in [\frac{3}{2}\pi, 2\pi]$,
5. $f(-x, y) = f(x, y)$.

On the basis of analysis in [15, Example 4.4], one has that for $h = [0, 1]$, $f_\infty(0)(h, h) = 0$, $f_+''(0)(h, h) = 2$.

Remark 3.1. *Theorem 1.6 generalizes Theorem 1.5.* Indeed, due to Proposition 1.5, $f \in C^{1,1}$. Setting $\lambda = 0$ in the definition of $f_{+u}^\circ(x, u)$, one has $\nabla f(0) = 0$. For the rest, it suffices to apply Proposition 3.1.

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References

- [1] J.M. Borwein, M. Fabián, *On generic second-order Gâteaux differentiability*, Nonlinear Anal. T.M.A **20** (1993), 1373–1382.
- [2] J.M. Borwein and D. Noll, *Second-order differentiability of convex functions in Banach spaces*, Trans. Amer. Math. Soc. **342** (1994), 43–81.
- [3] A. Ben-Tal and J. Zowe, *Directional derivatives in nonsmooth optimization*, J. Optim. Theory Appl. **47** (1985), 483–490.
- [4] F.H. Clarke, *Necessary conditions for nonsmooth problems in optimal control and the calculus of variations*, Ph.D. thesis, Univ. of Washington 1973.
- [5] F.H. Clarke, *Optimization and nonsmooth analysis*, J. Wiley, New York 1983.
- [6] R. Cominetti, *Equivalence between the classes of $C^{1,1}$ and twice locally Lipschitzian functions*, Ph.D. thesis, Université Blaise Pascal 1989.
- [7] R. Cominetti and R. Correa, *A generalized second-order derivative in nonsmooth optimization*, SIAM J. Control Optim. **28** (1990), 789–809.
- [8] W.L. Chan, L.R. Huang and K.F. Ng, *On generalized second-order derivatives and Taylor expansions in nonsmooth optimization*, SIAM J. Control Optim. **32** (1994), 789–809.
- [9] P.G. Georgiev and N.P. Zlateva, *Second-order subdifferentials of $C^{1,1}$ functions and optimality conditions*, Set-Valued Analysis **4** (1996), 101–117.

- [10] J.B. Hiriart-Urruty and C. Lemarchal, *Convex Analysis nad Minimization Algorithms*, Springer Verlag, Berlin 1993.
- [11] L.R. Huang and K.F. Ng, *On some relations between Chaney's generalized second-order directional derivative and that of Ben-Tal and Zowe*, SIAM J. Control Optim. **34** (1996), 1220–1234.
- [12] V. Jeyakumar and X.Q. Yang, *Approximate generalized Hessians and Taylor's expansions for continously Gâteaux differentiable functions*, Nonlinear Anal. T.M.A **36** (1999), 353–368.
- [13] G. Lebourg, *Generic differentiability of Lipschitz functions*, Trans. Amer. Math. Soc. **256** (1979), 125–144.
- [14] Y. Maruyama, *Second-order necessary conditions for nonlinear problems in Banach spases and their applications to optimal control problems*, Math. Programming **41** (1988), 73–96.
- [15] P. Michel and J.P. Penot, *Second-order moderate derivatives*, Nonlinear Anal. T.M.A **22** (1994), 809–821.
- [16] K. Pastor, *Generalized second-order directional derivatives for locally Lipschitz functions*, Preprint no. 17/2001 of Palacký Univerzity, Olomouc.
- [17] R.T. Rockafellar, *Characterization of the subdifferentials of convex functions*, Pacific J. Math. **17** (1966), 497–509.
- [18] R.T. Rockafellar, *Second-order optimality conditions in nonlinear programming obtained by way of epi-derivatives*, Math. Oper. Res. **14** (1989), 462–484.
- [19] X.Q. Yang, *On relations and applications of generalized second-order directional derivatives*, Non. Anal. T.M.A **36** (1999), 595–614.
- [20] X.Q. Yang, *On second-order directional derivatives*, Non. Anal. T.M.A **26**, 55–66.

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