

**EXISTENCE AND CONTROLLABILITY
OF FRACTIONAL-ORDER IMPULSIVE
STOCHASTIC SYSTEM WITH INFINITE DELAY**

TOUFIK GUENDOZI

Laboratory of Stochastic Models, Statistic and Applications
Tahar Moulay University PO. Box 138 En-Nasr, 20000 Saida, Algeria

e-mail: tf.guendouzi@gmail.com

Abstract

This paper is concerned with the existence and approximate controllability for impulsive fractional-order stochastic infinite delay integro-differential equations in Hilbert space. By using Krasnoselskii's fixed point theorem with stochastic analysis theory, we derive a new set of sufficient conditions for the approximate controllability of impulsive fractional stochastic system under the assumption that the corresponding linear system is approximately controllable. Finally, an example is provided to illustrate the obtained theory.

Keywords: existence result, approximate controllability, fractional stochastic differential equations, resolvent operators, infinite delay.

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1. INTRODUCTION

Fractional calculus is an emerging field in the area of the applied mathematics that deals with derivatives and integrals of arbitrary orders as well as with their applications. During the history of fractional calculus it was reported that the pure mathematical formulations of the investigated problems started to be dressed with more applications in various fields. As a result during the last decade fractional calculus has been applied successfully to almost every field of science and engineering. However, despite of the fact that several fields of application of fractional differentiation and integration are already well established, some others have just started.

Many applications of fractional calculus dynamics can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled

thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, astrophysics, etc. (see for more details Refs. [5, 14, 16] and the references therein).

On the other hand, there is also an increasing interest in the recent issue related to dynamical fractional systems oriented towards the field of control theory concerning heat transfer, lossless transmission lines [18], the use of discretizing devices supported by fractional calculus. In recent years, various controllability problems for different kinds of dynamical systems have been studied in many publications [1, 13, 24]. From the mathematical point of view, the problems of exact and approximate controllability are to be distinguished. However, the concept of exact controllability is usually too strong and has limited applicability. Approximate controllability is a weaker concept than complete controllability and it is completely adequate in applications [4, 12]. In particular, the fixed point techniques are widely used in studying the controllability problems for nonlinear control systems. Klamka studied the practical applicability of the fixed point theorem in solving various controllability problems for different types of dynamical control systems. Wang derived a set of sufficient conditions for the approximate controllability of differential equations with multiple delays by implementing some natural conditions such as growth conditions for the nonlinear term and compactness of the semigroup. Sakthivel and Anandhi [19] investigated the problem of approximate controllability for a class of nonlinear impulsive differential equations with state-dependent delay by using semigroup theory and fixed point technique.

Moreover, the study of stochastic differential equations has attracted great interest due to its applications in characterizing many problems in physics, biology, chemistry, mechanics, and so on. The deterministic models often fluctuate due to noise, so we must move from deterministic control to stochastic control problems. In the present literature there is only a limited number of papers that deal with the controllability of stochastic systems [10, 20]. Klamka [11] derived a set of sufficient conditions for constrained local relative controllability near the origin for semilinear finite-dimensional dynamical control systems by using the generalized open mapping theorem.

Sakthivel *et al.* [21] studied the approximate controllability of nonlinear deterministic and stochastic evolution systems with unbounded delay in abstract spaces. Muthukumar and Balasubramaniam [15] derived a set of sufficient conditions for the approximate controllability of mixed stochastic Volterra-Fredholm type integro-differential systems in Hilbert spaces by using the Banach fixed point theorem. More recently, the approximate controllability of fractional stochastic evolution equations has been studied in [22]. The authors obtained a new set of sufficient conditions for the approximate controllability of nonlinear fractional stochastic control system under the assumptions that the corresponding linear

system is approximately controllable. However, to the best of our knowledge, the approximate controllability problem for impulsive fractional stochastic integro-differential system with infinite delay has not been investigated yet. Motivated by this consideration, in this paper we will study the approximate controllability for impulsive fractional-order stochastic infinite delay integro-differential system in Hilbert space under the assumption that the associated linear system is approximately controllable. Our paper is organized as follows. Section 2 is devoted to a review of some essential results in fractional calculus and the resolvent operators that will be used in this work to obtain our main results. In Section 3, we state and prove the existence of mild solution and controllability result. Section 4 deals with an example to illustrate the abstract results.

2. PRELIMINARIES AND BASIC PROPERTIES

Let \mathcal{H}, \mathcal{K} be two separable Hilbert spaces and $\mathcal{L}(\mathcal{K}, \mathcal{H})$ be the space of bounded linear operators from \mathcal{K} into \mathcal{H} . For convenience, we will use the same notation $\|\cdot\|$ to denote the norms in \mathcal{H}, \mathcal{K} and $\mathcal{L}(\mathcal{K}, \mathcal{H})$, and use (\cdot, \cdot) to denote the inner product of \mathcal{H} and \mathcal{K} . Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying that \mathcal{F}_0 contains all \mathbb{P} -null sets of \mathcal{F} . $\omega = (\omega_t)_{t \geq 0}$ be a Q -Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with the covariance operator Q such that $TrQ < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k \geq 1}$ in \mathcal{K} , a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1, 2, \dots$, and a sequence of independent Brownian motions $\{\beta_k\}_{k \geq 1}$ such that

$$(\omega(t), e)_{\mathcal{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k} (e_k, e)_{\mathcal{K}} \beta_k(t), \quad e \in \mathcal{K}, t \geq 0.$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}\mathcal{K}, \mathcal{H})$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathcal{K}$ to \mathcal{H} with the inner product $\langle \varphi, \psi \rangle_{\mathcal{L}_2^0} = Tr(\varphi Q \psi^*)$.

The purpose of this paper is to investigate the existence of mild solution and the approximate controllability for the following impulsive fractional stochastic differential equations with infinite delay involving the Caputo derivative in the form

$$(1) \quad \left\{ \begin{array}{l} {}^c D_t^\alpha x(t) = Ax(t) + Bu(t) + f(t, x_t, Fx(t)) + \sigma(t, x_t, Gx(t)) \frac{d\omega(t)}{dt}, \\ \quad \quad \quad t \in J = [0, T], T > 0, t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \\ x(t) = \phi(t), \quad \phi(t) \in \mathcal{B}_h, \end{array} \right.$$

where ${}^c D_t^\alpha$ is the Caputo fractional derivative of order α , $0 < \alpha$; $x(\cdot)$ takes the value in the separable Hilbert space \mathcal{H} ; $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ is the infinitesimal generator of an α -resolvent family $\{S_\alpha(t)\}_{t \geq 0}$; the control function $u(\cdot)$ is given in $\mathcal{L}^2(J, \mathcal{U})$, \mathcal{U} is a Hilbert space; B is a bounded linear operator from \mathcal{U} into \mathcal{H} . The history $x_t : (-\infty, 0] \rightarrow \mathcal{H}$, $x_t(\theta) = x(t + \theta)$, $\theta \leq 0$, belongs to an abstract phase space \mathcal{B}_h ; $f : J \times \mathcal{B}_h \times \mathcal{H} \rightarrow \mathcal{H}$ and $\sigma : J \times \mathcal{B}_h \times \mathcal{L}_2^0 \rightarrow \mathcal{H}$ are appropriate functions to be specified later; $I_k : \mathcal{B}_h \rightarrow \mathcal{H}$, $k = 1, 2, \dots, m$, are appropriate functions. The terms $Fx(t)$ and $Gx(t)$ are given by $Fx(t) = \int_0^t K(t, s)x(s)ds$ and $Gx(t) = \int_0^t P(t, s)x(s)ds$ respectively, where $K, P \in \mathcal{C}(D, \mathbb{R}^+)$ are the set of all positive continuous functions on $D = \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$. Here $0 = t_0 \leq t_1 \leq \dots \leq t_m \leq t_{m+1} = T$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h \rightarrow 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h \rightarrow 0} x(t_k - h)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. The initial data $\phi = \{\phi(t), t \in (-\infty, 0]\}$ is an \mathcal{F}_0 -measurable, \mathcal{B}_h -valued random variable independent of ω with finite second moments.

Now, we assume that $h : (-\infty, 0] \rightarrow (0, \infty)$ with $l = \int_{-\infty}^0 h(t)dt < \infty$ is a continuous function. We define the abstract phase space \mathcal{B}_h by

$$\mathcal{B}_h = \left\{ \phi : (-\infty, 0] \rightarrow \mathcal{H}, \text{ for any } a > 0, (\mathbb{E}|\phi(\theta)|^2)^{1/2} \text{ is bounded and measurable function on } [-a, 0] \text{ with } \phi(0) = 0 \text{ and } \int_{-\infty}^0 h(s) \sup_{s \leq \theta \leq 0} (\mathbb{E}|\phi(\theta)|^2)^{1/2} ds < \infty \right\}.$$

If \mathcal{B}_h is endowed with the norm

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^0 h(s) \sup_{s \leq \theta \leq 0} (\mathbb{E}|\phi(\theta)|^2)^{1/2} ds, \quad \phi \in \mathcal{B}_h,$$

then $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space.

We consider the space

$$\mathcal{B}_b = \left\{ x : (-\infty, T] \rightarrow \mathcal{H} \text{ such that } x|_{J_k} \in \mathcal{C}(J_k, \mathcal{H}) \text{ and there exist } x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k) = x(t_k^-), x_0 = \phi \in \mathcal{B}_h, k = 1, 2, \dots, m \right\},$$

where $x|_{J_k}$ is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \dots, m$. The function $\|\cdot\|_{\mathcal{B}_b}$ defined by

$$\|x\|_{\mathcal{B}_b} = \|\phi\|_{\mathcal{B}_h} + \sup_{0 \leq s \leq T} (\mathbb{E}\|x(s)\|^2)^{1/2}, \quad x \in \mathcal{B}_b$$

is a seminorm in \mathcal{B}_b .

Lemma 2.1 [17]. Assume that $x \in B_h$. Then $x_t \in \mathcal{B}_h$ for $t \in J$. Moreover,

$$l(\mathbb{E}\|x(t)\|^2)^{1/2} \leq l \sup_{0 \leq s \leq T} (\mathbb{E}\|x(s)\|^2)^{1/2} + \|x_0\|_{\mathcal{B}_h},$$

where $l = \int_{-\infty}^0 h(s) ds < \infty$.

Let us recall the following known definitions. For details see [5].

Definition 2.2. The fractional integral of order α with the lower limit 0 for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \alpha > 0$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Definition 2.3. Riemann-Liouville derivative of order α with lower limit zero for a function $f : [0, \infty) \rightarrow \mathbb{R}$ can be written as

$$(2) \quad {}^L D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0, n-1 < \alpha < n.$$

Definition 2.4. The Caputo derivative of order α for a function $f : [0, \infty) \rightarrow \mathbb{R}$ can be written as

$$(3) \quad {}^c D^\alpha f(t) = {}^L D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, n-1 < \alpha < n.$$

If $f(t) \in C^n[0, \infty)$, then

$${}^c D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds = I^{n-\alpha} f^{(n)}(s), \quad t > 0, n-1 < \alpha < n.$$

Obviously, the Caputo derivative of a constant is equal to zero. The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given as

$$L\{{}^c D^\alpha f(t); s\} = s^\alpha \hat{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0); \quad n-1 \leq \alpha < n.$$

Definition 2.5. A two parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_C \frac{\mu^{\alpha-\beta} e^{\mu}}{\mu^{\alpha} - z} d\mu, \quad \alpha, \beta \in C, \mathcal{R}(\alpha) > 0,$$

where C is a contour which starts and ends at $-\infty$ end encircles the disc $|\mu| \leq |z|^{1/2}$ counter clockwise.

For short, $E_{\alpha}(z) = E_{\alpha,1}(z)$. It is an entire function which provides a simple generalization of the exponent function: $E_1(z) = e^z$ and the cosine function: $E_2(z^2) = \cosh(z)$, $E_2(-z^2) = \cos(z)$, and plays a vital role in the theory of fractional differential equations. The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

$$\int_0^{\infty} e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^{\alpha}) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha} - \omega}, \quad \operatorname{Re} \lambda > \omega^{\frac{1}{\alpha}}, \omega > 0,$$

and for more details see [5].

Definition 2.6 [25]. A closed and linear operator A is said to be sectorial if there are constants $\omega \in \mathbb{R}$, $\theta \in [\frac{\pi}{2}, \pi]$, $M > 0$, such that the following two conditions are satisfied:

- $\rho(A) \subset \Sigma_{\theta,\omega} = \{\lambda \in C : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}$,
- $\|\mathbf{R}(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|}$, $\lambda \in \Sigma_{\theta,\omega}$.

Definition 2.7. Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space H . Let $\rho(A)$ be the resolvent set of A . We say that A is the generator of an α -resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $S_{\alpha} : \mathbb{R}_+ \rightarrow L(H)$, where $L(H)$ is a Banach space of all bounded linear operators from H into H and the corresponding norm is denoted by $\|\cdot\|$, such that $\{\lambda^{\alpha} : \operatorname{Re} \lambda > \omega\} \subset \rho(A)$ and

$$(4) \quad (\lambda^{\alpha} I - A)^{-1} x = \int_0^{\infty} e^{\lambda t} S_{\alpha}(t) x dt, \quad \operatorname{Re} \lambda > \omega, x \in H,$$

where $S_{\alpha}(t)$ is called the α -resolvent family generated by A .

Definition 2.8. Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space H and $\alpha > 0$. We say that A is the generator of a solution operator if there exist $\omega \geq 0$ and a strongly continuous function

$S_\alpha : \mathbb{R}_+ \rightarrow L(H)$ such that $\{\lambda^\alpha : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$ and

$$(5) \quad \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{\lambda t} S_\alpha(t)x dt, \quad \operatorname{Re}\lambda > \omega, x \in H,$$

where $S_\alpha(t)$ is called the solution operator generated by A .

The concept of the solution operator is closely related to the concept of a resolvent family. For more details on α -resolvent family and solution operators, we refer the reader to [5].

Lemma 2.9 [3]. *If f satisfies the uniform Hölder condition with the exponent $\beta \in (0, 1]$ and A is a sectorial operator, then the unique solution of the Cauchy problem*

$$(6) \quad \begin{aligned} {}^c D_t^\alpha x(t) &= Ax(t) + f(t, x_t, Fx(t)), \quad t > t_0, t_0 \geq 0, 0 < \alpha < 1, \\ x(t) &= \phi(t), \quad t \leq t_0, \end{aligned}$$

is given by

$$(7) \quad x(t) = T_\alpha(t - t_0)(x(t_0^+)) + \int_{t_0}^t S_\alpha(t - s)f(s, x_s, Fx(s))ds,$$

where

$$(8) \quad T_\alpha(t) = E_{\alpha,1}(At^\alpha) = \frac{1}{2\pi i} \int_{\widehat{B}_r} e^{\lambda t} \frac{\lambda^{\alpha-1}}{\lambda^\alpha - A} d\lambda,$$

$$(9) \quad S_\alpha(t) = t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha) = \frac{1}{2\pi i} \int_{\widehat{B}_r} e^{\lambda t} \frac{1}{\lambda^\alpha - A} d\lambda,$$

here \widehat{B}_r denotes the Bromwich path; $S_\alpha(t)$ is called the α -resolvent family and $T_\alpha(t)$ is the solution operator generated by A .

Now, we present the definition of mild solutions for the system (1).

Definition 2.10 [23]. An \mathcal{F}_t -adapted stochastic process $x : (-\infty, T] \rightarrow \mathcal{H}$ is called a mild solution of the system (1) if $x_0 = \phi \in \mathcal{B}_h$ satisfying $x_0 \in \mathcal{L}_2^0(\Omega, \mathcal{H})$ and the following conditions:

- (i) $x(t)$ is \mathcal{B}_h -valued and the restriction of $x(\cdot)$ to $(t_k, t_{k+1}]$, $k = 1, 2, \dots, m$ is continuous.
- (ii) For each $t \in J$, $x(t)$ satisfies the following integral equation

$$(10) \quad x(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \int_0^t S_\alpha(t-s)[Bu(s) + f(s, x_s, Fx(s))]ds \\ + \int_0^t S_\alpha(t-s)\sigma(s, x_s, Gx(s))d\omega(s), & t \in [0, t_1], \\ T_\alpha(t-t_1)(x(t_1^-) + I_1(x(t_1^-))) + \int_{t_1}^t S_\alpha(t-s)[Bu(s) + f(s, x_s, Fx(s))]ds \\ + \int_{t_1}^t S_\alpha(t-s)\sigma(s, x_s, Gx(s))d\omega(s), & t \in (t_1, t_2], \\ \vdots \\ T_\alpha(t-t_m)(x(t_m^-) + I_m(x(t_m^-))) + \int_{t_m}^t S_\alpha(t-s)[Bu(s) + f(s, x_s, Fx(s))]ds \\ + \int_{t_m}^t S_\alpha(t-s)\sigma(s, x_s, Gx(s))d\omega(s), & t \in (t_m, T]. \end{cases}$$

(iii) $\Delta x|_{t=t_k} = I_k(x(t_k^-))$, $k = 1, 2, \dots, m$ the restriction of $x(\cdot)$ to the interval $[0, T] \setminus \{t_1, \dots, t_m\}$ is continuous.

3. MAIN RESULTS

In the present section, we shall formulate and prove sufficient conditions for the approximate controllability of the system (1). To do this, we first prove the existence of solutions for fractional impulsive control system. Then, we show that under certain assumptions, the approximate controllability of semilinear fractional impulsive control system (1) is implied by the approximate controllability of the associated linear system.

Definition 3.1 [22]. Let $x_T(\phi; u)$ be the state value of (1) at the terminal time T corresponding to the control u and the initial value ϕ . Introduce the set

$$\mathcal{R}(T, \phi) = \{x_T(\phi; u)(0); u(\cdot) \in \mathcal{L}^2(J, \mathcal{U})\},$$

which is called the reachable set of (1) at the terminal time T and its closure in \mathcal{H} is denoted by $\overline{\mathcal{R}(T, \phi)}$. The system (1) is said to be approximately controllable on the interval J if $\mathcal{R}(T, \phi) = \mathcal{H}$.

In order to study the approximate controllability for the impulsive fractional control system (1), we introduce the approximate controllability of its linear part

$$(11) \quad \begin{cases} {}^c D_t^\alpha x(t) = Ax(t) + (Bu)(t), & t \in J = [0, T], T > 0, t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)), & k = 1, 2, \dots, m, \\ x(0) = \phi(0). \end{cases}$$

For this purpose, we need to introduce the relevant operator

$$\Gamma_0^T = \int_0^T S_\alpha(T-s)BB^*S_\alpha^*(T-s)ds,$$

$$R(q, \Gamma_0^T) = (qI + \Gamma_0^T)^{-1},$$

where B^* denotes the adjoint of B and $S_\alpha^*(t)$ is the adjoint of $S_\alpha(t)$. It is straightforward that the operator Γ_0^T is a linear bounded operator.

(H_0) $qR(q, \Gamma_0^T) \rightarrow 0$ as $\alpha \rightarrow 0^+$ in the strong operator topology.

The hypothesis (H_0) is equivalent to the fact that the linear fractional control system (11) is approximately controllable on $[0, T]$ (see [13], Theorem 2).

In order to establish the result, we impose the following conditions.

(iv) If $\alpha \in (0, 1)$ and $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$, then for $x \in \mathcal{H}$ and $t > 0$ we have $\|T_\alpha(t)\| \leq Me^{\omega t}$ and $\|S_\alpha(t)\| \leq Ce^{\omega t}(1 + t^{\alpha-1})$, $\omega > \omega_0$. Thus we have

$$\|T_\alpha(t)\| \leq \widetilde{M}_T \quad \text{and} \quad \|S_\alpha(t)\| \leq t^{\alpha-1}\widetilde{M}_S,$$

where $\widetilde{M}_T = \sup_{0 \leq t \leq T} \|T_\alpha(t)\|$, and $\widetilde{M}_S = \sup_{0 \leq t \leq T} Ce^{\omega t}(1 + t^{1-\alpha})$ (for more details, see [25]).

(v) There exist $\mu_1, \mu_2 > 0$ such that

$$\mathbf{E}\|f(t, \gamma, x) - f(t, \psi, y)\|_{\mathcal{H}}^2 \leq \mu_1 \|\gamma - \psi\|_{\mathcal{B}_h}^2 + \mu_2 \mathbf{E}\|x - y\|_{\mathcal{H}}^2.$$

(vi) There exist $\nu_1, \nu_2 > 0$ such that

$$\mathbf{E}\|\sigma(t, \gamma, x) - \sigma(t, \psi, y)\|_{\mathcal{L}_2^0}^2 \leq \nu_1 \|\gamma - \psi\|_{\mathcal{B}_h}^2 + \nu_2 \mathbf{E}\|x - y\|_{\mathcal{H}}^2.$$

(vii) $f : J \times \mathcal{B}_h \times \mathcal{H} \rightarrow \mathcal{H}$ is continuous and there exist two continuous functions $\mu_1, \mu_2 : J \rightarrow (0, \infty)$ such that

$$\mathbf{E}\|f(t, \psi, x)\|_{\mathcal{H}}^2 \leq \mu_1(t)\|\psi\|_{\mathcal{B}_h}^2 + \mu_2(t)\mathbf{E}\|x\|_{\mathcal{H}}^2, \quad (t, \psi, x) \in J \times \mathcal{B}_h \times \mathcal{H},$$

and $\mu_1^* = \sup_{s \in [0, t]} \mu_1(s)$, $\mu_2^* = \sup_{s \in [0, t]} \mu_2(s)$.

(viii) $\sigma : J \times \mathcal{B}_h \times \mathcal{L}_2^0 \rightarrow \mathcal{H}$ is continuous and there exist two continuous functions $\nu_1, \nu_2 : J \rightarrow (0, \infty)$ such that

$$\mathbf{E}\|\sigma(t, \psi, x)\|_{\mathcal{L}_2^0}^2 \leq \nu_1(t)\|\psi\|_{\mathcal{B}_h}^2 + \nu_2(t)\mathbf{E}\|x\|_{\mathcal{H}}^2, \quad (t, \psi, x) \in J \times \mathcal{B}_h \times \mathcal{L}_2^0,$$

and $\nu_1^* = \sup_{s \in [0, t]} \nu_1(s)$, $\nu_2^* = \sup_{s \in [0, t]} \nu_2(s)$.

(ix) The functions $I_k : \mathcal{H} \rightarrow \mathcal{H}$ is continuous and there exists $\Lambda, \hat{\Lambda} > 0$ such that

$$\Lambda = \max_{1 \leq k \leq m, x \in \hat{B}_p} \{\mathbf{E}\|I_k(x)\|_{\mathcal{H}}^2\}, \quad \hat{\Lambda} = \max_{1 \leq k \leq m; x, \hat{x} \in \hat{B}_p} \{\mathbf{E}\|I_k(x) - I_k(\hat{x})\|_{\mathcal{H}}^2\},$$

where $\hat{B}_p = \{y \in \mathcal{B}_b^0, \|y\|_{\mathcal{B}_b^0}^2 \leq p, \quad p > 0\}$.

The set \hat{B}_p is clearly a bounded closed convex set in \mathcal{B}_b^0 for each p and for each $y \in \hat{B}_p$. From Lemma 2.1, we have

$$\begin{aligned} \|y_t + \bar{y}_t\|_{\mathcal{B}_h}^2 &\leq 2(\|y_t\|_{\mathcal{B}_h}^2 + \|\bar{y}_t\|_{\mathcal{B}_h}^2) \\ (12) \quad &\leq 4 \left(l^2 \sup_{0 \leq t \leq T} \mathbf{E}\|y(t)\|_{\mathcal{H}}^2 + \|y_0\|_{\mathcal{B}_h}^2 \right) + 4 \left(l^2 \sup_{0 \leq t \leq T} \mathbf{E}\|\bar{y}(t)\|_{\mathcal{H}}^2 + \|\bar{y}_0\|_{\mathcal{B}_h}^2 \right) \\ &\leq 4(\|\phi\|_{\mathcal{B}_h}^2 + l^2 p). \end{aligned}$$

The following lemma is required to define the control function. The reader can refer to [19] for the proof.

Lemma 3.2. *For any $\tilde{x}_T \in \mathcal{L}^2(\mathcal{F}_T, \mathcal{H})$, there exists $\tilde{g} \in \mathcal{L}^2_{\mathcal{F}}(\Omega; \mathcal{L}^2(0, T; \mathcal{L}_2^0))$ such that*

$$\tilde{x}_T = \mathbf{E}\tilde{x}_T + \int_0^T \tilde{g}(s) d\omega(s).$$

Now for any $q > 0$, $k = 1, 2, \dots, m$ and $\tilde{x} \in \mathcal{L}^2(\mathcal{F}_T, \mathcal{H})$, we define the control function

$$\begin{aligned} u(t) = u^q(t) &= B^* S_\alpha^*(T-t)(qI + \Gamma_{t_k}^T)^{-1} \\ &\times \left\{ \mathbf{E}\tilde{x}_T + \int_{t_k}^T \tilde{g}(s) d\omega(s) - T_\alpha(T-t_k)[x(t_k^-) + I_k(x(t_k^-))] \right\} \\ &- B^* S_\alpha^*(T-t) \int_{t_k}^T (qI + \Gamma_0^T)^{-1} S_\alpha(T-s) f(s, x_s, Fx(s)) ds \\ &- B^* S_\alpha^*(T-t) \int_{t_k}^T (qI + \Gamma_0^T)^{-1} S_\alpha(T-s) \sigma(s, x_s, Gx(s)) d\omega(s). \end{aligned}$$

Next, we mention the statement of Krasnoselskii's fixed point theorem [23].

Theorem 3.3. Let \hat{B} be a nonempty closed convex subset of a Banach space $(X, \|\cdot\|)$. Suppose that P and Q map \hat{B} into X and satisfy

- (a) $Px + Qy \in \hat{B}$ whenever $x, y \in \hat{B}$;
- (b) P is compact and continuous;
- (c) Q is a contraction mapping.

Then there exists $z \in \hat{B}$ such that $z = Pz + Qz$.

Theorem 3.4. Suppose that the assumptions (iv) – (ix) are satisfied with

$$(13) \quad p \geq 5\widetilde{M}_T^2(p + \Lambda)M_5 + 5\widetilde{M}_S^2T^{2\alpha}\left(\frac{\lambda_1}{\alpha^2} + \frac{\lambda_2}{T(2\alpha - 1)}\right)M_6 + M_7$$

and

$$(14) \quad \left[M_{11} + 3M_{10}\widetilde{M}_S^2T^{2\alpha}\left(\frac{1}{\alpha^2}(\mu_1l + \mu_2F^*) + \frac{1}{T(2\alpha - 1)}(\nu_1l + \nu_2G^*)\right) \right] < 1.$$

Then the impulsive stochastic fractional differential equation (1) has a mild solution on $(-\infty, T]$.

Proof. Let $\mathcal{P}_1 : \hat{B}_p \rightarrow \hat{B}_p$ and $\mathcal{P}_2 : \hat{B}_p \rightarrow \hat{B}_p$ be defined as

$$(15) \quad (\mathcal{P}_1 z)(t) = \begin{cases} 0, & t \in [0, t_1] \\ T_\alpha(t - t_1)(Z(t_1^-) + I_1(Z(t_1^-))), & t \in (t_1, t_2] \\ \vdots \\ T_\alpha(t - t_m)(Z(t_m^-) + I_m(Z(t_m^-))), & t \in (t_m, T] \end{cases}$$

and

$$(16) \quad (\mathcal{P}_2 z)(t) = \begin{cases} \int_0^t S_\alpha(t-s)[Bu^q(s) + f(s, g_s + \bar{z}_s, F(g(s) + \bar{z}(s)))]ds \\ \quad + \int_0^t S_\alpha(t-s)\sigma(s, g_s + \bar{z}_s, G(g(s) + \bar{z}(s)))d\omega(s), & t \in [0, t_1], \\ \int_{t_1}^t S_\alpha(t-s)[Bu^q(s) + f(s, g_s + \bar{z}_s, F(g(s) + \bar{z}(s)))]ds \\ \quad + \int_{t_1}^t S_\alpha(t-s)\sigma(s, g_s + \bar{z}_s, G(g(s) + \bar{z}(s)))d\omega(s), & t \in (t_1, t_2], \\ \vdots \\ \int_{t_m}^t S_\alpha(t-s)[Bu^q(s) + f(s, g_s + \bar{z}_s, F(g(s) + \bar{z}(s)))]ds \\ \quad + \int_{t_m}^t S_\alpha(t-s)\sigma(s, g_s + \bar{z}_s, G(g(s) + \bar{z}(s)))d\omega(s), & t \in (t_m, T]. \end{cases}$$

It will be shown that the impulsive stochastic fractional differential equation (1) is approximately controllable if for all $q > 0$ there exists a fixed point of the operator $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$. To prove this result, we use Krasnoselskii's fixed point theorem (Theorem 3.3). We shall show that the operator $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$ has a fixed point, which is a solution of (1).

In order to use Theorem 3.3, we will verify that \mathcal{P}_1 is compact and continuous while \mathcal{P}_2 is a contraction operator. For the sake of convenience, we divide the proof into several steps.

Step 1. We show that $\mathcal{P}_1 z + \mathcal{P}_2 z^* \in \hat{B}_p$ for $z, z^* \in \hat{B}_p$. For $t \in [0, t_1]$, we have

$$\begin{aligned}
& \mathbf{E} \|(\mathcal{P}_1 z)(t) + (\mathcal{P}_2 z^*)(t)\|_{\mathcal{H}}^2 \\
& \leq 3 \mathbf{E} \left\| \int_0^t S_\alpha(t-s) f(s, g_s + 4\bar{z}_s^*, F(g(s) + \bar{z}^*(s))) ds \right\|_{\mathcal{H}}^2 \\
& + 3 \mathbf{E} \left\| \int_0^t S_\alpha(t-s) B u^q(s) ds \right\|_{\mathcal{H}}^2 \\
& + 3 \mathbf{E} \left\| \int_0^t S_\alpha(t-s) \sigma(s, g_s + \bar{z}_s^*, G(g(s) + \bar{z}^*(s))) d\omega(s) \right\|_{\mathcal{H}}^2 \\
& \leq 3 \int_0^t \|S_\alpha(t-s)\| ds \int_0^t \|S_\alpha(t-s)\| \mathbf{E} \|f(s, g_s + \bar{z}_s^*, F(g(s) + \bar{z}^*(s)))\|_{\mathcal{H}}^2 ds \\
& + 3 \int_0^t \|S_\alpha(t-s)\|^2 \mathbf{E} \|B u^q(s)\|^2 ds \\
& + 3 \int_0^t \|S_\alpha(t-s)\|^2 \mathbf{E} \|\sigma(s, g_s + \bar{z}_s^*, G(g(s) + \bar{z}^*(s)))\|_{\mathcal{L}_2^0}^2 ds.
\end{aligned}$$

We have for $t \in [0, t_1]$ and $M_B = \|B\|$,

$$\begin{aligned}
\mathbf{E} \|u^q(s)\|^2 & \leq \frac{T^{2\alpha-2}}{q^2} M_B^2 \widetilde{M}_S^2 \left\{ 3 \left\| \mathbf{E} \tilde{x}_T + \int_0^t \tilde{g}(s) d\omega(s) \right\|^2 \right. \\
& + 3 \mathbf{E} \left\| \int_0^t S_\alpha(T-s) f(s, g_s + \bar{z}_s^*, F(g(s) + \bar{z}^*(s))) ds \right\|^2 \\
& \left. + 3 \mathbf{E} \left\| \int_0^t S_\alpha(T-s) \sigma(s, g_s + \bar{z}_s^*, G(g(s) + \bar{z}^*(s))) d\omega(s) \right\|^2 \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq 3\frac{T^{2\alpha-2}}{q^2}M_B^2\widetilde{M}_S^2\left[2\|\mathbf{E}\tilde{x}_T\|^2+2\int_0^t\mathbf{E}\|\tilde{g}(s)\|^2ds\right. \\
&+ \int_0^t\|S_\alpha(T-s)\|ds\int_0^t\|S_\alpha(T-s)\|\mathbf{E}\|f(s,g_s+\bar{z}_s^*,F(g(s)+\bar{z}^*(s)))\|^2ds \\
&\left. + \int_0^t\|S_\alpha(T-s)\|^2\mathbf{E}\|\sigma(s,g_s+\bar{z}_s^*,G(g(s)+\bar{z}^*(s)))\|^2ds\right].
\end{aligned}$$

By using (vii) and (viii), we get

$$\begin{aligned}
&\mathbf{E}\|u^q(s)\|^2 \\
&\leq 3\frac{T^{2\alpha-2}}{q^2}M_B^2\widetilde{M}_S^2\left[2\|\mathbf{E}\tilde{x}_T\|^2+2TM_{\tilde{g}}+\widetilde{M}_S^2\int_0^t(T-s)^{\alpha-1}ds\right. \\
&\times\int_0^t(T-s)^{\alpha-1}[\mu_1(s)\|g_s+\bar{z}_s^*\|_{\mathcal{B}_h}^2+\mu_2(s)\mathbf{E}\|F(g(s)+\bar{z}^*(s))\|_{\mathcal{H}}^2]ds \\
&+ \widetilde{M}_S^2\int_0^t(T-s)^{2\alpha-2}[\nu_1(s)\|g_s+\bar{z}_s^*\|_{\mathcal{B}_h}^2+\nu_2(s)\mathbf{E}\|G(g(s)+\bar{z}^*(s))\|_{\mathcal{H}}^2]ds \\
&\leq 3\frac{T^{2\alpha-2}}{q^2}M_B^2\widetilde{M}_S^2\left[2\|\mathbf{E}\tilde{x}_T\|^2+2TM_{\tilde{g}}+\widetilde{M}_S^2\frac{T^\alpha}{\alpha}\int_0^t(T-s)^{\alpha-1}\right. \\
&\times[4\mu_1^*(\|\phi\|_{\mathcal{B}_h}^2+l^2p)+\mu_2^*F^*\sup_{s\in[0,T]}\mathbf{E}\|z^*\|_{\mathcal{H}}^2]ds \\
&+ \widetilde{M}_S^2\int_0^t(T-s)^{2\alpha-2}[4\nu_1^*(\|\phi\|_{\mathcal{B}_h}^2+l^2p)+\nu_2^*G^*\sup_{s\in[0,T]}\mathbf{E}\|z^*\|_{\mathcal{H}}^2]ds \\
&\leq 3\frac{T^{2\alpha-2}}{q^2}M_B^2\widetilde{M}_S^2\left[2\|\mathbf{E}\tilde{x}_T\|^2+2TM_{\tilde{g}}+\widetilde{M}_S^2\frac{T^{2\alpha}}{\alpha^2}[4\mu_1^*(\|\phi\|_{\mathcal{B}_h}^2+l^2p)+\mu_2^*F^*p]\right. \\
&\times\left.\widetilde{M}_S^2\frac{T^{2\alpha-1}}{2\alpha-1}[4\nu_1^*(\|\phi\|_{\mathcal{B}_h}^2+l^2p)+\nu_2^*G^*p]\right] \\
&= M_1+3\frac{T^{2(2\alpha-1)}}{q^2}M_B^2\widetilde{M}_S^4\left(\frac{\lambda_1}{\alpha^2}+\frac{\lambda_2}{T(2\alpha-1)}\right),
\end{aligned}$$

where $M_{\tilde{g}} = \max\{\|\tilde{g}(s)\|^2; s \in [0, t]\}$ and $M_1 = 3\frac{T^{2\alpha-2}}{q^2}M_B^2\widetilde{M}_S^2[2\|\mathbf{E}\tilde{x}_T\|^2+2TM_{\tilde{g}}]$.

Now, we have

$$\begin{aligned}
& \mathbb{E}\|(\mathcal{P}_1 z)(t) + (\mathcal{P}_2 z^*)(t)\|_{\mathcal{H}}^2 \\
& \leq 3\widetilde{M}_S^2 T^{2\alpha} \left[\frac{\lambda_1}{\alpha^2} + \frac{\lambda_2}{T(2\alpha-1)} \right] \\
& + 3\widetilde{M}_S^2 M_B^2 \frac{T^{2\alpha-1}}{2\alpha-1} \left[M_1 + 3 \frac{T^{2(2\alpha-1)}}{q^2} M_B^2 \widetilde{M}_S^4 \left(\frac{\lambda_1}{\alpha^2} + \frac{\lambda_2}{T(2\alpha-1)} \right) \right] \\
& = 3\widetilde{M}_S^2 M_2 T^{2\alpha} \left[\frac{\lambda_1}{\alpha^2} + \frac{\lambda_2}{T(2\alpha-1)} \right] + M_3,
\end{aligned}$$

with $M_2 = \left(1 + 3\widetilde{M}_S^4 M_B^2 \frac{T^{4\alpha-3}}{q^2(2\alpha-1)}\right)$ and $M_3 = 3\widetilde{M}_S^2 M_B^2 M_1 \frac{T^{2\alpha-1}}{2\alpha-1}$. Thus, by the condition (13), we obtain $\|\mathcal{P}_1 z + \mathcal{P}_2 z^*\|_{\mathcal{B}_b^0} \leq p$.

Similarly, for $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$, we get the estimate

$$\begin{aligned}
& \mathbb{E}\|(\mathcal{P}_1 z)(t) + (\mathcal{P}_2 z^*)(t)\|_{\mathcal{H}}^2 \\
& \leq 5\|T_\alpha(t-t_i)\|^2 \mathbb{E}\|z(t_i^-)\|_{\mathcal{H}}^2 + 5\|T_\alpha(t-t_i)\|^2 \mathbb{E}\|I_i(z(t_i^-))\|_{\mathcal{H}}^2 \\
& + 5\mathbb{E}\left\| \int_{t_i}^t S_\alpha(t-s) B u^q(s) ds \right\|_{\mathcal{H}}^2 \\
& + 5\mathbb{E}\left\| \int_{t_i}^t S_\alpha(t-s) f(s, g_s + \bar{z}_s^*, F(g(s) + \bar{z}^*(s))) ds \right\|_{\mathcal{H}}^2 \\
& + 5\mathbb{E}\left\| \int_{t_i}^t S_\alpha(t-s) \sigma(s, g_s + \bar{z}_s^*, G(g(s) + \bar{z}^*(s))) d\omega(s) \right\|_{\mathcal{H}}^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
& \mathbb{E}\|u^q(s)\|^2 \\
& \leq \frac{T^{2\alpha-2}}{q^2} M_B^2 \widetilde{M}_S^2 \left\{ 5\|\mathbb{E}\tilde{x}_T + \int_{t_i}^t \tilde{g}(s) d\omega(s)\|^2 + 5\|T_\alpha(T-t_i)\|^2 \mathbb{E}\|z(t_i^-)\|_{\mathcal{H}}^2 \right. \\
& + 5\|T_\alpha(T-t_i)\|^2 \mathbb{E}\|I_i(z(t_i^-))\|_{\mathcal{H}}^2 \\
& + 3\mathbb{E}\left\| \int_{t_i}^t S_\alpha(T-s) f(s, g_s + \bar{z}_s^*, F(g(s) + \bar{z}^*(s))) ds \right\|^2 \\
& \left. + 3\mathbb{E}\left\| \int_{t_i}^t S_\alpha(T-s) \sigma(s, g_s + \bar{z}_s^*, G(g(s) + \bar{z}^*(s))) d\omega(s) \right\|^2 \right\}
\end{aligned}$$

$$\begin{aligned} &\leq 5 \frac{T^{2\alpha-2}}{q^2} M_B^2 \widetilde{M}_S^2 \widetilde{M}_T^2 \left[\|z\|_{\mathcal{B}_b^0}^2 + \mathbf{E} \|I_i(z(t_i^-))\|_{\mathcal{H}}^2 \right] \\ &\quad + M_4 + 5 \frac{T^{2(2\alpha-1)}}{q^2} M_B^2 \widetilde{M}_S^4 \left(\frac{\lambda_1}{\alpha^2} + \frac{\lambda_2}{T(2\alpha-1)} \right), \end{aligned}$$

$$M_4 = 5 \frac{T^{2\alpha-2}}{q^2} M_B^2 \widetilde{M}_S^2 \left[2 \|\mathbf{E} \dot{x}_T\|^2 + 2TM_{\bar{g}} \right].$$

Now, we have

$$\begin{aligned} &\mathbf{E} \|(\mathcal{P}_1 z)(t) + (\mathcal{P}_2 z^*)(t)\|_{\mathcal{H}}^2 \\ &\leq 5M_B^2 \widetilde{M}_S^2 \frac{T^{2\alpha-1}}{2\alpha-1} \left[M_4 + 5 \frac{T^{2\alpha-2}}{q^2} M_B^2 \widetilde{M}_S^2 \widetilde{M}_T^2 (p + \Lambda) \right. \\ &\quad \left. + 5 \frac{T^{2(2\alpha-1)}}{q^2} M_B^2 \widetilde{M}_S^4 \left(\frac{\lambda_1}{\alpha^2} + \frac{\lambda_2}{T(2\alpha-1)} \right) \right] \\ &\quad + 5\widetilde{M}_T^2 (p + \Lambda) + 5\widetilde{M}_S^2 T^{2\alpha} \left(\frac{\lambda_1}{\alpha^2} + \frac{\lambda_2}{T(2\alpha-1)} \right) \\ &= 5\widetilde{M}_T^2 (p + \Lambda) M_5 + 5\widetilde{M}_S^2 T^{2\alpha} \left(\frac{\lambda_1}{\alpha^2} + \frac{\lambda_2}{T(2\alpha-1)} \right) M_6 + M_7 \leq p, \end{aligned}$$

where $M_5 = (1 + 5M_B^4 \widetilde{M}_S^4 \frac{T^{4\alpha-3}}{q^2(2\alpha-1)})$, $M_6 = (1 + 5M_B^4 \widetilde{M}_S^4 \frac{T^{2(\alpha-2)}}{q^2(2\alpha-1)})$ and $M_7 = 5M_B^2 \widetilde{M}_S^2 \frac{T^{2\alpha-3}}{2\alpha-1} M_4$.

This implies that $\|\mathcal{P}_1 z + \mathcal{P}_2 z^*\|_{\mathcal{B}_b^0} \leq p$ with $\lambda_1 = 4\mu_1^*(\|\phi\|_{\mathcal{B}_h}^2 + l^2 p) + \mu_2^* F^* p$ and $\lambda_2 = 4\nu_1^*(\|\phi\|_{\mathcal{B}_h}^2 + l^2 p) + \nu_2^* G^* p$. Hence, we get $\mathcal{P}_1 z + \mathcal{P}_2 z^* \in B_p$.

Step 2. The map \mathcal{P}_1 is continuous on B_p .

Let $\{z^n\}_{n=1}^\infty$ be a sequence in B_p with $\lim z^n \rightarrow z \in B_p$. Then for $t \in (t_i, t_{i+1}]$, $i = 0, 1, \dots, m$, we have

$$\begin{aligned} &\mathbf{E} \|(\mathcal{P}_1 z^n)(t) - (\mathcal{P}_1 z)(t)\|_{\mathcal{H}}^2 \\ &\leq 2 \|T_\alpha(t - t_i)\|^2 \left[\mathbf{E} \|z^n(t_i^-) - z(t_i^-)\|_{\mathcal{H}}^2 + \mathbf{E} \|I_i(z^n(t_i^-)) - I_i(z(t_i^-))\|_{\mathcal{H}}^2 \right]. \end{aligned}$$

Since the functions I_i , $i = 0, 1, \dots, m$ are continuous, then

$\lim_{n \rightarrow \infty} \mathbf{E} \|\mathcal{P}_1 z^n - \mathcal{P}_1 z\|^2 = 0$ which implies that the mapping \mathcal{P}_1 is continuous on B_p .

Step 3. \mathcal{P}_1 maps bounded sets into bounded sets in B_p .

Let us prove that for $p > 0$ there exists a $\vartheta > 0$ such that for each $z \in B_p$, we have $\mathbf{E} \|(\mathcal{P}_1 z)(t)\|_{\mathcal{H}}^2 \leq \vartheta$ for $t \in (t_i, t_{i+1}]$, $i = 0, 1, \dots, m$. We obtain

$$\begin{aligned}\mathbf{E}\|(\mathcal{P}_1 z)(t)\|_{\mathcal{H}}^2 &\leq 2\|T_\alpha(t-t_i)\|^2 \left[\mathbf{E}\|z(t_i^-)\|_{\mathcal{H}}^2 + \mathbf{E}\|I_i(z(t_i^-))\|_{\mathcal{H}}^2 \right] \\ &\leq 2\widetilde{M}_T^2(p+\Lambda) = \vartheta,\end{aligned}$$

which proves the result.

Step 4. The map \mathcal{P}_1 is equicontinuous.

Let $\tau_1, \tau_2 \in (t_i, t_{i+1}]$, $t_i \leq \tau_1 < \tau_2 \leq t_{i+1}$, $i = 0, 1, \dots, m$, $z \in B_p$. We have

$$\begin{aligned}\mathbf{E}\|(\mathcal{P}_1 z)(\tau_2) - (\mathcal{P}_1 z)(\tau_1)\|_{\mathcal{H}}^2 &\leq 2\|T_\alpha(\tau_2-t_i) - T_\alpha(\tau_1-t_i)\|^2 \left[\mathbf{E}\|z(t_i^-)\|_{\mathcal{H}}^2 + \mathbf{E}\|I_i(z(t_i^-))\|_{\mathcal{H}}^2 \right] \\ &\leq 2(p+\Lambda)\|T_\alpha(\tau_2-t_i) - T_\alpha(\tau_1-t_i)\|^2.\end{aligned}$$

Since T_α is strongly continuous it allows us to conclude that $\lim_{\tau_2 \rightarrow \tau_1} \|T_\alpha(\tau_2-t_i) - T_\alpha(\tau_1-t_i)\|^2 = 0$, which implies that $\mathcal{P}_1(B_p)$ is equicontinuous. Finally, combining Step 1 to Step 4 together with Ascoli's theorem, we conclude that the operator \mathcal{P}_1 is compact.

Next, we show that the map \mathcal{P}_2 is a contraction mapping. Let $z, z^* \in B_p$ and $t \in (t_i, t_{i+1}]$, $i = 0, 1, \dots, m$. We have

$$\begin{aligned}\mathbf{E}\|(\mathcal{P}_2 z)(t) - (\mathcal{P}_2 z^*)(t)\|_{\mathcal{H}}^2 &\leq 3\mathbf{E}\left\| \int_{t_i}^t S_\alpha(t-s) \left[f(s, g_s + \bar{z}_s, F(g(s) + \bar{z}(s))) \right. \right. \\ &\quad \left. \left. - f(s, g_s + \bar{z}_s^*, F(g(s) + \bar{z}^*(s))) \right] ds \right\|_{\mathcal{H}}^2 \\ &+ 3\mathbf{E}\left\| \int_{t_i}^t S_\alpha(t-s) \left[\sigma(s, g_s + \bar{z}_s, G(g(s) + \bar{z}(s))) \right. \right. \\ &\quad \left. \left. - \sigma(s, g_s + \bar{z}_s^*, G(g(s) + \bar{z}^*(s))) \right] d\omega(s) \right\|_{\mathcal{H}}^2 \\ &+ 3\mathbf{E}\left\| \int_{t_i}^t S_\alpha(t-s) Bu^q(s) ds \right\|_{\mathcal{H}}^2 \\ &\leq 3 \int_{t_i}^t \|S_\alpha(t-s)\| ds \int_{t_i}^t \|S_\alpha(t-s)\| \\ &\quad \times \mathbf{E}\|f(s, g_s + \bar{z}_s, F(g(s) + \bar{z}(s))) - f(s, g_s + \bar{z}_s^*, F(g(s) + \bar{z}^*(s)))\|_{\mathcal{H}}^2 ds\end{aligned}$$

$$\begin{aligned}
& + 3 \int_{t_i}^t \|S_\alpha(t-s)\|^2 \mathbf{E} \|\sigma(s, g_s + \bar{z}_s, G(g(s) + \bar{z}(s))) \\
& \quad - \sigma(s, g_s + \bar{z}_s^*, G(g(s) + \bar{z}^*(s)))\|_{\mathcal{L}_2^0}^2 ds \\
& + 3 \int_{t_i}^t \|S_\alpha(t-s)\|^2 \mathbf{E} \|Bu^q(s)\|^2 ds.
\end{aligned}$$

Moreover,

$$\begin{aligned}
& \mathbf{E} \|u^q(s)\|^2 \\
& \leq \frac{T^{2\alpha-2}}{q^2} M_B^2 \widetilde{M}_S^2 \left\{ 5 \|\mathbf{E} \tilde{x}_T + \int_{t_i}^t \tilde{g}(s) d\omega(s)\|^2 + 5 \|T_\alpha(T-t_i)\|^2 \mathbf{E} \|z(t_i^-) - z^*(t_i^-)\|_{\mathcal{H}}^2 \right. \\
& + 5 \|T_\alpha(T-t_i)\|^2 \mathbf{E} \|I_i(z(t_i^-)) - I_i(z^*(t_i^-))\|_{\mathcal{H}}^2 \\
& + 5 \mathbf{E} \left\| \int_{t_i}^t S_\alpha(T-s) \left[f(s, g_s + \bar{z}_s, F(g(s) + \bar{z}(s))) \right. \right. \\
& \quad \left. \left. - f(s, g_s + \bar{z}_s^*, F(g(s) + \bar{z}^*(s))) \right] ds \right\|_{\mathcal{H}}^2 \\
& + 5 \mathbf{E} \left\| \int_{t_i}^t S_\alpha(T-s) \left[\sigma(s, g_s + \bar{z}_s, G(g(s) + \bar{z}(s))) \right. \right. \\
& \quad \left. \left. - \sigma(s, g_s + \bar{z}_s^*, G(g(s) + \bar{z}^*(s))) \right] d\omega(s) \right\|_{\mathcal{H}}^2 \left. \right\} \\
& \leq M_4 + 5 \frac{T^{2\alpha-2}}{q^2} M_B^2 \widetilde{M}_S^2 \widetilde{M}_T^2 \left[\|z - z^*\|_{\mathcal{B}_b^0}^2 + \hat{\Lambda} \right] + 5 \frac{T^{2\alpha-2}}{q^2} M_B^2 \widetilde{M}_S^4 \int_{t_i}^t (T-s)^{\alpha-1} ds \\
& \quad \times \int_{t_i}^t (T-s)^{\alpha-1} \left[\mu_1 \|\bar{z}_s - \bar{z}_s^*\|_{\mathcal{B}_h}^2 + \mu_2 \mathbf{E} \|F(g(s) + \bar{z}(s)) - F(g(s) + \bar{z}^*(s))\|_{\mathcal{H}}^2 \right] ds \\
& + 5 \frac{T^{2\alpha-2}}{q^2} M_B^2 \widetilde{M}_S^4 \int_{t_i}^t (T-s)^{2(\alpha-1)} \left[\nu_1 \|\bar{z}_s - \bar{z}_s^*\|_{\mathcal{B}_h}^2 \right. \\
& \quad \left. + \nu_2 \mathbf{E} \|G(g(s) + \bar{z}(s)) - G(g(s) + \bar{z}^*(s))\|_{\mathcal{H}}^2 \right] ds \\
& \leq M_4 + 5 \frac{T^{2\alpha-2}}{q^2} M_B^2 \widetilde{M}_S^2 \widetilde{M}_T^2 \left[\|z - z^*\|_{\mathcal{B}_b^0}^2 + \hat{\Lambda} \right] + 5 \frac{T^{2\alpha-2}}{q^2} M_B^2 \widetilde{M}_S^4 \frac{T^\alpha}{\alpha} \int_{t_i}^t (T-s)^{\alpha-1} \\
& \quad \times \left[\mu_1 l \sup \mathbf{E} \|z(s) - z^*(s)\|_{\mathcal{H}}^2 + \mu_2 F^* \sup \mathbf{E} \|z(s) - z^*(s)\|_{\mathcal{H}}^2 \right] ds
\end{aligned}$$

$$\begin{aligned}
& + 5 \frac{T^{2\alpha-2}}{q^2} M_B^2 \widetilde{M}_S^4 \frac{T^\alpha}{\alpha} \int_{t_i}^t (T-s)^{2(\alpha-1)} \\
& \quad \times \left[\nu_1 l \sup \mathbf{E} \|z(s) - z^*(s)\|_{\mathcal{H}}^2 + \nu_2 G^* \sup \mathbf{E} \|z(s) - z^*(s)\|_{\mathcal{H}}^2 \right] ds \\
& \leq M_4 + 5 \frac{T^{2\alpha-2}}{q^2} M_B^2 \widetilde{M}_S^2 \widetilde{M}_T^2 \left[\|z - z^*\|_{\mathcal{B}_b^0}^2 + \hat{\Lambda} \right] \\
& + 5 \frac{T^{2\alpha-2}}{q^2} M_B^2 \widetilde{M}_S^4 T^{2\alpha} \left[\frac{1}{\alpha^2} (\mu_1 l + \mu_2 F^*) + \frac{1}{T(2\alpha-1)} (\nu_1 l + \nu_2 G^*) \right] \|z - z^*\|_{\mathcal{B}_b^0}^2.
\end{aligned}$$

Thus

$$\begin{aligned}
& \mathbf{E} \|(\mathcal{P}_2 z)(t) - (\mathcal{P}_2 z^*)(t)\|_{\mathcal{H}}^2 \\
& \leq 3M_B^2 \widetilde{M}_S^2 \frac{T^{2\alpha-1}}{2\alpha-1} M_4 + 15M_B^4 \widetilde{M}_S^4 \frac{T^{4\alpha-3}}{(2\alpha-1)q^2} \left(\|z - z^*\|_{\mathcal{B}_b^0}^2 + \hat{\Lambda} \right) \\
& + 3\widetilde{M}_S^2 T^{2\alpha} \left(1 + 5M_B^4 \widetilde{M}_S^2 \frac{T^{4\alpha-3}}{(2\alpha-1)q^2} \right) \left(\frac{1}{\alpha^2} (\mu_1 l + \mu_2 F^*) \right. \\
& \quad \left. + \frac{1}{T(2\alpha-1)} (\nu_1 l + \nu_2 G^*) \right) \|z - z^*\|_{\mathcal{B}_b^0}^2 \\
& = M_8 + M_9 \left(\|z - z^*\|_{\mathcal{B}_b^0}^2 + \hat{\Lambda} \right) \\
& + 3M_{10} \widetilde{M}_S^2 T^{2\alpha} \left(\frac{1}{\alpha^2} (\mu_1 l + \mu_2 F^*) + \frac{1}{T(2\alpha-1)} (\nu_1 l + \nu_2 G^*) \right) \|z - z^*\|_{\mathcal{B}_b^0}^2 \\
& \leq \left[M_{11} + 3M_{10} \widetilde{M}_S^2 T^{2\alpha} \left(\frac{1}{\alpha^2} (\mu_1 l + \mu_2 F^*) + \frac{1}{T(2\alpha-1)} (\nu_1 l + \nu_2 G^*) \right) \right] \|z - z^*\|_{\mathcal{B}_b^0}^2,
\end{aligned}$$

$$M_{11} = M_8 + (1 + \hat{\Lambda})M_9.$$

By the condition (14), we deduce that \mathcal{P}_2 is a contraction mapping. Hence, by Krasnoselskii's fixed point theorem we conclude that problem (1) has at least one solution on $(-\infty, T]$. This completes the proof of the theorem. \blacksquare

Theorem 3.5. *Assume that the assumptions of Theorem 3.4, hold and, in addition, the functions f and σ are uniformly bounded on their respective domains. Further, if $S_\alpha(t)$ is compact, then the fractional stochastic impulsive control system (1) is approximately controllable on $(-\infty, T]$.*

Proof. Let $x^q \in \hat{B}_p$ be a fixed point of the operator $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$. By the stochastic Fubini theorem, it is easy to see that for all $i = 1, 2, \dots, m$

$$\begin{aligned}
& x^q(T) \\
& = \tilde{x}_T - q(qI + \Gamma_{t_i}^T)^{-1} \left[\mathbf{E} \hat{x}_T + \int_{t_i}^T \tilde{g}(s) d\omega(s) - T_\alpha(T - t_i) [x^q(t_i^-) + I_i(x^q(t_i^-))] \right]
\end{aligned}$$

$$\begin{aligned}
& + q \int_{t_i}^T (qI + \Gamma_s^T)^{-1} S_\alpha(T-s) f(s, x_s^q, Fx^q(s)) ds \\
& + q \int_{t_i}^T (qI + \Gamma_s^T)^{-1} S_\alpha(T-s) \sigma(s, x_s^q, Gx^q(s)) d\omega(s).
\end{aligned}$$

It follows from the assumption on f and σ that there exists N such that

$$\|f(s, x_s^q, Fx^q(s))\|^2 + \|\sigma(s, x_s^q, Gx^q(s))\|^2 \leq N.$$

Then there is a subsequence denoted by $\{f(s, x_s^q, Fx^q(s)), \sigma(s, x_s^q, Gx^q(s))\}$ weakly converging to some $\{f(s), \sigma(s)\}$ in $\mathcal{H} \times \mathcal{L}_2^0$. Thus from the above equation, we have

$$\begin{aligned}
& \mathbb{E}\|x^q(T) - \tilde{x}_T\|^2 \\
& \leq 6\|q(qI + \Gamma_{t_i}^T)^{-1}[\mathbb{E}\tilde{x}_T - T_\alpha(T-t_i)(x^q(t_i^-) + I_i(x^q(t_i^-)))]\|^2 \\
& + 6\mathbb{E}\left(\int_{t_i}^T \|q(qI + \Gamma_s^T)^{-1}\tilde{g}(s)\|_{\mathcal{L}_2^0}^2 ds\right) \\
& + 6\mathbb{E}\left(\int_{t_i}^T \|q(qI + \Gamma_s^T)^{-1}S_\alpha(T-s)[f(s, x_s^q, Fx^q(s)) - f(s)]\| ds\right)^2 \\
& + 6\mathbb{E}\left(\int_{t_i}^T \|q(qI + \Gamma_s^T)^{-1}S_\alpha(T-s)f(s)\| ds\right)^2 \\
& + 6\mathbb{E}\left(\int_{t_i}^T \|q(qI + \Gamma_s^T)^{-1}S_\alpha(T-s)\sigma(s)\|_{\mathcal{L}_2^0}^2 ds\right) \\
& + 6\mathbb{E}\left(\int_{t_i}^T \|q(qI + \Gamma_s^T)^{-1}S_\alpha(T-s)[\sigma(s, x_s^q, Gx^q(s)) - \sigma(s)]\|_{\mathcal{L}_2^0}^2 ds\right).
\end{aligned}$$

On the other hand, by assumption (H_0) for all $t_i \leq s \leq T$, $i = 1, \dots, m$, the operator $q(qI + \Gamma_s^T)^{-1} \rightarrow 0$ strongly as $\alpha \rightarrow 0^+$, and moreover $\|q(qI + \Gamma_s^T)^{-1}\| \leq 1$. Thus, by the Lebesgue dominated convergence theorem and the compactness of S_α , we obtain $\mathbb{E}\|x^q(T) - \tilde{x}_T\|^2 \rightarrow 0$ as $\alpha \rightarrow 0^+$. This gives the approximate controllability of (1). \blacksquare

4. EXAMPLE

In this section, we consider an example to illustrate our main theorem. We first examine the existence of solutions for the fractional stochastic partial differential equation of the form

(17)

$$\begin{aligned}
{}^c D_t^\alpha u(t, x) &= \frac{\partial^2}{\partial x^2} u(x, t) + \mu(t, x) + \int_{-\infty}^t H(t, x, s-t) Q(u(s, x)) ds \\
&\quad + \int_0^t k(s, t) e^{-u(s, x)} ds \\
&\quad + \left[\int_{-\infty}^t V(t, x, s-t) U(u(s, x)) ds + \int_0^t p(s, t) e^{-u(s, x)} ds \right] \frac{d\beta(t)}{dt}, \\
&\quad x \in [0, \pi], \quad t \in [0, T], t \neq t_k
\end{aligned}$$

$$u(t, 0) = 0 = u(t, \pi), \quad t \geq 0,$$

$$u(t, x) = \phi(t, x), \quad t \in (-\infty, 0], \quad x \in [0, \pi]$$

$$\Delta u(t_k)(x) = \int_{-\infty}^t q_k(t_k - s) u(s, x) ds, \quad x \in [0, \pi],$$

where $\beta(t)$ is a standard cylindrical Wiener process in \mathcal{H} defined on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$; ${}^c D_t^\alpha$ is the Caputo fractional derivative of order $0 < \alpha < 1$; $0 < t_1 < t_2 < \dots < t_n < T$ are prefixed numbers; H, Q, V and U are continuous; $\phi \in \mathcal{B}_h$.

Let $\mathcal{H} = L^2([0, \pi])$ with the norm $\|\cdot\|$. Define $A : \mathcal{H} \rightarrow \mathcal{H}$ by $Az = z''$ with the domain $D(A) = \{z \in \mathcal{H}; z, z'' \text{ are absolutely continuous, } z'' \in \mathcal{H}, \text{ and } z(0) = z(\pi) = 0\}$. Then

$$Az = \sum_{n=1}^{\infty} n^2 (z, z_n) z_n, \quad z \in D(A),$$

where $z_n(x) = \sqrt{\frac{2}{n}} \sin(nx)$, $n \in \mathbb{N}$ is the orthogonal set of eigenvectors of A . It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t \geq 0}$ in \mathcal{H} and is given by

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2 t} (z, z_n) z_n, \quad \text{for all } z \in \mathcal{H}, t > 0.$$

It follows from the above expressions that $(T(t))_{t \geq 0}$ is a uniformly bounded compact semigroup, so that, $R(\lambda, A) = (\lambda - A)^{-1}$ is a compact operator for all $\lambda \in \rho(A)$ i.e. $A \in \mathcal{A}^\alpha(\theta_0, \omega_0)$. Let $h(s) = e^{2s}$, $s < 0$, then $l = \int_{-\infty}^0 h(s) ds = \frac{1}{2}$ and define

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^0 h(s) \sup_{s \leq \theta \leq 0} (\mathbb{E}|\phi(\theta)|^2)^{1/2} ds.$$

Hence for $(t, \phi) \in [0, T] \times \mathcal{B}_h$, where $\phi(\theta)(z) = \phi(\theta, z)$, $(\theta, z) \in (-\infty, 0] \times [0, \pi]$. Put $u(t) = u(t, \cdot)$, that is $u(t)(x) = u(t, x)$. Define the bounded linear operator $B : \mathcal{U} \rightarrow \mathcal{H}$ by $Bv(t)(x) = \mu(t, x)$, $0 \leq x \leq \pi$, $v \in \mathcal{U}$; $f : J \times \mathcal{B}_h \times L^2([0, \pi]) \rightarrow L^2([0, \pi])$ and $\sigma : J \times \mathcal{B}_h \times \mathcal{L}_2^0 \rightarrow L^2([0, \pi])$ as follows:

$$\begin{aligned} f(t, \phi, Fu(t))(x) &= \int_{-\infty}^0 H(t, x, \theta)Q(\phi(\theta))(x)d\theta + Fu(t)(x), \\ \sigma(t, \phi, Gu(t))(x) &= \int_{-\infty}^0 V(t, x, \theta)U(\phi(\theta))(x)d\theta + Fu(t)(x), \end{aligned}$$

where $Fu(t)(x) = \int_0^t k(s, t)e^{-u(s, x)}ds$ and $Gu(t)(x) = \int_0^t p(s, t)e^{-u(s, x)}ds$. Then, with the above settings the considered equation (17) can be written in the abstract form of equation (1). All conditions of Theorem 3.4 are now fulfilled, so we deduce that the system (17) has a mild solution on $(-\infty, T]$. On the other hand, the linear system corresponding to (17) is approximately controllable on $[0, T]$. Hence all the conditions of Theorem 3.5 are satisfied. Thus by Theorem 3.5, the fractional stochastic impulsive control system (17) is approximately controllable on $[0, T]$.

5. CONCLUSION

This paper has investigated the existence and approximate controllability for impulsive fractional-order stochastic infinite delay integro-differential equations in Hilbert space. New sufficient conditions for the approximate controllability of the considered system have been formulated. As the differential inclusion system can be considered as a generalization of the system described by differential equations, it should be pointed out that under some suitable conditions on f and σ , one can establish the approximate controllability with constrained controls for fractional stochastic differential inclusions with nonlocal conditions. It can be done by adapting the techniques and ideas established in this paper and the papers [6, 7, 9], and suitably introducing the technique of single valued maps defined in [2]. This is one of our future goals.

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