Discussiones Mathematicae
Differential Inclusions, Control and Optimization 33 (2013) 41–45
doi:10.7151/dmdico.1145

A NOTE ON VARIATIONAL-TYPE INEQUALITIES FOR (η, θ, δ)- PSEUDOMONOTONE-TYPE SET-VALUED MAPPINGS IN NONREFLEXIVE BANACH SPACES

Magdalena Nockowska-Rosiak

Institute of Mathematics
Technical University of Łódź
Wólczańska 215, 90–924 Łódź, Poland

e-mail: magdalena.nockowska@p.lodz.pl

Abstract

In this paper the existence of solutions to variational-type inequalities problems for (η, θ, δ)- pseudomonotone-type set-valued mappings in nonreflexive Banach spaces introduced in [4] is considered. Presented theorem does not require a compact set-valued mapping, but requires a weaker condition 'locally bounded' for the mapping.

Keywords: variational-type inequalities, (η, θ, δ)- pseudomonotone-type, nonreflexive Banach spaces.

2010 Mathematics Subject Classification: 49J40, 65K10.

1. Introduction

Variational inequalities for monotone single-valued mappings in nonreflexive Banach spaces was firstly considered by Chang, Lee and Chen in [1]. After that Watson in [7] and Verma [6] studied variational inequalities for pseudomonotonicity and strong pseudomonotonicity for single-valued mappings in nonreflexive Banach spaces, respectively. Lee and Lee in [3] introduced a notion of (η, θ)- pseudomonotonicity for single valued mappings in nonreflexive Banach spaces which extends previously mentioned notions. (θ, η)- pseudomonotonicity for set-valued mappings in nonreflexive Banach spaces was considered by Lee and Noh in [5]. Lee, Lee and Lee in [4] introduced (η, θ, δ)- pseudomonotone-type which generalizes (η, θ)- pseudomonotonicity to set-valued case in nonreflexive Banach spaces. In this paper we show the existence of solutions to variational-type inequality problems for (η, θ, δ)- pseudomonotone-type set-valued mapping in nonreflexive

Banach spaces without the assumption of a compact set-valued mapping. We change this assumption into locally bounded mapping and the assumption which is quite similar to one of conditions in the definition of pseudomonotonicity in reflexive Banach spaces.

Definition [4]**.** Let X be a real nonreflexive Banach space with the dual $X^\star$ and $X^{\star\star}$ the dual of $X^\star$. Let $T : K \to 2^{X^\star} \setminus \{\emptyset\}$ be a set-valued mapping, $\eta, \theta : K \times K \to X^{\star\star}$ be operators and $\delta : K \times K \to \mathbb{R}$ a function, where $K \subset X^{\star\star}$. T is said to be (η, θ, δ)- pseudomonotone-type, if there exists a constant r (called (η, θ, δ)- pseudomonotone-type constant of T) such that for all $x, y \in K$ and $v \in T(y)$ there exists $u \in T(x)$ such that

$$\big\langle v, \eta(x,y)\big\rangle + \delta(x,y) \geq 0 \text{ implies } \big\langle u, \eta(x,y)\big\rangle + \delta(x,y) \geq r\|\theta(x,y)\|^2.$$

Definition [4]**.** A set-valued mapping $T : X^{\star\star} \supset K \to 2^{X^\star}$ is said to be finite-dimensional u.s.c. if for any finite-dimensional subspace F of $X^{\star\star}$ with $K_F = K \cap F \neq \emptyset$, $T : K_F \to 2^{X^\star}$ is u.s.c. in the norm topology.

Definition. A set-valued mapping $T : X^{\star\star} \supset K \to 2^{X^\star}$ is said to be locally bounded set-valued mapping if for any $x \in K$ there exist $M > 0$, $r > 0$ that for any $y \in B(x,r)$ and $v \in T(y)$ we have $\|v\| \leq M$.

Definition [2]**.** Let $K \subset X$, where X is a topological vector space. A set-valued mapping $T : K \to 2^X$ is called a Knaster-Kuratowski-Mazurkiewicz mapping (in short KKM mapping) if for each nonempty finite subset N of K, $\operatorname{conv}(N) \subset T(N)$, where conv denotes convex hull and $T(N) = \bigcup\{T(x) : x \in N\}$.

Theorem 1 (KKM Theorem, [2])**.** *Let K be an arbitrary nonempty subset of a Hausdorff topological vector space X. Let a set-valued mapping $T : K \to 2^X$ be KKM mapping such that $T(x)$ is closed for all $x \in K$ and compact for a least one $x \in K$. Then*

$$\bigcap_{x \in K} T(x) \neq \emptyset.$$

2. Main results

Let us remind the main result from [4].

Theorem 2 (Theorem 2.3, [4])**.** *Let X be a real nonreflexive Banach space and K a nonempty bounded closed convex subset of $X^{\star\star}$. Let $T : K \to 2^{X^\star}$ be an (η, θ, δ)-pseudomonotone-type, finite dimensional u.s.c., compact set-valued mapping, and $\eta, \theta : K \times K \to X^{\star\star}$ be operators and $\delta : K \times K \to \mathbb{R}$ be a function such that*

(i) *for all* $x \in K$, $\eta(x,x) = \overline{0}$, $\delta(x,x) = 0$;

(ii) *for all* $y \in K$ $x \to \eta(x,y)$, $x \to \theta(x,y)$ *are affine and* $x \to \delta(x,y)$ *is convex;*

(iii) *for all* $x \in K$ $y \to \eta(x,y)$, $y \to \theta(x,y)$ *and* $y \to \delta(x,y)$ *are continuous.*

Then there exists $x_0 \in K$ *such that for all* $x \in K$ *there exists* $v_0 \in T(x_0)$ *such that*

$$\big\langle v_0, \eta(x,x_0)\big\rangle + \delta(x,x_0) \geq 0.$$

The assumption of a compact set-valued mapping T is quite restrictive. In the following theorem it will be proved that this assumption can be weaken.

Theorem 3. *Let* X *be a real nonreflexive Banach space and* K *a nonempty bounded closed, convex subset of* $X^{\star\star}$. *Let* $T : K \to 2^{X^\star}$ *be an* (η,θ,δ)*- pseudo-monotone-type, finite dimensional u.s.c., locally bounded and such that the following condition is fulfilled*

(a) *for any* $\{x_n\} \subset K$, *the conditions* $x_n \rightharpoonup x$ *in* $X^{\star\star}$ *and*

$$\liminf_{n\to\infty}(\big\langle v_n, \eta(x,x_n)\big\rangle + \delta(x,x_n)) \geq 0,$$

with $v_n \in T(x_n)$ *imply that for any* $y \in K$ *there exists* $v(y) \in T(x)$ *such that*

$$\limsup_{n\to\infty}(\big\langle v_n, \eta(y,x_n)\big\rangle + \delta(x_n,y)) \leq \big\langle v(y), \eta(y,x)\big\rangle + \delta(y,x).$$

Moreover, let $\eta, \theta : K \times K \to X^{\star\star}$ *be operators and* $\delta : K \times K \to \mathbb{R}$ *be a functional such that*

(i) *for all* $x \in K$, $\eta(x,x) = 0$, $\delta(x,x) = 0$;

(ii) *for all* $y \in K$ $x \to \eta(x,y)$, $x \to \theta(x,y)$ *are affine and* $x \to \delta(x,y)$ *is convex;*

(iii) *for all* $x \in K$ $y \to \eta(x,y)$, $y \to \theta(x,y)$ *and* $y \to \delta(x,y)$ *are continuous.*

Then there exists $x_0 \in K$ *such that for all* $x \in K$ *there exists* $v_0 \in T(x_0)$ *such that*

$$\big\langle v_0, \eta(x,x_0)\big\rangle + \delta(x,x_0) \geq 0.$$

Proof. The proof is very similar to the proof of the Theorem 2.3 in [4]. We prove that for each finite dimensional subspace F of $X^{\star\star}$ with $K_F = F \cap K \neq \emptyset$ there exists $x_0 \in K_F$ such that for all $x \in K_F$ there exists $v_0 \in T(x_0)$ such that

$$\big\langle v_0, \eta(x,x_0)\big\rangle + \delta(x,x_0) \geq 0.$$

The rest of the proof is the same as in [4].

Let F be a subspace of $X^{\star\star}$ with $K_F = F \cap K \neq \emptyset$. Define a set valued mapping $G : K_F \to 2^F$ for any $y \in K_F$ by

$$G(y) := \{x \in K_F : \text{there exists } v \in T(x) \text{ such that } \langle v, \eta(y,x)\rangle + \delta(y,x) \geq 0\}.$$

In [4] it was proved that the mapping G is a KKM mapping.

Now, we show that G is a closed valued mapping. Let $y \in K_F$ and $\{x_n\} \in G(y)$ be such that $\lim_{n\to\infty} x_n = x_0$ in F. Then for any $n \in \mathbb{N}$ there exists $v_n \in T(x_n)$ such that

$$\langle v_n, \eta(y,x_n)\rangle + \delta(y,x_n) \geq 0. \tag{1}$$

From the fact that T is a locally bounded mapping, the continuity of η, δ and the assumption (i) we get that

$$\liminf_{n\to\infty}(\langle v_n, \eta(x_0,x_n)\rangle + \delta(x_0,x_n)) = 0.$$

Then from the assumption (a) we get there exists $v(y) \in T(x_0)$ such that

$$\limsup_{n\to\infty}(\langle v_n, \eta(y,x_n)\rangle + \delta(x_n,y)) \leq \langle v(y), \eta(y,x_0)\rangle + \delta(y,x_0).$$

From (1) we obtain

$$0 \leq \langle v(y), \eta(y,x_0)\rangle + \delta(y,x_0),$$

which means that $x_0 \in G(y)$. Hence $G(y)$ is closed in F.

G is compact valued mapping, because of the compactness of K_F. Consequently, from the KKM theorem $\bigcap_{x\in K_F} G(x) \neq \emptyset$. ∎

Remark 4. It is quite obvious that if $T : K \to X^\star$, $K \subset X^{\star\star}$ is a compact set-valued mapping then T is locally bounded and fulfilled the condition

(a) for any $\{x_n\} \subset K$, the conditions $x_n \rightharpoonup x$ in $X^{\star\star}$ and

$$\liminf_{n\to\infty}(\langle v_n, \eta(x,x_n)\rangle + \delta(x,x_n)) \geq 0,$$

with $v_n \in T(x_n)$ imply that for any $y \in K$ there exists $v(y) \in T(x)$ such that

$$\limsup_{n\to\infty}(\langle v_n, \eta(y,x_n)\rangle + \delta(x_n,y)) \leq \langle v(y), \eta(y,x)\rangle + \delta(y,x).$$

These conditions seem to be easier to check than the compactness of graph of mapping T.

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Received 16 July 2012