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ON DERIVO-PERIODIC MULTIFUNCTIONS

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Abstract

The problem of linearity of a multivalued derivative and consequently the problem of necessary and sufficient conditions for derivo-periodic multifunctions are investigated. The notion of a derivative of multivalued functions is understood in various ways. Advantages and disadvantages of these approaches are discussed.

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1. Introduction

More than one hundred years ago, the problem of finding a solution which need not be periodic itself, but which has a periodic derivative was mentioned by H. Poincaré. Physically, such solutions can correspond, for example, to a periodic velocity, a subsynchronous level of performance of the motor, a "slalom orbit" of an electron beam [8] or a motion of particles in a sinusoidal potential related to a free-electron laser [12]. So, one can find many applications in astronomy, engineering, laser physics, quantum physics, etc. For an extensive list of references and contributions concerning this problem, see e.g. [1, 11].

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The multivalued approach can make further progress, especially in such a case, when the explicitness of a process is lost. For example, in quantum physics, according to the Heisenberg principle of uncertainty, the position of particles can be detected only in a certain domain, but with the same probability. Hence, the multivalued approach seems to be much more appropriate than the single-valued approximations.

On the other hand, several concepts of differentiability for multifunctions have been considered by many authors from different points of view (see e.g. [3, 5, 6, 9, 10]). This work investigates a suitable definition of derivo-periodic multifunctions and leads to an analogous version of the following well-known theorem for related derivo-periodic single-valued functions (see [7], p. 235).

Theorem 1. *The function $\varphi \in C^1(\mathbb{R}, \mathbb{R}^n)$ is derivo-periodic with period $T > 0$ if and only if there exist a constant vector $a \in \mathbb{R}^n$ and a periodic function $\psi \in C^1(\mathbb{R}, \mathbb{R}^n)$ with period T such that $\varphi(t) \equiv at + \psi(t)$.*

2. Differentiability of multifunctions

Let us recall the following considerations in [10]. Let F be a multivalued map from a linear space X into a linear space Y. Assume that the following linearity condition holds $F(\alpha x + \beta y) = \alpha F(x) + \beta F(y)$, for any $x, y \in X$, and $\alpha, \beta \in \mathbb{R}$. Then F is single-valued. In fact, for $\alpha = -\beta = 1$ and $x = y$, we have $F(x) - F(x) = F(0)$. Since $F(0) = 0$, it follows that $F(x)$ is a singleton, for any $x \in X$.

Therefore, the definition of differentiability for single-valued maps cannot be generalized to the multivalued case in a word by a word fashion.

The majority of approaches to the problem of differentiability of multifunctions has common features. The mostly frequent ones are

(i) applying the usual differentiability in special spaces, especially in a real normed linear space, whose elements are subsets of Y,

or

(ii) reducing the requirements on the differential, namely the multivalued differential is required to be positively homogeneous, or using the tangency concept, following the fact that the tangent space to the graph of a function $f : \mathbb{R} \to \mathbb{R}$, at any point (x, y) of its graph, is the line of the slope $f'(x)$.

Let B denote the unit ball $B(0,1)$ in a Banach space X and $\mathcal{B}(X)(\mathcal{B}_c(X))$ denote the family of all non-empty bounded, closed (bounded, closed, convex) subsets of X. Let $\overline{A}$ denote the closure of $A \subset X$. Define

$$d_H(M,N) := \inf\{t > 0 : M \subset N + tB,\ N \subset M + tB\}$$

or, equivalently

$$d_H(M,N) := \max\{\sup_{x\in M} \operatorname{dist}(x,N), \sup_{x\in N} \operatorname{dist}(x,M)\}.$$

Note that d_H is the Hausdorff metric in the space $\mathcal{B}(X)$.

Definition 1. A multivalued map $F : X \rightsquigarrow Y$ is said to be *upper semicontinuous* (u.s.c.) at $x \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $F(x+h) \subset F(x) + \varepsilon B$, when $\|h\| < \delta$.

A multivalued map $F : X \rightsquigarrow Y$ is said to be *lower semicontinuous* (l.s.c.) at $x \in X$ if for any $y \in F(x)$ and for any sequence of elements $x_n \in X$ convergent to x, there exists a sequence of elements $y_n \in F(x_n)$ converging to y.

A multivalued map $F : X \rightsquigarrow Y$ is said to be *positively homogeneous* if $F(tx) = tF(x)$, $t \geq 0$, $x \in X$.

Definition 2 ([6]). A multivalued map $F : X \rightsquigarrow Y$ is said to be *differentiable* at $x \in X$ if there exist a map $D_x : X \to \mathcal{B}_c(Y)$, which is u.c.s. and positively homogeneous, and a number $\delta > 0$ such that

$$d_H(F(x+h), F(x) + D_x(h)) = o(h), \quad \text{when } \|h\| \leq \delta. \tag{1}$$

(Here $o(h)$ denotes a nonnegative function such that $\lim_{h\to 0} o(h)/\|h\| = 0$.) D_x is called the (*multivalued*) *differential* of F at x.

In the sequel, we call this type of differential the De Blasi-like differential.

Remark 1. The motivation of defining such a type of multivalued differential is based on the incrementary property. The differential is required to be only u.s.c. and positively homogeneous (instead of continuous and linear). If a multivalued differential exists, it is unique ([6]). On the other hand, the class of multivalued mappings having such a differential at a point $x_0 \in \operatorname{Dom} F$ (or even on some interval $\langle a,b\rangle$) is quite small in comparison with other definitions.

For this reason, the notion of such a differential can be weakened, for example, to the *upper differential* φ *of* F *at* x, when φ is u.s.c. and positively homogeneous and $F(x+h) \subset F(x) + \varphi(h)$, for $\|h\| < \delta$, $\delta > 0$. This differential is not unique. Denote by $\mathbb{F}$ the sets of all upper differentials of F at x, note that $\mathbb{F}$ can be empty. If F is Lipschitzean, i.e. $d_H(F(x+h), F(x)) \leq L\|h\|$, for small h, then $\mathbb{F} \neq \emptyset$, and it leads to the Lasota-Straus notion of a differential Δ_x, defined by $\Delta_x(h) = \cap_{\varphi \in \mathbb{F}} \varphi(h)$.

The Lasota-Straus differential is not the only possibility how to weaken requirements laid to the differential introduced by (1), cf. [10]. In [9] the presented multivalued differential (at a point) is not necessarily unique. The above mentioned definitions of a differential extend the class of differentiable multifunctions. For our reasons, the differential introduced in Definition 2 is convenient for the sake of lucidity.

Another definition of differentiability of multivalued maps can be found in [5] as a usual derivative in a special space.

When X has reasonable properties, i.e. when X is reflexive, then there is a real normed linear space $\mathbb{B}(X)$ (or for simplicity $\mathbb{B}$) and isometric mapping $\pi : \mathcal{B}_c \to \mathbb{B}$, where $\mathcal{B}_c$ is metrized by d_H, such that $\pi(\mathcal{B})$ is a convex cone in $\mathbb{B}$ and $\mathbb{B}$ is minimal.

Definition 3. A multivalued map $F : X \to \mathcal{B}_c(X)$ is said to be π-*differentiable* at $x_0 \in X$ if the mapping $\widehat{F} : X \to \mathbb{B}(X)$ is differentiable at $x_0 \in X$, i.e. there exists a continuous linear mapping $\widehat{F}'(x_0) : X \to \mathbb{B}$ such that

$$\widehat{F}(x) - \widehat{F}(x_0) - \widehat{F}'(x_0)(x - x_0) = o(\|x - x_0\|).$$

This definition is not "real" multivalued definition of differentiability. For this reason, Definition 3 is presented here for the sake of completeness and we will not work with it.

The third possible approach is based on a property of tangency. In the single-valued case the graph of the derivative at a point x_0 is also a tangent space to the graph at that point. So we can define the multivalued differential of a multivalued map at (x, y) to be a multivalued map whose graph is a tangent cone to the original graph at (x, y).

Definition 4. Let K be nonempty subset of a Hilbert space X. We define the *Bouligand* (*contingent*) *cone* $T_K(x)$ to K at x as follows

$$v \in T_K(x) \quad \text{if} \quad \liminf_{h \to 0+} \frac{d_K(x + hv)}{h} = 0.$$

Definition 5. Let $F : X \rightsquigarrow X$ be a multivalued map. Denote $DF(x, y)$ the multivalued map from X into X, whose graph is the contingent cone $T_{\mathrm{graph}(F)}(x, y)$ to the graph of F at (x, y). We shall say that $DF(x, y)$ is the *contingent derivative* of F at $x \in X$ and $y \in F(x)$.

Lemma 1 ([4])**.** *Let $F : X \rightsquigarrow X$ be a multivalued map. $v \in DF(x, y)(u)$ holds if and only if there exist sequences of strictly positive numbers h_n and of elements $u_n \in X, v_n \in X$ satisfying* $\lim_{n\to\infty} h_n = 0$, $\lim_{n\to\infty} u_n = u$, $\lim_{n\to\infty} v_n = v$ *and* $y + h_n v_n \in F(x + h_n u_n)$.

Remark 2. Contingent cones, and subsequently contingent derivatives, are not the only possibilities how to define multivalued derivatives. Other cones (for example Dubovitskij-Milijutin's, adjacent, Clarke's tangent cones) can be considered and corresponding derivatives have to be treated (see e.g. [4]).

3. Multivalued derivo-periodicity

Let us start with one of the possible definitions of a periodic multifunction. This definition is the most convenient one for our purposes.

Definition 6. A multifunction $F : \mathbb{R} \rightsquigarrow \mathbb{R}$ is called *periodic* with period T if $F(t) \equiv F(t + T)$, for all $t \in \mathbb{R}$.

The following definition and theorem investigate the case concerning De Blasi-like differentiable multifunctions (Definition 2).

Note that if there is a continuously differentiable function $a : I \to \mathbb{R}$ and an interval $B \subset \mathbb{R}$, B is bounded such that $F(x) = a(x) + B$, $\forall x \in I$, it is easy to compute that the differential D_x takes the form of $D_x(h) = a'(x)h$, for all $x \in I$.

First, we recall the notion of Dini derivatives.

Definition 7. Let $a(x) : \mathbb{R} \to \mathbb{R}$ be a continuous function. The notation $D^+a(x), D^-a(x)$, $D_+a(x), D_-a(x)$ will be used for the right and left, upper and lower Dini derivatives of $a(x)$

$$D^+a(x) = \limsup_{h\to 0+} \tfrac{a(x+h)-a(x)}{h}, \quad D_+a(x) = \liminf_{h\to 0+} \tfrac{a(x+h)-a(x)}{h},$$
$$D^-a(x) = \limsup_{h\to 0-} \tfrac{a(x+h)-a(x)}{h}, \quad D_-a(x) = \liminf_{h\to 0-} \tfrac{a(x+h)-a(x)}{h},$$

respectively.

Theorem 2. *A multifunction $F : \mathbb{R} \rightsquigarrow \mathbb{R}$ is De Blasi-like differentiable on an interval I, $I \subset \mathbb{R}$, if and only if there exist a function $a : I \to \mathbb{R}$, $a \in \mathcal{A}$, where $\mathcal{A}$ denotes the set of all continuous functions*

$$\mathcal{A} = \{a(x) \mid a(x) \in C(I), D^+a(x) = D_+a(x) = a'_+(x) \in \mathbb{R},$$
$$D^-a(x) = D_-a(x) = a'_-(x) \in \mathbb{R},\ \forall x \in I\},$$

and an interval $B \subset \mathbb{R}$, B is bounded, such that $F(x) = a(x) + B$ $\forall x \in I$.

Proof. It follows immediately that if $F(x)$ cannot be written in the form of $F(x) = a(x) + B$, $\forall x \in I$, B is bounded, then $F(x)$ is not De Blasi-like differentiable on some interval I. (Note that if $\operatorname{diam}(F(x))$ is not constant on I, then $F(x)$ is not De Blasi-like differentiable. Similarly, if $F(x+\varepsilon) \subsetneq F(x)$, for $\varepsilon > 0$ small enough, and $\operatorname{diam}(F(x))$ is constant, then $F(x)$ is not De Blasi-like differentiable, too.)

If $a(x) : I \to \mathbb{R}$ is not continuous (at $x \in I$), such that D_x exists satisfying condition (1), then D_x is not positively homogeneous and $F(x)$ is not De Blasi-like differentiable.

Let $a(x)$ be continuous. Let D^+, D_+, D^-, D_- denote the Dini derivatives of $a(\cdot)$ at x. Suppose that D_x exists, it is u.s.c., positively homogeneous and satisfies (1). Let $D^+ \neq D_+$, for example. Then there exist $h_n \to 0+$ such that $D^+ = \lim_{n\to\infty} \frac{a(x+h_n)-a(x)}{h_n}$ and $h'_n \to 0+$ such that $D_+ = \lim_{n\to\infty} \frac{a(x+h'_n)-a(x)}{h'_n}$. Since $o(h) = d_H(a(x+h)+B, a(x)+B+D_x(h)) = d_H(a(x)+h_nD^+ + o(h_n)+B, a(x)+B+D_x(h_n)) = d_H(a(x)+h_nD^+ + B, a(x)+B+D_x(h_n)) + o(h_n)$, we have $D_x(h_n) = D^+h_n$, where $\{z_n\} \in o(h_n)$, whenever $\lim_{n\to\infty} \frac{z_n}{h_n} = 0$ for $h_n \to 0$, $n \to \infty$. Similarly, $D_x(h'_n) = D_+h'_n$.

Since D_x is positively homogeneous and since $h_n = h'_n \frac{h_n}{h'_n}$, we have

$$D^+h_n = D_x(h_n) = D_x(h'_n \frac{h_n}{h'_n}) = \frac{h_n}{h'_n} D_x(h'_n) = \frac{h_n}{h'_n} D_+h'_n = D_+h_n,$$

which implies $D^+ = D_+$.

The cases $D^+ = D_+ = \pm\infty$ and $D^- = D_- = \pm\infty$ are not possible, because then $D_x(h)$, satisfying the assumptions in Definition 2, does not exist (it can be shown, for example, that $D_x(h)$ has an unbounded value for $h = 0$).

If $D^+ = D_+ = D+ \in \mathbb{R}$ and $D^- = D_- = D- \in \mathbb{R}$, then it immediately follows that

$$D_x(h) = \begin{cases} (D+)h, & h > 0, \\ (D-)h, & h < 0, \\ 0, & h = 0. \end{cases}$$

Definition 8. Let $F : \mathbb{R} \rightsquigarrow \mathbb{R}$ be a De Blasi-like differentiable multifunction. We say that F is derivo-periodic with period T if the mapping D, defined by $D(x) = D_x$, is T-periodic.

Theorem 3. *Let $F : \mathbb{R} \rightsquigarrow \mathbb{R}$ be a De Blasi-like differentiable multifunction. Then F is derivo-periodic with period T if and only if $a(x) \in \mathcal{A}$ satisfies*

$$(2) \qquad a'_+(x) = a'_+(x+T) \quad \textit{and} \quad a'_-(x) = a'_-(x+T), \quad \textit{for all} \quad x \in \mathbb{R},$$

($i.e.$ *$a(x)$ is derivo-periodic with period T in case that $a(x)$ is continuously differentiable*).

Proof. Let F be a T-derivo-periodic multifunction. Since F is differentiable, we have $F(x) = a(x) + B$, $a(x)$ satisfies (2). Since $D_x(h) = a'_+(x)h$, $h \geq 0$ and $D_x(h) = a'_-(x)h$, $h < 0$ is T-periodic (in x), we obtain $a'_+(x)h = D_x(h) = D_{x+T}(h) = a'_+(x+T)h$, $h \geq 0$, and similarly for $h < 0$.

Let $a(x)$, satisfying (2), be a T-derivo-periodic function. Since $D_x(h) = a'_+(x)h = a'_+(x+T)h = D_{x+T}(h)$, $h \geq 0$, and similarly for $h < 0$, the multifunction $F(x)$ is derivo-periodic with period T. ■

The presented differential is a common single-valued function and the above mentioned class of differentiable multifunctions is small. The usage of contingent cones leads to another convenient definition of multifunctions derivo-periodicity.

Definition 9. A multifunction $F : \mathbb{R} \to \mathbb{R}$ is called *derivo-periodic* with period T if differential $DF(t,x)$ is periodic (in t) with period T, i.e. $DF(t,x) = DF(t+T,y)$, for all $(t,x) \in \mathbb{R} \times \mathbb{R}$, where $y \in F(t+T)$ is a value corresponding to x (they both are boundary points, for example).

Remark 3. Note that the derivative of a multifunction is defined at any point of its graph and not in a point, which belongs to the closure of the graph and not to the graph. For this reason, different multivalued maps having the same closure of their graphs, have different derivatives.

Remark 4. As usual, the main difficulty arising when we are working with a differential, is additivity of the differential. It is not difficult to find, for example, u.s.c. and l.s.c. mappings, for which the linearity of differential does not hold true, so additional conditions have to be fulfilled.

Continuous multifunctions with closed values are required in the following. If the continuity assumption reduces only to a l.s.c. assumption or the values can be open, then the definition of a contingent derivative leads to a complicated behaviour of the mapping $\Omega : (x, y, u) \rightsquigarrow DG(x, y)(u)$.

The additivity of the derivative at $(x, y) \in \operatorname{graph}(F + G)$ can be defined by

$$D(F+G)(x,y)(u) = \cup_{z \in Z_{(x,y)}} (DF(x,z)(u) + DG(x, y-z)(u)), \tag{3}$$

where $Z_{(x,y)}$ is the set containing all points $z \in \mathbb{R}$ such that $(x, z) \in \operatorname{graph} F$, $(x, y - z) \in \operatorname{graph} G$.

As an example, see marked points $\bullet$ in the following figure.

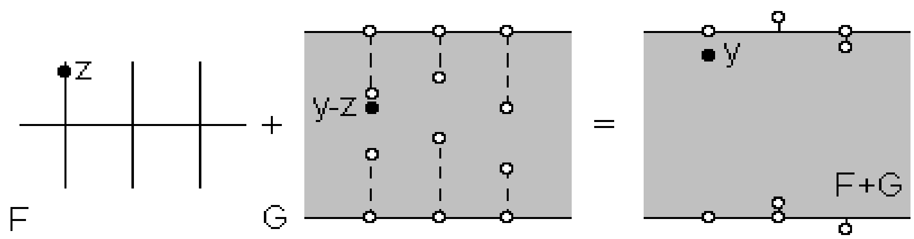

Figure 1

It is true that

$$DF(x,z)(u) = \begin{cases} \emptyset, & u < 0, \\ \mathbb{R}, & u = 0, \\ \emptyset, & u > 0, \end{cases}$$

and $DG(x, y - z)(u) = \mathbb{R}$, for all $u \in \mathbb{R}$, so

$$\cup_{z \in Z_{(x,y)}} (DF(x,z)(u) + DG(x,y-z)(u)) = \begin{cases} \emptyset, & u < 0, \\ \mathbb{R}, & u = 0, \\ \emptyset, & u > 0, \end{cases}$$

where the fact $\emptyset + A = \emptyset$ was used. This differs from the mapping $D(F+G)$ $(x, y)(u) = \mathbb{R}$, for all $u \in \mathbb{R}$.

Theorem 4. (see [4]) *Let $F : \mathbb{R} \rightsquigarrow \mathbb{R}$ be an (arbitrary) multifunction and $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable at $x \in \mathbb{R}$. Then, for all $y \in (f + F)(x)$, the equality*

$$D(f + F)(x, y)(u) = f'(x)(u) + DF(x, y - f(x))(u)$$

holds true.

Proof. "$\subset$" Letting $v \in D(f + F)(x, y)(u)$, then there exist $h_n \to 0+$, $u_n \to u$ and $v_n \to v$ such that $y + h_n v_n \in (f + F)(x + h_n u_n)$. Since f is continuously differentiable, $f(x + h_n u_n) = f(x) + h_n(f'(x)(u_n) + \widetilde{\varepsilon}(h_n)) = f(x) + h_n(f'(x)(u) + f'(x)(u_n - u) + \widetilde{\varepsilon}(h_n)) = f(x) + h_n(f'(x)(u) + \varepsilon(h_n))$, where $\varepsilon(h_n) \to 0$. Then $y - f(x) + h_n(v_n - f'(x)(u) - \varepsilon(h_n)) \in F(x + h_n u_n)$. This means $v - f'(x)(u) \in DF(x, y - f(x))(u)$.

"$\supset$" Letting $v \in f'(x)(u) + DF(x, y - f(x))(u)$, then $v - f'(x)(u) \in DF(x, y - f(x))(u)$. So, there exist $w_n \to v - f'(x)(u)$, $u_n \to u$ and $h_n \to 0+$ such that $y - f(x) + h_n w_n \in F(x + h_n u_n)$. Then, there exists $\varepsilon(h_n) \to 0$ such that $w_n + f'(x)(u) + \varepsilon(h_n) \to v$. From the definition of the continuous differentiability $f'(x)(u) = \frac{1}{h_n}(f(x + h_n u_n) - f(x)) - \varepsilon(h_n)$, and from the condition imposed on w_n, we have $w_n + \frac{1}{h_n}(f(x+h_n u_n) - f(x)) \to v$. Defining $v_n = w_n + \frac{1}{h_n}(f(x + h_n u_n) - f(x))$, then $y - f(x) - h_n \frac{1}{h_n}(f(x + h_n u_n) - f(x)) + h_n v_n = y - f(x + h_n u_n) + h_n v_n \in F(x + h_n u_n)$. Consequently, $y + h_n v_n \in (f + F)(x + h_n u_n)$ and $v \in D(f + F)(x, y)(u)$, because $h_n \to 0+$, $u_n \to u$, $v_n \to v$. ■

Theorem 5. *Let $F : \mathbb{R} \rightsquigarrow \mathbb{R}$ be a multifunction periodic with the period T. Then $DF(t, x)$ is T-periodic (in t), for all $x \in F(t)$.*

Proof. Let $v \in DF(t, x)(u)$, then there exist $v_n \to v$, $u_n \to u$ and $h_n \to 0+$ such that $x + h_n v_n \in F(t + h_n u_n)$. Since F is T-periodic, we have $x + h_n v_n \in F(t + T + h_n u_n)$, so $v \in DF(t + T, x)$, and the conclusion holds true. ■

The following theorem is our first result on derivo-periodicity of multivalued functions, when contingent derivatives are used.

Theorem 6. *Let $F(x) = G(x) + ax$, where $a \in \mathbb{R}$ and $G(x) : \mathbb{R} \rightsquigarrow \mathbb{R}$ is T-periodic. Then $F(x)$ is derivo-periodic with the period T and $DF(x, y) = a + DG(x, y - ax)$.*

Proof. From the additivity of a multivalued derivative, proved for such a class of multifunctions, we have $DF(x, y) = a + DG(x, y - ax)$, where

$y \in G(x)+ax$, and also $DF(x+T,z) = a + DG(x+T, z-ax-aT)$, where $z \in G(x+T)+ax+aT$. Since G is T-periodic, we have $z - aT \in G(x)+ax$. Since there is one to one correspondence between y and $z = y + aT$, and because of the periodicity of $DG(x,y)$, we obtain that, $DF(x+T, y+aT) = DF(x+T,z) = a+DG(x+T, z-ax-aT) = a+DG(x, y-ax) = DF(x,y)$. ∎

The main disadvantage of this theorem is that the "linear" part is strictly single-valued. The following theorem allows using some "zone" map, instead of a linear function. As usual, some additional requirements need to be fulfilled.

By a zone map we understand a multivalued mapping $G : \mathbb{R} \rightsquigarrow \mathbb{R}$ such that $G(x) = \langle a_1(x), a_2(x)\rangle$, where $a_1, a_2 \in \mathcal{A}$, $a_1(x) \le a_2(x)$, for all $x \in \mathbb{R}$.

Lemma 2. *Let $G : \mathbb{R} \rightsquigarrow \mathbb{R}$ be a zone multifunction. If $y \in \operatorname{int} G(x)$, then $DG(x,y)(u) = \mathbb{R}$, for all $u \in \mathbb{R}$. If $y = a_1(x) \neq a_2(x)$, resp. $y = a_1(x) = a_2(x)$, then*

$$DG(x,y)(u) = \begin{cases} \langle a'_{1-}(x)u, \infty), & u < 0, \\ \langle 0, \infty), & u = 0, \\ \langle a'_{1+}(x)u, \infty), & u > 0, \end{cases}$$

resp.

$$DG(x,y)(u) = \begin{cases} \langle a'_{1-}(x)u, a'_{2-}(x)u\rangle, & u < 0, \\ 0, & u = 0, \\ \langle a'_{1+}(x)u, a'_{2+}(x)u\rangle, & u > 0, \end{cases}$$

The proof requires only a simple computation and the result can be easily extended for arbitrary continuous functions $a_1(x)$ and $a_2(x)$.

Theorem 7. *Let $F : \mathbb{R} \rightsquigarrow \mathbb{R}$ be an (arbitrary) multifunction. Let $G : \mathbb{R} \rightsquigarrow \mathbb{R}$ be a zone multifunction. Then*

$$\text{(4)} \qquad \cup_{z\in Z_{(x,y)}} (DF(x,z)(u) + DG(x, y-z)(u)) \subset D(F+G)(x,y)(u).$$

Proof. Let $z \in Z_{(x,y)}$ be arbitrary and let $v \in DF(x,z)(u)$, then there exist $h_n \to 0+$, $u_n \to u$ and $v_n \to v$ such that $z + h_n v_n \in F(x + h_n u_n)$.

Let $w \in DG(x, y-z)(u)$.

If $y - z \in \operatorname{int} G(x)$, then $\forall h_n \to 0+, \forall u_n \to u$ and $\forall w_n \to w$ it holds, that $y - z + h_n w_n \in G(x + h_n u_n)$, for n large enough, and we are done, since $v + w \in D(F+G)(x,y)(u)$.

If $y-z \in \partial G(x)$ ($y-z=a_1(x)$, for example), then $\exists h_n \to 0+, \exists u_n' \to u$ and $\exists w_n \to w$ such that $y-z+h_n w_n \in G(x+h_n u_n')$. It is easy to see that a sequence $h_n \to 0+$ can be taken arbitrarily, since the left ($a_{1-}'(x)$) and right ($a_{1+}'(x)$) derivatives exist (if also $y-z=a_2(x)$ the same holds true). So we take the same sequence h_n, as for $v \in DF(x,z)(u)$.

Note that $G(x+h_n u_n') = G(x+h_n u_n + h_n(u_n'-u_n))$ and $u_n'-u_n \to 0$, for $n \to \infty$. Since G is continuous and corresponding derivatives are bounded, we have $G(x+h_n u_n) \subset G(x+h_n u_n') + B(0, o(h_n))$, where $\{z_n\} \in o(h_n)$, whenever $\lim_{n\to\infty} \frac{z_n}{h_n} = 0$ for $h_n \to 0$, $n \to \infty$.

For that reason, there exists $\varepsilon_n \to 0$, for $n \to \infty$, such that $y-z+h_n(w_n+\varepsilon_n) \in G(x+h_n u_n)$, and the conclusion follows immediately. ∎

Definition 10. We say that a continuous multivalued map $F(x)$ has a smooth boundary if for every open connected interval $C \subset \mathbb{R}$ there exist an open subset $U \subset \mathbb{R}$ and smooth functions $b_i : U \to \mathbb{R}, i = 1, \ldots, n$ such that

$$\partial \operatorname{graph} F|_C = \cup_{i=1}^n \operatorname{graph} b_i.$$

The function b_k, having at a point t a vertical tangent and, for example, for $t_1 > t$, having two values $b_{k1}(t_1), b_{k2}(t_1)$ and, for $t_2 < t$, empty set of values, is allowed. To avoid such a situation some rotated coordinate systems are appropriate. However, such a definition and the proof of the following theorem can become confusing.

So note that the boundary of such a graph consists of smooth functions.

Theorem 8. *Let $F : \mathbb{R} \rightsquigarrow \mathbb{R}$ be a closed continuous multi-valued map with the smooth boundary. Let $G : \mathbb{R} \rightsquigarrow \mathbb{R}$ have the form of a zone map $G(x) = \langle a_1(x), a_2(x) \rangle$, where $a_1, a_2 \in \mathcal{A}$, $a_1(x) \le a_2(x)$, for all $x \in \mathbb{R}$. Then*

$$D(F+G)(x,y)(u) = \cup_{z \in Z_{(x,y)}} (DF(x,z)(u) + DG(x, y-z)(u)),$$

where $Z_{(x,y)}$ is the set containing all points $z \in \mathbb{R}$ such that $(x,z) \in \operatorname{graph} F$, $(x, y-z) \in \operatorname{graph} G$.

The proof will be done, for simplicity, for continuously differentiable functions $a_1(x), a_2(x)$. For $a_1(x), a_2(x) \in \mathcal{A}$, the proof is the same.

Proof. If $y \in \operatorname{int}(F+G)(x)$, then from the continuity assumptions, we have $(x,y) \in \operatorname{int} \operatorname{graph}(F+G)$ and $D(F+G)(x,y)(u) = \mathbb{R}$, for all $u \in \mathbb{R}$. Now, three situations are possible. At first, there exists a point $z \in Z_{(x,y)}$, satisfying $z \in \operatorname{int} F(x)$ and $y - z \in \operatorname{int} G(x)$ (if one exists, infinitely many such points exist). Then $DF(x,z)(u) = \mathbb{R} = DG(x, y-z)(u)$, for all $u \in \mathbb{R}$, and equality (3) holds true.

Or, there exists a point $z \in Z_{(x,y)}$ satisfying $z \in \partial F(x)$ and $y - z \in \operatorname{int} G(x)$. Then $DF(x,y)(u) \neq \emptyset$, $DG(x, y - z(u)) = \mathbb{R}$ and equality (3) holds true.

Or, if such a point z does not exist, then there exist two points $z_1, z_2 \in \mathbb{R}$ such that $z_1, z_2 \in \partial F(x)$, and $a_1(x) = y - z_1$, $a_2(x) = y - z_2$. From the assumptions we have $(b_i + a_2)'(x) = 0$ and $(b_j + a_1)'(x) = 0$, where b_i, b_j are smooth functions according to Definition 10 above. Suppose, for example, that z_1 satisfy: if $w \in F(x)$, $w \in V$, where V is a sufficiently small neighbourhood of z_1, then $w \leq z_1$, i.e. z_1 is a "local maximum" of the value $F(x)$. Let us compute $DF(x, z_1)$. The tangent to the boundary of F at a point (x, z_1) has $b_i'(x) \in \mathbb{R}$ as its directive. Then $DF(x, z_1)(u) = (-\infty, b_i'(x)u\rangle$, similarly $DG(x, y - z_1)(u) = (-\infty, a_2'(x)u\rangle$, and since $y - z_1$ is a similar "maximum" of the value $G(x)$, $DF(x, z_1)(u) + DG(x, y - z_1)(u) = (-\infty, 0\rangle$. The next point z_2 can be treated similarly $DF(x, z_2)(u) + DG(x, y - z_2)(u) = \langle 0, \infty)$. And finally, equality (3) holds true.

The following figure gives us some information about this approach. If the zone map is thicker (after collapsing points k_1, k_2 into one point k, which is just the point y, in this proof), points l_1, l_2 will correspond to points z_1, z_2, in the same part of the proof.

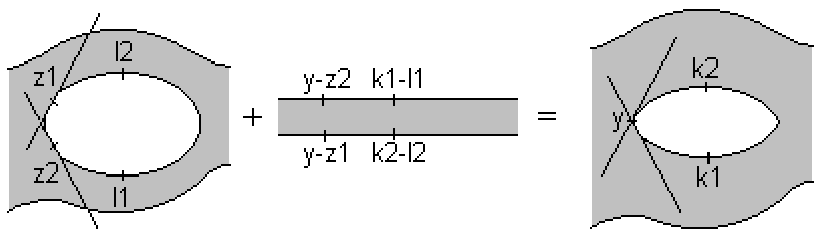

Figure 2. Illustration to the idea of the proof

If $y \in \partial(F + G)(x)$, then there can exist two points $z_1, z_2 \in Z$, satisfying similar properties, as mentioned in the past paragraph (see Figure 2 and points y, z_1, z_2, $y - z_1, y - z_2$ for illustration). The difference from the proof presented before is that $(b_i + a_2)'(x), (b_j + a_1)'(x)$ can take arbitrary values, also values $\pm\infty$ are now accepted. The procedure is similar. The only difference arises, when $(b_i + a_2)'(x) = \infty$ holds, for example. Then the corresponding derivative takes the values $D(F+G)(x,y)(u) = \emptyset$, for example, for $u < 0$ and $D(F+G)(x,y)(u) = \mathbb{R}$ for $u \geq 0$ and the fact that $\emptyset + A = \emptyset$, is used.

The last possibility is that there is exactly one point $z \in Z$ and the conclusion is immediate. In Figure 2, see, for example, points k_1, l_1, $k_1 - l_1$, (when the zone map is not thick). ■

Theorem 9. *Let $F(x) : \mathbb{R} \rightsquigarrow \mathbb{R}$ be a T-periodic continuous set-valued map with a smooth boundary. Let $G : \mathbb{R} \rightsquigarrow \mathbb{R}$, having the form of $G(x) = a(x)+B$, be a zone map, where B is a closed connected subset of $\mathbb{R}$ and $a(x)$ is a smooth single-valued function, satisfying* (2). *Then $(F+G)(x)$ is T-derivo-periodic.*

Proof. Since F is T-periodic, then $DF(x+T,z) = DF(x,z)$, for every $z \in Z_{(x,y)}$. For inner points $y \in (F+G)(x)$ the conclusion follows easily. For boundary points the proof follows.

Since $G(x) = ax + B$, then if B is not a singleton we have $DG(x, y + aT - z)(u) = (-\infty, a'_{+}(x+T)(u)\rangle$ for $n \geq 0$ and similarly, $DG(x, y + aT - z) = \langle a'_{+}(x+T)(u), \infty)$ for $u < 0$. Otherwise we obtain $DG(x, y + aT - z)(u) = \{a'_{+}(x+T)(u)\}$ for $u \geq a$ and a similar formula for $u < 0$. From condition (2) (i.e. from the T-derivo- periodicity of the function $a(x) = b(x) + cx$, where $b(x)$ is periodic and $c \in \mathbb{R}$, for continuously differentiable function), we know that $a'_{\pm}(x+T)(u) = a'_{\pm}(x)(u)$. Note that the point corresponding to (x,y) is $(x+T, y+cT)$.

Then, from (3), we have

$$\begin{aligned}
&D(F+G)(x+T, y+cT)(u) \\
&= \cup_{z \in Z_{(x+T,y+cT)}} (DF(x+T,z)(u) + DG(x+T, y+cT-z)(u)) \\
&= \cup_{z \in Z_{(x,y)}} (DF(x,z)(u) + DG(x, y-z)(u)) = D(F+G)(x,y)(u). \quad ■
\end{aligned}$$

Another definition of periodicity follows. Definition 6 and the following Definition 6' coincide for single-valued maps.

Definition 6'. A multifunction $F : \mathbb{R} \rightsquigarrow \mathbb{R}$ is called *periodic* with a period T if $F(t) \subset F(t+T)$, for all $t \in \mathbb{R}$.

The disadvantage of this definition is that the periodicity condition holds only in a forward direction and not in the opposite, backward, direction.

With respect to this definition, Theorem 3 holds true in the same meaning, because the classes of differentiable multifunctions are identical. Theorem 5, and consequently Theorem 6 and Theorem 9, hold for a larger class of maps. The proofs are similar, only convenient signs $=$ are replaced by $\subset$.

Remark 5. Sometimes, the closed convex processes, i.e. multivalued mappings having closed convex cones as their graphs, are considered as a multivalued analogy of linear maps. For such maps the Open Mapping Theorem and the Closed Graph Theorem hold, for example, (see [4]). For our purposes, the derivative of such a map is not constant. The convenient candidate is the above mentioned zone map (which is not a process).

Remark 6. The mapping $(x, y, u) \in \operatorname{graph} F \times \mathbb{R} \rightsquigarrow DF(x, y)(u)$ is, under some assumptions, l.s.c. The continuity of a multivalued derivative in nearly impossible. Note that if there exist some inner points $(x, y) \in \operatorname{int} \operatorname{graph} F$, then $DF(x, y)(u) = \mathbb{R}$, for all $u \in \mathbb{R}$. So, a modification of Theorem 8 for l.s.c. maps is convenient, for example, for computing the second contingent derivatives and studying second (or higher) order derivo-periodic multifunctions.

The stated theorems (Theorem 6, Theorem 9) give us sufficient conditions of derivo-periodicity of multifunctions (with nonconvex values). The necessary conditions are weaker. For example, we have constructed u.s.c. map plus a zone map, giving u.s.c. map such that the additivity condition of a derivative holds true, and consequently the statement of Theorem 9 holds true.

It seems that for a larger class than only for continuous functions F, Theorem 8 holds true. Precise statements and possible counterexamples will be discussed elsewhere.

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