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ON WEAK SHARP MINIMA FOR A SPECIAL CLASS OF NONSMOOTH FUNCTIONS

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Abstract

We present a characterization of weak sharp local minimizers of order one for a function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) := \max\{f_i(x)|i = 1, ..., p\}$, where the functions f_i are strictly differentiable. It is given in terms of the gradients of f_i and the Mordukhovich normal cone to a given set on which f is constant. Then we apply this result to a smooth nonlinear programming problem with constraints.

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1 Introduction

Weak sharp minima are some special types of possibly non-isolated minima, where the objective function is constant on a given set of minimizers and satisfies a certain "growth condition" outside this set. To give a precise definition (taken from [12]), let us consider the following mathematical program:

$$\min\{f(x)|x \in C\}, \tag{1}$$

where $f : \mathbb{R}^n \to \overline{\mathbb{R}} := [-\infty, +\infty]$ and C is a nonempty subset of $\mathbb{R}^n$.

Definition 1. Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^n$. Suppose that f is finite and constant on the set $S \subset \mathbb{R}^n$, and let $\bar{x} \in S \cap C$ and $m \geq 1$. For $x \in \mathbb{R}^n$, let

$$\operatorname{dist}^m(x, S) := \inf\{\|y - x\|^m \,|y \in S\}.$$

(a) We say that $\bar{x}$ is a *weak sharp minimizer of order* m for (1) if there exists $\beta > 0$ such that

$$f(x) - f(\bar{x}) \geq \beta \operatorname{dist}^m(x, S) \text{ for all } x \in C.$$

(b) For $\varepsilon > 0$, let $B(x, \varepsilon) := \{y \in \mathbb{R}^n \mid \|y - x\| \leq \varepsilon\}$. We say that $\bar{x}$ is a *weak sharp local minimizer of order* m for (1) if there exist $\beta > 0$ and $\varepsilon > 0$ such that

$$f(x) - f(\bar{x}) \geq \beta \operatorname{dist}^m(x, S) \text{ for all } x \in C \cap B(\bar{x}, \varepsilon). \tag{2}$$

The notion of a weak sharp minimum (of order one) was studied by Burke and Ferris in [3]. It is an extension of a sharp or strongly unique minimum to include the possibility of a nonunique solution set. Weak sharp minima of order m occur in many optimization problems and have important consequences for the study of optimization algorithms and for sensitivity analysis in nonlinear programming. Various characterizations of weak sharp minimizers of order m (local or global) in nonconvex optimization were obtained in [1] and [10] – [12].

In the unconstrained case, ($C = \mathbb{R}^n$) we will refer to a weak sharp (local) minimizer of order m for (1) as a *weak sharp* (*local*) *minimizer of order* m for f.

It should be noted that a function possessing a weak sharp minimizer of order m is a particular case of a *well-conditioned* function, the notion which was studied in the general metric space context (see [5] and references therein).

An outline of this paper is as follows: We begin in Section 2 by reviewing the characterizations of weak sharp local minimizers of order m obtained in [12] for the unconstrained case. We then point out certain difficulties which arise in the process of practical application of these results. In Section 3, we consider a special class of nonsmooth functions which are pointwise maxima of finite collections of strictly differentiable functions (see Definition 2 below). For this class, and for $m = 1$, we prove another characterization of weak sharp local minimizers, which avoids some of the difficulties stated before. We also show that this characterization cannot be easily reduced to a simpler condition. Finally, in Section 4 we show how to apply the result of Section 3 to a standard smooth nonlinear programming problem.

The following notation will be useful in the sequel: for a set $S \subset \mathbb{R}^n$, we denote the closure of S by $\operatorname{cl} S$, and the boundary of S by $\operatorname{bd} S$. The symbol $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$.

Definition 2. A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is *strictly differentiable* at a point $\bar{x}$ if $f(\bar{x})$ is finite and there is a vector $\nabla f(\bar{x})$ (called the *gradient* of f at $\bar{x}$) such that

$$\lim_{\substack{(x,y)\to(\bar{x},\bar{x})\\ x\neq y}} \frac{f(x)-f(y)-\langle \nabla f(\bar{x}), x-y\rangle}{\|x-y\|} = 0.$$

For subdifferential characterizations of strict differentiability, see [6, Theorem 9.18].

2 Review of results for the unconstrained case

The following concept of normal cone will play a major role in our optimality conditions:

Definition 3. Let $S \subset \mathbb{R}^n$ be nonempty.
(a) For $x \in \mathbb{R}^n$, call the subset

$$P(S,x) := \{w \in \operatorname{cl} S |\, \|x-w\| = \operatorname{dist}(x,S)\}$$

the *metric projection* of x onto S.

(b) Let $\bar{x} \in \operatorname{cl} S$. The *normal cone* to S at $\bar{x}$ is defined by

$$N(S,\bar{x}) := \{y | \exists \{y_j\} \to y, \{x_j\} \to \bar{x}, \{t_j\} \subset (0,+\infty), \{s_j\} \subset \mathbb{R}^n$$
$$\text{with } s_j \in P(S,x_j) \text{ and } y_j = (x_j - s_j)/t_j\}.$$

The normal cone $N(S,\bar{x})$ is often called the Mordukhovich normal cone or limiting proximal normal cone. In terms of $N(S,\bar{x})$, we can obtain a general characterization of weak sharp local minimality of order m.

Theorem 4 [12]**.** *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be finite and constant on a closed set $S \subset \mathbb{R}^n$, and let $\bar{x} \in S$ and $m \geq 1$. The following conditions are equivalent:*

(a) *$\bar{x}$ is a weak sharp local minimizer of order m for f;*

(b) *for all $y \in N(S,\bar{x})$ with $\|y\| = 1$ and for all sequences $\{x_j\}$, $\{s_j\}$ such that $s_j \in P(S,x_j)$, $x_j \to \bar{x}$, $(x_j - s_j)/\|x_j - s_j\| \to y$ and*

$$\liminf_{j\to\infty} \frac{f(x_j)-f(s_j)}{\|x_j - s_j\|^{m-1}} \leq 0,$$

we have

(3)
$$\liminf_{j\to\infty} \frac{f(x_j)-f(s_j)}{\|x_j - s_j\|^{m}} > 0;$$

(c) *for all* $y \in N(S,\bar{x})$ *with* $\|y\| = 1$ *and for all sequences* $\{x_j\}$, $\{s_j\}$ *such that* $s_j \in P(S,x_j)$, $x_j \to \bar{x}$ *and* $(x_j - s_j)/\|x_j - s_j\| \to y$, *inequality* (3) *holds.*

Theorem 4 gives very general characterizations of weak sharp local minima of order m for f. However, conditions (b) and (c) are rather difficult for practical use. More easily verifiable conditions can be developed in terms of certain generalized directional derivatives of f.

Definition 5. Let S be a nonempty closed subset of $\mathbb{R}^n$, and let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be finite on $\operatorname{bd} S$. For $x \in \operatorname{bd} S$ and $y \in \mathbb{R}^n$, define

$$\underline{d}_S^m f(x;y) := \liminf_{\substack{\operatorname{bd} S \ni s \to x \\ (t,v)\to(0^+,y)}} \frac{f(s+tv)-f(s)}{t^m}.$$

(In particular, (x,y) is an allowable choice of (s,v).)

Definition 5 gives a sort of generalization of the directional derivative

$$\underline{d}^m f(x;y) := \liminf_{(t,v)\to(0^+,y)} \frac{f(s+tv)-f(s)}{t^m}$$

which was used in [7] to study strict local minima of order m. Observe that when $S = \{x\}$, then $\underline{d}_S^m f(x;y)$ reduces to $\underline{d}^m f(x;y)$. It is not difficult to give other examples where these two limits are equal (see [12] for details). In fact, as we see below, this is a necessary requirement for the next characterization to be valid.

Definition 6. The *contingent cone* to a set $S \subset \mathbb{R}^n$ at $x \in S$ is defined by

$$K(S,x) := \{y \in \mathbb{R}^n | \exists \{t_j\} \to 0^+, \exists \{y_j\} \to y \text{ such that } x + t_j y_j \in S, \forall j\}.$$

Theorem 7 [12]**.** *Let* $f : R^n \to \overline{\mathbb{R}}$ *be finite and constant on a closed set* $S \subset R^n$*, and let* $\bar{x} \in \operatorname{bd} S$*. Suppose that*

(4) $$K(S,\bar{x}) \cap N(S,\bar{x}) = \{0\}$$

and

(5) $$\underline{d}^m f(\bar{x};y) = \underline{d}_S^m f(\bar{x};y) \text{ for all } y \in N(S,\bar{x})\backslash\{0\}.$$

Then $\bar{x}$ *is a weak sharp local minimizer of order* m *for* f *if and only if*

(6) $$\underline{d}^m f(\bar{x};y) > 0 \text{ for all } y \in N(S,\bar{x})\backslash\{0\}.$$

Condition (6) is much easier to verify than (b) or (c) of Theorem 4. Unfortunately, this characterization is valid only when assumptions (4) and (5) hold simultaneously. For $m = 1$, this is difficult to attain even for very simple nonsmooth functions (see Example 8 below). It is shown in [12] that (4) holds if S is convex and that (5) (with $m = 1$) holds if f is strictly differentiable at $\bar{x}$. However, the latter assumption is never satisfied if $\bar{x}$ is a weak sharp local minimizer of order one for f.

Example 8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x_1, x_2) := \max\{x_1, x_2, -x_2\}.$$

Then f attains a weak sharp minimum of order one at $\bar{x} = (0, 0)$ with $S = \{(x_1, 0)|x_1 \leq 0\}$. We have $K(S, \bar{x}) = S$ and $N(S, \bar{x}) = \{(y_1, y_2)|y_1 \geq 0\}$, hence (4) is satisfied. But (5) does not hold for $y = (1, 0)$ since $\underline{d}^1 f(\bar{x}; y) = 1$, while $\underline{d}^1_S f(\bar{x}; y) = 0$.

3 Finite maxima of differentiable functions

To be able to deal with such situations as in Example 8, we now derive another characterization of weak sharp local minimizers of order one for a special class of nonsmooth functions which are defined as finite maxima of strictly differentiable functions. A similar class (under stronger differentiability assumptions) was considered in [1], where necessary and sufficient conditions for weak sharp minima of order two were obtained.

We consider the problem of minimizing the function

$$f(x) := \max\{f_i(x)|i \in I\}, \tag{7}$$

where $I := \{1, ..., p\}$ is a finite index set and the functions $f_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, are strictly differentiable. For any $x \in \mathbb{R}^n$, define $I(x) := \{i \in I|f_i(x) = f(x)\}$.

Lemma 9. *For each $x \in \mathbb{R}^n$, there exists $\varepsilon > 0$ such that $I(u) \subset I(x)$ for all $u \in B(x, \varepsilon)$.*

Proof. Let $x \in \mathbb{R}^n$. Since $f_i(x) < f(x)$ for all $i \in I \backslash I(x)$ and the functions f_i and f are continuous, there exists $\varepsilon > 0$ such that $f_i(u) < f(u)$ for all $i \in I \backslash I(x)$ and $u \in B(x, \varepsilon)$. This means that $I \backslash I(x) \subset I \backslash I(u)$ for all $u \in B(x, \varepsilon)$, which gives the desired conclusion. ∎

Lemma 10 [4]. *Let S be a nonempty subset of $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$, $s \in S$. The following conditions are equivalent:*

(a) $s \in P(S, x)$;
(b) $s \in P(S, s + \lambda(x - s))$ *for all* $\lambda \in [0, 1]$.

The following theorem characterizes weak sharp local minimizers of order one for f.

Theorem 11. *Suppose that f is constant on a closed subset S of $\mathbb{R}^n$, and let $\bar{x} \in S$. The following conditions are equivalent:*

(a) *$\bar{x}$ is a weak sharp local minimizer of order one for f;*

(b) *for each $y \in N(S, \bar{x})$ with $\|y\| = 1$, there exists $\delta > 0$ such that, for each $x \in B(\bar{x}, \delta) \backslash S$ and $s \in P(S, x)$ satisfying*

$$\left\| \frac{x - s}{\|x - s\|} - y \right\| \leq \delta, \tag{8}$$

there exists $i \in I(s)$ such that $\langle \nabla f_i(\bar{x}), y \rangle > 0$.

Proof. (b) $\Longrightarrow$ (a). Suppose that (b) holds. To prove (a), we will verify condition (c) of Theorem 4 (for $m = 1$). Let $y \in N(S, \bar{x})$ with $\|y\| = 1$ and let δ be chosen according to (b). Take any sequences $\{x_j\}$, $\{s_j\}$ such that $s_j \in P(S, x_j)$, $x_j \to \bar{x}$ and

$$\frac{x_j - s_j}{\|x_j - s_j\|} \to y \tag{9}$$

(note that (9) implies $x_j \notin S$). By the definition of $P(S, x_j)$, we have $\|x_j - s_j\| \leq \|x_j - \bar{x}\|$ which gives $s_j \to \bar{x}$. For each j sufficiently large, we have $x_j \in B(\bar{x}, \delta)$ and

$$\left\| \frac{x_j - s_j}{\|x_j - s_j\|} - y \right\| \leq \delta,$$

hence, by our assumption, there exists $i(j) \in I(s_j)$ such that $\langle \nabla f_{i(j)}(\bar{x}), y \rangle > 0$. Since $f_{i(j)}(s_j) = f(s_j)$, we obtain

$$\liminf_{j \to \infty} \frac{f(x_j) - f(s_j)}{\|x_j - s_j\|} \geq \liminf_{j \to \infty} \frac{f_{i(j)}(x_j) - f_{i(j)}(s_j)}{\|x_j - s_j\|}. \tag{10}$$

Now, observe that all $I(s_j)$ are subsets of the same finite set I. Therefore, we can find a strictly increasing sequence $\{j_k\}$ of positive integers such that the subsequence $\{i(j_k)\}$ is constant (i.e. $i(j_k) = i_0$ for all k) and

$$\liminf_{j\to\infty} \frac{f_{i(j)}(x_j) - f_{i(j)}(s_j)}{\|x_j - s_j\|} = \lim_{k\to\infty} \frac{f_{i_0}(x_{j_k}) - f_{i_0}(s_{j_k})}{\|x_{j_k} - s_{j_k}\|}. \tag{11}$$

Using (9) – (11) and the strict differentiability of f_{i_0} at $\bar{x}$, we obtain

$$\begin{aligned}
&\liminf_{j\to\infty} \frac{f(x_j) - f(s_j)}{\|x_j - s_j\|} \\
&\geq \lim_{k\to\infty} \frac{f_{i_0}(x_{j_k}) - f_{i_0}(s_{j_k}) - \langle \nabla f_{i_0}(\bar{x}), x_{j_k} - s_{j_k}\rangle}{\|x_{j_k} - s_{j_k}\|} \\
&+ \lim_{k\to\infty} \left\langle \nabla f_{i_0}(\bar{x}), \frac{x_{j_k} - s_{j_k}}{\|x_{j_k} - s_{j_k}\|} \right\rangle \\
&= \lim_{\substack{(x,s)\to(\bar{x},\bar{x})\\ x\neq s}} \frac{f_{i_0}(x) - f_{i_0}(s) - \langle \nabla f_{i_0}(\bar{x}), x - s\rangle}{\|x - s\|} + \langle \nabla f_{i_0}(\bar{x}), y\rangle \\
&= \langle \nabla f_{i_0}(\bar{x}), y\rangle > 0,
\end{aligned} \tag{12}$$

which means that condition (c) of Theorem 4 is satisfied.

(a) $\Longrightarrow$ (b) (by contraposition). Suppose that (b) does not hold. Then there exists $y \in N(S, \bar{x})$ with $\|y\| = 1$ such that for each positive integer j, there exist $u_j \in \mathbb{R}^n \backslash S$ and $s_j \in P(S, u_j)$ satisfying the conditions

$$u_j \in B(\bar{x}, 1/j), \tag{13}$$

$$\left\| \frac{u_j - s_j}{\|u_j - s_j\|} - y \right\| \leq \frac{1}{j} \tag{14}$$

and

$$\langle \nabla f_i(\bar{x}), y\rangle \leq 0 \text{ for all } i \in I(s_j). \tag{15}$$

Applying Lemma 9 to each s_j, we can find $\lambda_j \in (0, 1]$ so small that, for x_j defined by $x_j := s_j + \lambda_j(u_j - s_j)$, we have $I(x_j) \subset I(s_j)$. Moreover, the condition $s_j \in P(S, u_j)$ implies $s_j \in P(S, x_j)$ by Lemma 10. Since $\bar{x} \in S$ and $s_j \in P(S, u_j)$ it follows that $\|u_j - s_j\| \leq \|u_j - \bar{x}\|$. Therefore

$$\begin{aligned}
\|x_j - \bar{x}\| &= \|s_j + \lambda_j(u_j - s_j) - \bar{x}\| \\
&\leq \|s_j - u_j\| + \|u_j - \bar{x}\| + \lambda_j \|u_j - s_j\| \\
&= (1 + \lambda_j) \|u_j - s_j\| + \|u_j - \bar{x}\| \\
&\leq 3 \|u_j - \bar{x}\| \leq 3/j,
\end{aligned}$$

where the last inequality follows from (13). Thus $x_j \to \bar{x}$, and by (14),

$$\frac{x_j - s_j}{\|x_j - s_j\|} = \frac{\lambda_j(u_j - s_j)}{\|\lambda_j(u_j - s_j)\|} = \frac{u_j - s_j}{\|u_j - s_j\|} \underset{j\to\infty}{\longrightarrow} y. \tag{16}$$

For each j, we now choose an arbitrary $i(j) \in I(x_j)$. Then we also have $i(j) \in I(s_j)$, and so

$$f(x_j) - f(s_j) = f_{i(j)}(x_j) - f_{i(j)}(s_j). \tag{17}$$

As noted in the first part of this proof, we can find a strictly increasing sequence $\{j_k\}$ such that $i(j_k) = i_0$ for all k, and equality (11) holds. Now, using (11) and (15) – (17), we can repeat the argument of the first part, replacing the first inequality in (12) by equality. This way we get

$$\liminf_{j\to\infty} \frac{f(x_j) - f(s_j)}{\|x_j - s_j\|} = \langle \nabla f_{i_0}(\bar{x}), y\rangle \le 0.$$

This inequality, together with (16) and the conditions $x_j \to \bar{x}$, $s_j \in P(S, x_j)$, gives a contradiction with condition (c) of Theorem 4. Thus, $\bar{x}$ is not a weak sharp local minimizer of order one for f. ■

Example 12. Consider the function f defined in Example 8. We will show that condition (b) of Theorem 11 is satisfied at $\bar{x} = (0, 0)$. Using the notation $f_1(x) := x_1$, $f_2(x) := x_2$, $f_3(x) := -x_2$, for all $x = (x_1, x_2)$, we can compute $I(\bar{x}) = \{1, 2, 3\}$ and $I(s) = \{2, 3\}$ for all $s \in S\backslash\{\bar{x}\}$. Observe that, for each $x \in \mathbb{R}^2$, there is a unique point $s \in P(S, x)$ given by

$$s = \begin{cases} \bar{x} & \text{if } x_1 \ge 0, \\ (x_1, 0) & \text{if } x_1 < 0. \end{cases} \tag{18}$$

Now, take any $y = (y_1, y_2) \in N(S, \bar{x})$ with $\|y\| = 1$. One of the following situations must occur:

(i) $y_1 \ge 0$ and $y_2 > 0$. Then $2 \in I(s)$ for all $s \in S$ and $\langle \nabla f_2(\bar{x}), y\rangle = y_2 > 0$.

(ii) $y_1 \ge 0$ and $y_2 < 0$. Then $3 \in I(s)$ for all $s \in S$ and $\langle \nabla f_3(\bar{x}), y\rangle = -y_2 > 0$.

(iii) $y = (1, 0)$. Then, for each $x \in \mathbb{R}^2$ and $s \in P(S, x)$, if $(x - s)/\|x - s\|$ is sufficiently close to y, we have $x_1 > 0$, hence $s = \bar{x}$ by (18). Therefore, $1 \in I(s)$ and $\langle \nabla f_1(\bar{x}), y\rangle = 1 > 0$.

The analysis of cases (i) – (iii) shows that the desired condition (b) is always fulfilled.

The next example shows that (b) in Theorem 11 cannot be replaced by the simpler condition

$$\max\{\langle \nabla f_i(\bar{x}), y\rangle \,|i \in I(\bar{x})\} > 0 \text{ for all } y \in N(S,\bar{x})\backslash\{0\}. \tag{19}$$

Example 13. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x_1, x_2) := \max\{0, x_1^3, x_2^3, x_1 + x_2\}.$$

This function is constant (equal to 0) on the set $S := \{(x_1, x_2)|x_1 \le 0,\ x_2 \le 0\}$. For $x = (x_1, x_2)$, define $f_1(x) := 0$, $f_2(x) := x_1^3$, $f_3(x) := x_2^3$, $f_4(x) := x_1 + x_2$. Observe that condition (19) holds at the point $\bar{x} = (0, 0)$. Indeed, we have $I(\bar{x}) = \{1, 2, 3, 4\}$, $N(S, \bar{x}) = \{(y_1, y_2)|y_1 \ge 0,\ y_2 \ge 0\}$ and $\nabla f_4(\bar{x}) = (1, 1)$. Hence,

$$\langle \nabla f_4(\bar{x}), (y_1, y_2)\rangle = y_1 + y_2 > 0$$

for all $(y_1, y_2) \in N(S, \bar{x})\backslash\{(0, 0)\}$.

However, $\bar{x}$ is not a weak sharp local minimizer of order one for f. To see this, consider a point $x = (x_1, x_2)$ sufficiently close to $\bar{x}$ and such that $x_1 < 0$, $0 < x_2 < -x_1$. For such x, we have $x_2^3 > 0 > x_1 + x_2$, and so $f(x) = f_3(x) = x_2^3$. Suppose that (2) holds for some $\beta > 0$ (with $C = \mathbb{R}^2$); then, for x as above, we have

$$f(x) - f(\bar{x}) = x_2^3 \ge \beta \operatorname{dist}(x, S) = \beta x_2.$$

Dividing this inequality by x_2, and taking the limit as $x_2 \to 0^+$, we obtain $\beta \le 0$ – a contradiction.

4 Smooth problems with constraints

In this section, we apply an argument similar to that of [1, Proposition 6] to obtain a reformulation of Theorem 11 for the following constrained nonlinear program:

$$\min\{f_0(x)|x \in C\}, \tag{20}$$

where

$$C := \{x \in \mathbb{R}^n|f_i(x) \le 0,\ i = 1, ..., p;\ f_i(x) = 0,\ i = p + 1, ..., q\}. \tag{21}$$

To simplify the notation, let $I := \{1, ..., p\}$ and $J := \{p+1, ..., q\}$. We assume that the functions f_i, $i \in I \cup J$, are continuously differentiable on $\mathbb{R}^n$.

Let S be a closed set of feasible points for (20) – (21) (i.e. $S \subset C$) such that f_0 is constant on S. Suppose that $\bar{x} \in S$ and define $I(\bar{x}) := \{i \in I | f_i(\bar{x}) = f(\bar{x})\}$. We will need the following *constraint qualification* to be satisfied at the given point $\bar{x}$:

(CQ) For each vector $y = (y_1, ..., y_q) \in \mathbb{R}^q$ such that $y_i = 0$ for $i \in I \backslash I(\bar{x})$, and $y_i \geq 0$ for $i \in I(\bar{x})$, the following implication holds:

$$\sum_{i=1}^{q} y_i \nabla f_i(\bar{x}) = 0 \quad \Longrightarrow \quad y = 0.$$

It can be verified that the well-known Mangasarian-Fromovitz constraint qualification implies condition (CQ) (see Remark (c) on p. 306 of [8]).

In order to apply the results of Section 3, we define

$$f(x) := \max\{f_0(x) - f_0(\bar{x}), f_i(x), |f_j(x)| \, | i \in I, \ j \in J\}. \tag{22}$$

Observe that f is constant (actually, equal to 0) on S. Moreover, it can be represented as a finite maximum of smooth functions as follows:

$$f(x) := \max\{f_0(x) - f_0(\bar{x}), f_i(x), f_j(x), -f_j(x) | i \in I, \ j \in J\}. \tag{23}$$

The following proposition clarifies the relationship between weak sharp local minima for the constrained problem (20) – (21) and the unconstrained problem of minimizing f on $\mathbb{R}^n$.

Proposition 14. *Let $\bar{x} \in S$ and suppose that condition* (CQ) *is satisfied at $\bar{x}$. The following conditions are equivalent:*

(a) *$\bar{x}$ is a weak sharp local minimizer of order one for f, that is, there exist $\beta > 0$ and $\varepsilon > 0$ such that*

$$f(x) \geq \beta \operatorname{dist}(x, S) \text{ for all } x \in B(\bar{x}, \varepsilon). \tag{24}$$

(b) *$\bar{x}$ is a weak sharp local minimizer of order one for* (20) – (21), *that is, there exist $\beta_0 > 0$ and $\varepsilon_0 > 0$ such that*

$$f_0(x) - f_0(\bar{x}) \geq \beta_0 \operatorname{dist}(x, S) \text{ for all } x \in C \cap B(\bar{x}, \varepsilon_0). \tag{25}$$

Proof. (a) $\Longrightarrow$ (b): Suppose that (24) holds and let $x \in C \cap B(\bar{x}, \varepsilon)$. Since x is feasible for (20) – (21), it follows from (22) that

$$f(x) \leq \max\{f_0(x) - f_0(\bar{x}), 0\}. \tag{26}$$

We claim that (26) can actually be replaced by a stronger condition

$$f(x) = f_0(x) - f_0(\bar{x}). \tag{27}$$

Indeed if $x \in S$, then $f_0(x) = f_0(\bar{x})$ by the assumption that f_0 is constant on S. In this case, formula (22) and the feasibility of x imply $f(x) = 0$, and so (27) holds. In the case when $x \notin S$, we have $\operatorname{dist}(x, S) > 0$. Hence, from (24) and (26), we deduce $f_0(x) - f_0(\bar{x}) > 0$. Using this inequality, (22) and the feasibility of x, we obtain (27) again. Conditions (24) and (27) mean that (25) holds for $\beta_0 := \beta$ and $\varepsilon_0 := \varepsilon$.

(b) $\Longrightarrow$ (a): Suppose that (25) holds. Since f_0 is continuously differentiable we can find $\varepsilon_1 \in (0, \varepsilon_0/2)$ such that f_0 is Lipschitzian of rank $L > 0$ on $B(\bar{x}, \varepsilon_1)$. It follows from [9, Theorem 1] (which is a specification of Borwein's regularity theorem [2, Theorem 3.2(a)]) that, under condition (CQ), there exist $K > 0$ and $\varepsilon_2 \in (0, \varepsilon_1)$ such that

$$\operatorname{dist}(x, C) \leq K \max\{f_i^+(x), |f_j(x)| \,|\, i \in I, j \in J\} \tag{28}$$

for all $x \in B(\bar{x}, \varepsilon_2)$, where $f_i^+(x) := \max\{0, f_i(x)\}$. To prove (24), we define $\beta := \beta_0/(K\beta_0 + KL + 1)$ and $\varepsilon := \varepsilon_2$. Let $x \in B(\bar{x}, \varepsilon)$. We consider separately two cases: (i) $x \in C$ and (ii) $x \notin C$.

Case (i). From (22) and (25) we get

$$f(x) \geq f_0(x) - f_0(\bar{x}) \geq \beta_0 \operatorname{dist}(x, S) \geq \beta \operatorname{dist}(x, S).$$

Case (ii). Since C is closed and nonempty we have $P(C, x) \neq \emptyset$. Choose any $u \in P(C, x)$. Because x is not feasible, we have either $f_i(x) > 0$ for some $i \in I$ or $|f_j(x)| > 0$ for some $j \in J$. In both cases, the maximum in (28) is a positive number which does not change if $f_i^+(x)$ are replaced by $f_i(x)$ for all $i \in I$. Therefore, from (28) and (22) we deduce

$$\|x - u\| = \operatorname{dist}(x, C) \leq K \max\{f_i(x), |f_j(x)| \,|\, i \in I,\ j \in J\} \leq K f(x). \tag{29}$$

We now verify that $u \in B(\bar{x}, \varepsilon_0)$. Indeed, since $u \in P(C, x)$ and $\bar{x} \in C$ we have $\|x - u\| \leq \|x - \bar{x}\|$. Hence,

$$\|u - \bar{x}\| \leq \|u - x\| + \|x - \bar{x}\| \leq 2\,\|x - \bar{x}\| \leq 2\varepsilon \leq \varepsilon_0.$$

This means that (25) may be applied for $x = u$:

$$f_0(u) - f_0(\bar{x}) \geq \beta_0 \operatorname{dist}(u, S). \tag{30}$$

It is known that the function $\operatorname{dist}(\cdot, S)$ is Lipschitzian of rank 1. Using successively the Lipschitz condition for $\operatorname{dist}(\cdot, S)$, inequality (30), the Lipschitz condition for f_0, and condition (29), we estimate

$$\begin{aligned}
\operatorname{dist}(x, S) &\leq \|x - u\| + \operatorname{dist}(u, S) \leq \|u - x\| + \beta_0^{-1}(f_0(u) - f_0(\bar{x})) \\
&\leq \|u - x\| + \beta_0^{-1}(f_0(x) + L\,\|u - x\| - f_0(\bar{x})) \\
&= (1 + \beta_0^{-1}L)\,\|u - x\| + \beta_0^{-1}(f_0(x) - f_0(\bar{x})) \\
&\leq (1 + \beta_0^{-1}L)Kf(x) + \beta_0^{-1}f(x) = \beta^{-1}f(x),
\end{aligned}$$

which completes the proof of (24) in this case. ∎

Using the representation of f given by (23), we can easily combine Theorem 11 with Proposition 14 to obtain the following result:

Theorem 15. *Let S be a closed set of feasible points for* (20) – (21) *such that f_0 is constant on S. Suppose that* (CQ) *holds at a given point $\bar{x} \in S$. Then the following conditions are equivalent:*

(a) *$\bar{x}$ is a weak sharp local minimizer of order one for* (20) – (21);

(b) *for each $y \in N(S, \bar{x})$ with $\|y\| = 1$, there exists $\delta > 0$ such that, for each $x \in B(\bar{x}, \delta)\backslash S$ and $s \in P(S, x)$ satisfying*

$$\left\| \frac{x - s}{\|x - s\|} - y \right\| \leq \delta,$$

we have either $\langle \nabla f_i(\bar{x}), y \rangle > 0$ for some $i \in \{0\} \cup I(\bar{x})$ or $\langle \nabla f_j(\bar{x}), y \rangle \neq 0$ for some $j \in J$.

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