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RATES OF CONVERGENCE OF CHLODOVSKY-KANTOROVICH POLYNOMIALS IN CLASSES OF LOCALLY INTEGRABLE FUNCTIONS

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Dedicated to Professor Michał Kisielewicz on his 70^{th} *birthday*

Abstract

In this paper we establish an estimation for the rate of pointwise convergence of the Chlodovsky-Kantorovich polynomials for functions f locally integrable on the interval $[0, \infty)$. In particular, corresponding estimation for functions f measurable and locally bounded on $[0, \infty)$ is presented, too.

Keywords and phrases: Chlodovsky polynomial, Kantorovich polynomial, rate of convergence.

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1. Introduction

Let f be a function defined on the interval $[0, \infty)$ and let $N = \{1, 2 \ldots\}$. The Bernstein-Chlodovsky polynomials $C_n f$ of the function f are defined as

$$(1) \quad C_n f(x) := \sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right) P_{n,k}\left(\frac{x}{b_n}\right) \qquad \text{for} \quad x \in [0, b_n], \quad n \in N,$$

where $P_{n,k}(t) := \binom{n}{k} t^k (1-t)^{n-k}$ for $t \in [0,1]$ and (b_n) is a positive increasing sequence satisfying the properties

$$(2) \qquad \lim_{n\to\infty} b_n = \infty, \qquad \lim_{n\to\infty} \frac{b_n}{n} = 0.$$

These polynomials were first introduced by I. Chlodovsky in 1937 as a generalization of the classical Bernstein polynomials

$$B_nf(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) P_{n,k}(x), \qquad 0 \le x \le 1,$$

of functions f defined on the interval $[0,1]$ (see [5] or [8], Chap. II). The well-known Chlodovsky theorem states that if

$$\lim_{n\to\infty} \sup_{0\le t\le b_n} |f(t)| \exp\left(-\alpha\frac{n}{b_n}\right) = 0 \qquad \text{for} \quad \text{every} \quad \alpha > 0, \tag{3}$$

then $\lim_{n\to\infty} C_nf(x) = f(x)$ at every point x of continuity of f. In 1960 J. Albrycht and J. Radecki [1] proved the Voronovskaya-type theorem for operators (1). Some other approximation properties of the Chlodovsky polynomials can be found e.g. in [3, 7].

For functions f Lebesgue-integrable on the interval $[0,1]$ the classical Kantorovich polynomial of order n is defined as

$$B_n^\star f(x) := \frac{\mathrm{d}}{\mathrm{d}x} B_{n+1}F(x) \equiv (n+1) \sum_{k=0}^{n} P_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \qquad 0 \le x \le 1,$$

where F is an indefinite integral of f. It is well known that $\lim_{n\to\infty} B_n^\star f(x) = f(x)$ at any point x of $(0,1)$ where f is the derivative of its indefinite integral (see e.g. [8], Chap. II).

In this paper we consider the Kantorovich-type modification of the Chlodovsky operators (1). Namely, assuming that $f \in L_{loc}[0,\infty)$, that is f is locally integrable on $[0,\infty)$, and denoting

$$F(x) = \int_0^x f(t)dt \qquad \text{for} \quad x > 0,$$

we define the Chlodovsky-Kantorovich polynomial of degree $n-1$ as

$$K_{n-1}f(x) := \frac{\mathrm{d}}{\mathrm{d}x} C_nF(x), \qquad n \in N.$$

It is easy to verify that

$$K_{n-1}f(x) = \frac{n}{b_n} \sum_{k=0}^{n-1} P_{n-1,k}\left(\frac{x}{b_n}\right) \int_{\frac{kb_n}{n}}^{\frac{(k+1)b_n}{n}} f(t)dt, \qquad 0 \le x \le b_n, \tag{4}$$

(see [3], Section 4).

In order to formulate our first result let us consider those points $x \in (0,\infty)$ at which

$$\lim_{h\to 0}\frac{1}{h}\int_0^h (f(x+t)-f(x))\,dt = 0 \tag{5}$$

and let us introduce the pointwise characteristic

$$w_x(\delta; f) := \sup_{0<|h|\le\delta}\left|\frac{1}{h}\int_0^h (f(x+t)-f(x))\,dt\right|, \qquad \delta>0. \tag{6}$$

Clearly, $w_x(\delta; f)$ is a non-decreasing function of $\delta > 0$ and $\lim_{\delta\to 0^+} w_x(\delta; f) = 0$ almost everywhere on $[0,\infty)$, that is at every point $x \in (0,\infty)$ at which (5) is satisfied.

Theorem 1. *Let $f \in L_{loc}[0,\infty)$ and let at a fixed point $x \in (0,\infty)$ condition (5) be fulfilled. Then, for all integers n such that $b_n > 2x$, $\sqrt{n/b_n} \ge 3$, we have*

$$\begin{aligned}
&|K_{n-1}f(x) - f(x)| \\
&\le\ c(q)\left(1+\frac{\varphi_n^{q/2}(x)}{x^q}\right)\left(\frac{b_n}{n}\right)^{\frac{q-1}{2}}\sum_{k=1}^{[n/b_n]} k^{\frac{q-3}{2}} w_x\left(\frac{x}{\sqrt{k}}; f\right) \\
&+\frac{c(r)}{x^r}\varphi_n^{r/2}(x)\left(\frac{b_n}{n}\right)^{r/2}|f(x)| \\
&+\frac{c(r)}{x^r}\sqrt{\frac{b_n}{x(b_n-x)}}\varphi_n^{r/2}(x)\left(\frac{b_n}{n}\right)^{\frac{r-1}{2}}\int_0^{b_n}|f(t)|dt\,\exp\left(-\frac{nx}{8b_n}\right),
\end{aligned}$$

where q, r are arbitrary positive integers, $c(q)$ and $c(r)$ are positive numbers depending only on the indicated parameter q and r, respectively, $\varphi_n(x) = x(1-\frac{x}{b_n})+\frac{b_n}{n}$ and $[n/b_n]$ denotes the greatest integer not greater than n/b_n.

Taking into acount fundamental assumptions (2) and choosing in Theorem 1, $q = 3$, $r = 2$ we easily get

Corollary 1. *If $f \in L_{loc}[0,\infty)$ and if*

$$\lim_{n\to\infty}\int_0^{b_n}|f(t)|dt\,\exp\left(-\alpha\frac{n}{b_n}\right) = 0 \qquad \text{for} \quad \text{every} \quad \alpha > 0,$$

then

$$\lim_{n\to\infty} K_{n-1}f(x) = f(x) \qquad \text{almost everywhere on} \quad [0,\infty).$$

Now, let us consider the subclass $M_{loc}[0,\infty)$ consisting of all measurable functions f locally bounded on $[0,\infty)$. In this case

$$w_x(\delta; f) \le osc(f; I_x(\delta)) \equiv \sup_{u,v\in I_x(\delta)} |f(u) - f(v)|,$$

where $0 \le \delta \le x$, $I_x(\delta) := [x-\delta, x+\delta]$.

Theorem 2. *Let $f \in M_{loc}[0,\infty)$ and let at a fixed point $x \in (0,\infty)$ the one-sided limits $f(x+)$, $f(x-)$ exist. Then, for all integers n such that $b_n > 2x$, $\sqrt{n/b_n} \ge 3$, we have*

$$\begin{aligned}
&|K_{n-1}f(x) - \frac{1}{2}\left(f(x+) + f(x-)\right)| \\
&\le\; c(q)\left(1 + \frac{\varphi_n^{q/2}(x)}{x^q}\right)\left(\frac{b_n}{n}\right)^{\frac{q-1}{2}} \sum_{k=1}^{[n/b_n]} k^{\frac{q-3}{2}} osc\left(g_x; I_x\left(\frac{x}{\sqrt{k}}\right)\right) \\
&+ c\sqrt{\frac{b_n}{x(b_n-x)}}\varphi_n^{1/2}(x) M(b_n; f)\exp\left(-\frac{nx}{8b_n}\right) \\
&+ 2\sqrt{\frac{b_n}{n}}\sqrt{\frac{b_n}{x(b_n-x)}}\,|f(x+) - f(x-)|,
\end{aligned}$$

where $M(b_n; f) = \sup_{0\le t\le b_n} |f(t)|$, $\varphi_n(x) = x\left(1 - \frac{x}{b_n}\right) + \frac{b_n}{n}$,

$$g_x(t) := \begin{cases} f(t) - f(x+) & \text{if} \quad t > x, \\ 0 & \text{if} \quad t = x, \\ f(t) - f(x-) & \text{if} \quad 0 \le t < x, \end{cases} \tag{7}$$

q is an arbitrary positive integer, $c(q)$ is a positive constant depending only on q and c is a positive absolute constant.

The function g_x is continuous at x. Hence $\lim_{\delta\to 0+} osc(g_x; I_x(\delta)) = 0$. Consequently, Theorem 2 yields the following

Corollary 2. *If $f \in M_{loc}[0,\infty)$ and if at a fixed point $x \in (0,\infty)$ the limits $f(x+)$, $f(x-)$ exist, then under the Chlodovsky assumption* (3)*, we have*

$$\lim_{n\to\infty} K_{n-1}f(x) = \frac{1}{2}\left(f(x+) + f(x-)\right). \tag{8}$$

Remark. In particular, let us consider the class $BV_\Phi[0,\infty)$ of functions of bounded variation in the Young sense on the interval $[0,\infty)$ (for the definition see e.g. [4, 10]). If $f \in BV_\Phi[0,\infty)$, then $M(b_n; f) \le M$ ($M =$ const.). The estimation given in Theorem 2 and the relation (8) hold true at every point $x \in (0,\infty)$.

2. Auxiliary results

We now present certain results which will be used in the proof of our main theorems. For this, let us introduce the notation: given any fixed $x \in [0, b_n]$ and any non-negative integer q, we will write

$$\mu_{n,q}(x) := \sum_{k=0}^{n} \left(\frac{kb_n}{n} - x\right)^q P_{n,k}\left(\frac{x}{b_n}\right),$$

$$|\mu_{n,q}|(x) := \sum_{k=0}^{n} \left|\frac{kb_n}{n} - x\right|^q P_{n,k}\left(\frac{x}{b_n}\right).$$

Moreover, we will use the notation $c_j(p)$, $j = 1, 2, \ldots$, for positive constants, not necessarily the same at each occurrence, depending only on the indicated parameter p.

Lemma 1. *Let $n \in N$, $x \in [0, b_n]$.*

(i) $$\mu_{n,0}(x) = 1, \quad \mu_{n,1}(x) = 0, \quad \mu_{n,2}(x) = \frac{b_n}{n} x \left(1 - \frac{x}{b_n}\right).$$

(ii) *If $s \in N$, $n \ge 2$, then*

$$\mu_{n,2s}(x) \le c_1(s) \left(\frac{b_n}{n}\right)^s x\left(1 - \frac{x}{b_n}\right)\left(x(1 - \frac{x}{b_n}) + \frac{b_n}{n}\right)^{s-1}.$$

(iii) *If* $q \in N$, $q \geq 2$, $n \geq 2$, *then*

$$|\mu_{n,q}|(x) \leq c_2(q) \left(\frac{b_n}{n}\right)^{\frac{q}{2}} x \left(1 - \frac{x}{b_n}\right) \left(x \left(1 - \frac{x}{b_n}\right) + \frac{b_n}{n}\right)^{\frac{q}{2}-1}.$$

Proof. Formulas (i) follow by easy calculation. Suppose $s > 1$ and put $y := x/b_n$. Then $y \in [0,1]$ and

$$\mu_{n,2s}(x) = \left(\frac{b_n}{n}\right)^{2s} \sum_{k=0}^{n} (k - ny)^{2s} P_{n,k}(y).$$

Applying the known represetation formula for the above sum (see [6], Lemma 3.6 with $c = -1$) we obtain

$$\mu_{n,2s}(x) = \left(\frac{b_n}{n}\right)^{2s} \sum_{j=1}^{s} \beta_{j,s} \left(ny(1-y)\right)^j,$$

where $\beta_{j,s}$ are real numbers independent of y and bounded uniformly in n. Now, let us observe that for $y \in [0, \frac{1}{n}]$ or $y \in [1 - \frac{1}{n}, 1]$ one has $ny(1-y) \leq \frac{n-1}{n} < 1$ and

$$\left| \sum_{j=1}^{s} \beta_{j,s} (ny(1-y))^j \right| \leq\ ny(1-y) \sum_{j=1}^{s} |\beta_{j,s}|.$$

If $y \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ then $\left(ny(1-y)\right)^{-1} \leq \frac{n}{n-1} \leq 2$ and

$$\begin{aligned} \left| \sum_{j=1}^{s} \beta_{j,s} (ny(1-y))^j \right| &\leq\ \left(ny(1-y)\right)^s \sum_{j=1}^{s} |\beta_{j,s}| \left(ny(1-y)\right)^{j-s} \\ &\leq\ \left(ny(1-y)\right)^s \sum_{j=1}^{s} 2^{s-j} |\beta_{j,s}|. \end{aligned}$$

Consequently, for all $y \in [0,1]$ (that is for all $x \in [0, b_n]$) we have

$$\mu_{n,2s}(x) \leq c_1(s) \left(\frac{b_n}{n}\right)^{2s} ny(1-y) \left(1 + ny(1-y)\right)^{s-1}$$

with

$$c_1(s) \geq \sum_{j=1}^{s} 2^{s-j} |\beta_{j,s}|.$$

Inequality (ii) follows by taking $y = x/b_n$. The same estimation holds true for $|\mu_{n,q}|(x)$ with even q $(q = 2s)$. If q is odd $(q = 2s+1)$, then

$$|\mu_{n,q}|(x) \leq (\mu_{n,4s}(x))^{\frac{1}{2}}(\mu_{n,2}(x))^{\frac{1}{2}}$$

by Cauchy-Schwarz inequality, and the proof is complete. ■

Lemma 2. *If* $n \in N$, $0 < x < b_n$, *then*

$$\begin{aligned} &\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)K_{n-1}f(x) \\ &= \frac{n}{b_n^2}\sum_{k=0}^{n}\left(\frac{kb_n}{n}-x\right)P_{n,k}\left(\frac{x}{b_n}\right)\int_0^{\frac{kb_n}{n}-x} f(x+t)dt. \end{aligned} \tag{9}$$

Proof. By (4) and by partial summation, we find that

$$\begin{aligned} K_{n-1}f(x) &= \frac{n}{b_n}\sum_{k=0}^{n-1}P_{n-1,k}\left(\frac{x}{b_n}\right)\int_{\frac{kb_n}{n}}^{\frac{(k+1)}{n}b_n} f(t)dt = \\ &= \frac{n}{b_n}P_{n-1,n-1}\left(\frac{x}{b_n}\right)\int_0^{b_n} f(t)dt \\ &+ \frac{n}{b_n}\sum_{k=1}^{n-1}\left(P_{n-1,k-1}\left(\frac{x}{b_n}\right)-P_{n-1,k}\left(\frac{x}{b_n}\right)\right)\int_0^{\frac{kb_n}{n}} f(t)dt. \end{aligned}$$

Putting $y = x/b_n$ and observing that

$$y(1-y)\left(P_{n-1,k-1}(y)-P_{n-1,k}(y)\right) = \left(\frac{k}{n}-y\right)P_{n,k}(y)$$

for $k = 1, 2, \ldots, n-2$ and

$$y(1-y)n\,P_{n-1,n-1}(y) = yP_{n,n-1}(y) = n(1-y)P_{n,n}(y),$$

we easily get

$$\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)K_{n-1}f(x) = \frac{n}{b_n}\sum_{k=0}^{n}\left(\frac{k}{n}-\frac{x}{b_n}\right)P_{n,k}\left(\frac{x}{b_n}\right)\int_0^{\frac{kb_n}{n}} f(t)dt.$$

Now, it is enough to recall that

$$\sum_{k=0}^{n}\left(\frac{kb_n}{n}-x\right)P_{n,k}\left(\frac{x}{b_n}\right)=\mu_{n,1}(x)=0$$

(Lemma 1 (i)). Consequently,

$$\frac{x}{b_n}(1-\frac{x}{b_n})K_{n-1}f(x)=\frac{n}{b_n^2}\sum_{k=0}^{n}\left(\frac{kb_n}{n}-x\right)P_{n,k}\left(\frac{x}{b_n}\right)\int_x^{\frac{kb_n}{n}}f(t)dt$$

and the proof is complete. ■

Note that a corresponding representation like in the formula (9) for the classical Kantorovich polynomials is given in [2].

Lemma 3. *If* $0<\delta\le x<b_n$ *then*

$$\sum_{|\frac{kb_n}{n}-x|\ge\delta}P_{n,k}\left(\frac{x}{b_n}\right)\le 2\exp\left(-\frac{n\delta^2}{4xb_n}\right)$$

for all $n\in N$ *such that* $b_n\ge\frac{3x^2}{3x-\delta}$.

The proof of Lemma 3 runs as in [1] and is based on the known Chlodovsky inequality ([8], Theorem 1.5.3)):

$$\sum_{|k-nt|\ge 2z\sqrt{nt(1-t)}}P_{n,k}(t)\le 2\exp\left(-z^2\right),$$

provided that $0\le t\le 1$, $0\le z\le\frac{3}{2}\sqrt{nt(1-t)}$.

Lemma 4. *Let* $0<x<b_n$ *and let* $n\ge 2$.
(i) *If* $0\le k\le n-1$, *then*

$$P_{n-1,k}(\frac{x}{b_n})\le\frac{1}{\sqrt{e}}\sqrt{\frac{b_n}{n}}\sqrt{\frac{b_n}{x(b_n-x)}}.$$

(ii)

$$\left| \sum_{\frac{nx}{b_n} < k \le n} P_{n-1,k}\left(\frac{x}{b_n}\right) - \frac{1}{2} \right| \le 0.82\sqrt{2}\sqrt{\frac{b_n}{n}}\sqrt{\frac{b_n}{x(b_n - x)}}.$$

Proof. Estimation (i) follows from the result by X.M. Zeng [11] (Theorem 1): if $0 \le k \le n$ and $y \in (0,1)$, then

$$P_{n,k}(y) \le \frac{1}{\sqrt{2e}} \frac{1}{\sqrt{ny(1-y)}}.$$

Inequality (ii) is an immediate consequence of the Berry-Esséen Theorem:

$$\left| \sum_{\frac{k}{n} > y} P_{n,k}(y) - \frac{1}{2} \right| < \frac{0.82}{\sqrt{ny(1-y)}}, \quad 0 < y < 1$$

(see e.g., [12], Lemma 2). ■

3. PROOFS OF THEOREMS

***Proof of Theorem* 1**. In view of Lemma 1 (i) one can write

$$\begin{aligned}
&\frac{x}{b_n}\left(1 - \frac{x}{b_n}\right) f(x) = \frac{n}{b_n^2}\mu_{n,2}(x) f(x) \\
&= \frac{n}{b_n^2}\sum_{k=0}^{n}\left(\frac{kb_n}{n} - x\right) P_{n,k}\left(\frac{x}{b_n}\right)\int_0^{\frac{kb_n}{n}-x} dt f(x).
\end{aligned}$$

The above identity and the representation (9) lead to

$$\begin{aligned}
&\frac{x}{b_n}\left(1 - \frac{x}{b_n}\right)(K_{n-1}f(x) - f(x)) \\
&= \frac{n}{b_n^2}\sum_{k=0}^{n}\left(\frac{kb_n}{n} - x\right) P_{n,k}\left(\frac{x}{b_n}\right)\int_0^{\frac{kb_n}{n}-x} (f(x+t) - f(x))\, dt \qquad (10) \\
&\equiv \sum_{k\in\Lambda} + \sum_{k\in\Omega},
\end{aligned}$$

where Λ and Ω are the sets of indices $k \in \{0, 1, \dots, n\}$ such that $|\frac{kb_n}{n} - x| \le x$ and $\frac{kb_n}{n} - x > x$, respectively.

For the sake of brevity let us introduce the notation: $d_n = \sqrt{b_n/n}$, $m = [\sqrt{n/b_n}]$, $w_x(\delta; f) = w_x(\delta)$. Consider the sum $\sum_{k\in\Lambda}$ and divide the set Λ in the following manner: $\Lambda = \bigcup_{j=0}^{m} \Lambda_j$, where Λ_j are the sets of indices k such that

$$0 \le \left| \frac{kb_n}{n} - x \right| \le x d_n \quad \text{if} \quad j = 0,$$

$$j x d_n < \left| \frac{kb_n}{n} - x \right| \le (j+1) x d_n \quad \text{if} \quad j = 1, 2, \dots, m-1,$$

$$m x d_n < \left| \frac{kb_n}{n} - x \right| \le x \quad \text{if} \quad j = m.$$

In view of definition (6),

$$\left| \sum_{k\in\Lambda} \right| \le \sum_{j=0}^{m-1} T_{n,j}(x) w_x((j+1)xd_n) + T_{n,m}(x) w_x(x),$$

where

$$T_{n,j}(x) := \frac{n}{b_n^2} \sum_{k\in\Lambda_j} \left(\frac{kb_n}{n} - x \right)^2 P_{n,k}\left(\frac{x}{b_n} \right).$$

From Lemma 1 (i) one has

$$T_{n,0}(x) \le \frac{n}{b_n^2} \mu_{n,2}(x) = \frac{x}{b_n} \left(1 - \frac{x}{b_n} \right).$$

Next, given any positive integer q, we have

$$T_{n,j}(x) \le \frac{n}{b_n^2} \frac{1}{(j x d_n)^q} \sum_{k=0}^{n} \left| \frac{kb_n}{n} - x \right|^{q+2} P_{n,k}\left(\frac{x}{b_n} \right)$$

for $j = 1, 2, \dots, m$. Hence Lemma 1 (iii) yields

$$T_{n,j}(x) \le \frac{c_1(q)}{j^q x^q} \frac{x}{b_n} \left(1 - \frac{x}{b_n} \right) \varphi_n^{q/2}(x)$$

where $\varphi_n(x) = x\left(1-\frac{x}{b_n}\right)+\frac{b_n}{n}$. Consequently,

$$\left|\sum_{k\in\Lambda}\right| \le \frac{x}{b_n}\left(1-\frac{x}{b_n}\right)\left(1+\frac{c_1(q)}{x^q}\varphi_n^{q/2}(x)\right)\left(\sum_{j=1}^{m-1}\frac{w_x((j+1)xd_n)}{j^q}+\frac{w_x(x)}{m^q}\right).$$

Clearly,

$$\begin{aligned}\sum_{j=1}^{m-2}\frac{w_x((j+1)xd_n)}{j^q} &\le 3^q d_n^{q-1}\int_{2d_n}^{md_n}\frac{w_x(xt)}{t^q}dt\\ &\le 3^q d_n^{q-1}\int_1^{m^2}(\sqrt{s})^{q-3}w_x\left(\frac{x}{\sqrt{s}}\right)ds\\ &\le c_2(q)d_n^{q-1}\sum_{k=1}^{m^2-1}(\sqrt{k+1})^{q-3}w_x\left(\frac{x}{\sqrt{k}}\right)\end{aligned}$$

and

$$\frac{w_x(x)}{(m-1)^q}+\frac{w_x(x)}{m^q}\le\frac{2}{(m-1)^q}w_x(x)\le 3^q d_n^{q-1}w_x(x).$$

Hence

$$\left|\sum_{k\in\Lambda}\right|\le c_3(q)\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)\left(1+\frac{\varphi_n^{q/2}(x)}{x^q}\right)d_n^{q-1}\sum_{k=1}^{m^2-1}\left(\sqrt{k}\right)^{q-3}w_x\left(\frac{x}{\sqrt{k}}\right). \tag{11}$$

Now, let us consider the sum $\sum_{k\in\Omega}$ in formula (10). Given any positive integer r, we have

$$\begin{aligned}\left|\sum_{k\in\Omega}\right| &\le \frac{n}{b_n^2x^r}\sum_{k\in\Omega}\left|\frac{kb_n}{n}-x\right|^{r+2}P_{n,k}\left(\frac{x}{b_n}\right)|f(x)|\\ &+\frac{n}{b_n^2x^r}\sum_{k\in\Omega}\left|\frac{kb_n}{n}-x\right|^{r+1}P_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n}|f(t)|dt\\ &\le \frac{n}{b_n^2x^r}|\mu_{n,r+2}|(x)\,|f(x)|\\ &+\frac{n}{b_n^2x^r}\int_0^{b_n}|f(t)|dt\,(\mu_{n,2r+2}(x))^{1/2}\left(\sum_{k\in\Omega}P_{n,k}\left(\frac{x}{b_n}\right)\right)^{1/2}.\end{aligned}$$

Applying Lemmas 1 and 3 we then get

$$\left|\sum_{k\in\Omega}\right| \le \frac{c_4(r)}{x^r}\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)\left(\frac{b_n}{n}\right)^{r/2}\varphi_n^{r/2}(x)|f(x)|$$

$$+\frac{c_4(r)}{x^r}\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)\left(\frac{b_n}{n}\right)^{\frac{r}{2}-\frac{1}{2}}\sqrt{\frac{b_n}{x(b_n-x)}}\,\varphi_n^{r/2}(x)\int_0^{b_n}|f(t)|dt\,\exp\left(-\frac{nx}{8b_n}\right).$$

This gives the desired conclusion when combined with (10) and (11). ■

***Proof of Theorem* 2**. Let $f \in M_{loc}[0,\infty)$ and let the limits $f(x+)$, $f(x-)$ exist at a fixed point $x > 0$. Consider the function g_x defined by (7). It is easily seen that

$$f(t) - \frac{f(x+)+f(x-)}{2} = g_x(t) + \frac{f(x+)-f(x-)}{2}\operatorname{sgn}_x(t)$$
$$+\left(f(x) - \frac{f(x+)+f(x-)}{2}\right)\delta_x(t),$$

where $\operatorname{sgn}_x(t) = \operatorname{sgn}(t-x)$, $\delta_x(t) = 1$ if $t = x$, $\delta_x(t) = 0$ otherwise (see e.g. [9]). Hence

$$\begin{aligned} &K_{n-1}f(x) - \frac{f(x+)+f(x-)}{2} \\ &= K_{n-1}\,g_x(x) + \frac{f(x+)-f(x-)}{2}K_{n-1}\operatorname{sgn}_x(x). \end{aligned} \tag{12}$$

The function g_x is continuous at x and $g_x(x) = 0$. So, $K_{n-1}\,g_x(x) = K_{n-1}\,g_x(x) - g_x(x)$ can be estimated as in the proof of Theorem 1. Namely, using formula (10) in which f is replaced by g_x and observing that

$$w_x(\delta; g_x) \le osc\,(g_x; I_x(\delta)) \quad \text{for} \quad 0 < \delta \le x$$

we get the estimation for $\left|\sum_{k\in\Lambda}\right|$ as in (11) with $w_x\left(\frac{x}{\sqrt{k}}\right)$ replaced by $osc\left(g_x; I_x\left(\frac{x}{\sqrt{k}}\right)\right)$. Indeed, we estimate the sum $\sum_{k\in\Omega}$ as follows:

$$\left|\sum_{k\in\Omega}\right| \le \frac{2n}{b_n^2}M(b_n;f)\sum_{k\in\Omega}\left(\frac{kb_n}{n}-x\right)^2 P_{n,k}\left(\frac{x}{b_n}\right),$$

where $M(b_n; f) = \sup_{0\le t\le b_n} |f(t)|$. Next, the Cauchy-Schwarz inequality and Lemmas 1, 3 lead to

$$\begin{aligned}\left|\sum_{k\in\Omega}\right| &\le 2M(b_n;f)\frac{n}{b_n^2}\left(\mu_{n,4}(x)\right)^{1/2}\left(2\exp\left(-\frac{nx}{4b_n}\right)\right)^{1/2}\\ &\le c\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)\sqrt{\frac{b_n}{x(b_n-x)}}\varphi_n^{1/2}(x)M(b_n;f)\exp\left(-\frac{nx}{8b_n}\right),\end{aligned}$$

where c is an absolute positive constant. Consequently,

$$\begin{aligned}&|K_{n-1}g_x(x)| \le\\ &\le c(q)\left(1+\frac{\varphi_n^{q/2}(x)}{x^q}\right)\left(\frac{b_n}{n}\right)^{\frac{q-1}{2}}\sum_{k=1}^{[n/b_n]}(\sqrt{k})^{q-3}osc\left(g_x; I_x\left(\frac{x}{\sqrt{k}}\right)\right)\\ &+c\sqrt{\frac{b_n}{x(b_n-x)}}\varphi_n^{1/2}(x)M(b_n;f)\exp\left(-\frac{nx}{8b_n}\right),\end{aligned}$$

where q is arbitrary positive integer, $c(q)$ is a positive constant depending only on q and c is an absolute constant.

Now it is enough to estimate the term $K_{n-1}\mathrm{sgn}_x(x)$. Choose the integer l such that $x \in [\frac{l}{n}b_n, \frac{l+1}{n}b_n)$. It is clear that

$$\begin{aligned}K_{n-1}\mathrm{sgn}_x(x) &= \sum_{k>l}P_{n-1,k}\left(\frac{x}{b_n}\right)-\sum_{k<l}P_{n-1,k}\left(\frac{x}{b_n}\right)\\ &+\frac{n}{b_n}P_{n-1,l}\left(\frac{x}{b_n}\right)\left(2\frac{l}{n}b_n+\frac{b_n}{n}-2x\right)\\ &= 2\sum_{k>l}P_{n-1,k}\left(\frac{x}{b_n}\right)-1+2P_{n-1,l}\left(\frac{x}{b_n}\right)\frac{n}{b_n}\left(\frac{l+1}{n}b_n-x\right).\end{aligned}$$

Therefore,

$$\begin{aligned}|K_{n-1}\,\mathrm{sgn}_x(x)| &\le 2\left|\sum_{k>l}P_{n-1,k}\left(\frac{x}{b_n}\right)-\frac{1}{2}\right|+2P_{n-1,l}\left(\frac{x}{b_n}\right)\\ &\le 4\sqrt{\frac{b_n}{n}}\sqrt{\frac{b_n}{x(b_n-x)}},\end{aligned}$$

by Lemma 4. Combining the above estimations for $|K_{n-1}g_x(x)|$ and $|K_{n-1}\mathrm{sgn}_x(x)|$ with (12) we obtain the desired conclusion. Thus the proof of Theorem 2 is complete. ■

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