

POSITIVITY AND STABILIZATION OF 2D LINEAR SYSTEMS

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Abstract

The problem of finding a gain matrix of the state-feedback of 2D linear system such that the closed-loop system is positive and asymptotically stable is formulated and solved. Necessary and sufficient conditions for the solvability of the problem are established. It is shown that the problem can be reduced to suitable linear programming problem. The proposed approach can be extended to 2D linear system described by the 2D Roesser model.

Keywords: linear 2D systems, general model, positivity, stabilization, state-feedback.

1. INTRODUCTION

The most popular models of two-dimensional (2D) linear systems are the models introduced by Roesser [25], Fornasini-Marchesini [8, 9] and Kurek [24]. The models have been extended for positive systems in [21, 27, 22, 14]. An overview of 2D linear systems theory is given in [1, 2, 10, 11, 13], and some recent results in positive systems have been given in monographs [6, 14]. Reachability and minimum energy control of positive 2D systems with one delay in states have been considered in [22]. The choice of the Lyapunov functions for positive 2D Roesser model has been investigated in [20]. The internal stability and asymptotic behavior of 2D positive systems have been investigated by Valcher in [27]. The stability of 2D positive systems described by the Roesser model and synthesis of state-feedback controllers have been considered in the paper [12]. The asymptotic stability of positive

1D and 2D linear systems has been investigated in [4, 15–18, 26] and the stability of fractional systems with delays in [3]. The concept of practical stability of positive fractional discrete-time linear systems has been introduced in [19].

In this paper it will be shown that the problem of finding a gain matrix of the state-feedback such that the closed-loop system is positive and asymptotically stable can be reduced to suitable linear programming problem. The paper is organized as follows. In Section 2 the basic definitions and theorem concerning the 2D general model and Roesser model are recalled. The main result of the paper is presented in Section 3. Necessary and sufficient conditions for the solvability of the problem are established and a procedure for computation of the gain matrix is given and illustrated by a numerical example. Concluding remarks are given in Section 4.

The following notation will be used.

Let $\mathfrak{R}^{n \times m}$ be the set of real matrices with n rows and m columns and $\mathfrak{R}^n = \mathfrak{R}^{n \times 1}$. The set of real $n \times m$ matrices with nonnegative entries will be denoted by $\mathfrak{R}_+^{n \times m}$ and $\mathfrak{R}_+^n = \mathfrak{R}_+^{n \times 1}$. A matrix $A = [a_{ij}] = \mathfrak{R}_+^{n \times m}$ (a vector $x = [x_i] = \mathfrak{R}_+^n$) will be called strictly positive and denoted by $A > 0$ if $a_{ij} > 0$ for $i = 1, 2, \dots, n, j = 1, 2, \dots, m$ (by $x > 0$ if $x_i > 0$ for $i = 1, 2, \dots, n$). The set of nonnegative integers will be denoted by Z_+ .

2. POSITIVE 2D SYSTEMS

Consider the general model of 2D linear systems

$$(1a) \quad x_{i+1,j+1} = A_0 x_{i,j} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_0 u_{i,j} + B_1 u_{i+1,j} + B_2 u_{i,j+1}$$

$$(1b) \quad y_{i,j} = C x_{i,j} + D u_{i,j} \quad i, j \in Z_+,$$

where $x_{i,j} \in \mathfrak{R}^n, u_{i,j} \in \mathfrak{R}^m, y_{i,j} \in \mathfrak{R}^p$ are the state, input and output vectors at the point (i, j) and $A_k \in \mathfrak{R}^{n \times n}, B_k \in \mathfrak{R}^{n \times m}, k = 0, 1, 2, C \in \mathfrak{R}^{p \times n}, D \in \mathfrak{R}^{p \times m}$. Boundary conditions for (1a) have the form

$$(2) \quad x_{i,0} \in \mathfrak{R}^n, \quad i \in Z_+ \quad \text{and} \quad x_{0,j} \in \mathfrak{R}^n, \quad j \in Z_+.$$

The model (1) is called (internally) positive if $x_{i,j} \in \mathfrak{R}_+^n$ and $y_{i,j} \in \mathfrak{R}_+^p, i, j \in Z_+$ for all boundary conditions $x_{i,0} \in \mathfrak{R}_+^n, i \in Z_+, x_{0,j} \in \mathfrak{R}_+^n, j \in Z_+$ and every input sequence $u_{i,j} \in \mathfrak{R}_+^m, i, j \in Z_+$.

Theorem 1 [14]. *The system (1) is positive if and only if*

$$(3) \quad A_k \in \mathfrak{R}_+^{n \times m}, B_k \in \mathfrak{R}_+^{n \times m}, k = 0, 1, 2, C \in R_+^{p \times n}, D \in \mathfrak{R}_+^{p \times m}.$$

Substituting in (1a) $B_1 = B_2 = 0$ and $B_0 = 0$, we obtain the first Fornasini-Marchesini model (FF-MM) and substituting in (1a) $A_0 = 0$ and $B_0 = 0$ we obtain the second Fornasini-Marchesini model (SF-MM). The Roesser model of 2D linear systems has the form

$$(4a) \quad \begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} u_{i,j}$$

$$(4b) \quad y_{i,j} = [C_1 \ C_2] \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix} + D u_{i,j}, \quad i, j \in Z_+$$

where $x_{i,j}^h \in \mathfrak{R}^{n_1}$ and $x_{i,j}^v \in \mathfrak{R}^{n_2}$ are the horizontal and vertical state vectors at the point (i, j) , $u_{i,j} \in \mathfrak{R}^m$ and $y_{i,j} \in \mathfrak{R}^p$ are the input and output vectors and $A_{kl} \in \mathfrak{R}^{n_k \times n_l}$, $k, l = 1, 2$, $B_{11} \in \mathfrak{R}^{n_1 \times m}$, $B_{22} \in \mathfrak{R}^{n_2 \times m}$, $C_1 \in \mathfrak{R}^{p \times n_1}$, $C_2 \in \mathfrak{R}^{p \times n_2}$, $D \in \mathfrak{R}^{p \times m}$. Boundary conditions for (4a) have the form

$$(5) \quad x_{0,j}^h \in \mathfrak{R}^{n_1}, j \in Z_+ \quad \text{and} \quad x_{i,0}^v \in \mathfrak{R}^{n_2}, i \in Z_+.$$

The model (4) is called (internally) positive Roesser model if $x_{i,j}^h \in \mathfrak{R}_+^{n_1}$, $x_{i,j}^v \in \mathfrak{R}_+^{n_2}$, and $y_{i,j} \in \mathfrak{R}_+^p$, $i, j \in Z_+$ for any nonnegative boundary conditions

$$(6) \quad x_{0,j}^h \in \mathfrak{R}_+^{n_1}, j \in Z_+ \quad \text{and} \quad x_{i,0}^v \in \mathfrak{R}_+^{n_2}, i \in Z_+$$

and all input sequences $u_{i,j} \in \mathfrak{R}_+^m$, $i, j \in Z_+$.

Theorem 2 [14]. *The system Roesser model is positive if and only if*

$$(7) \quad \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathfrak{R}_+^{n \times n}, \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} \in \mathfrak{R}_+^{n \times m}, [C_1 \ C_2] \in \mathfrak{R}_+^{p \times n}, \\ D \in \mathfrak{R}_+^{p \times m}, \quad n = n_1 + n_2.$$

Defining

$$(8) \quad x_{i,j} = \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad A_2 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ B_{22} \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}$$

we may write the Roesser model in the form of SF-MM

$$(9) \quad x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_1 u_{i+1,j} + B_2 u_{i,j+1}.$$

The positive general model (1a) is called asymptotically stable if for any bounded boundary conditions $x_{i,0} \in \mathfrak{R}_+^n$, $i \in Z_+$, $x_{0,j} \in \mathfrak{R}_+^n$, $j \in Z_+$ and zero inputs $u_{i,j} = 0$, $i, j \in Z_+$ the following condition holds

$$(10) \quad \lim_{i,j \rightarrow \infty} x_{i,j} = 0 \quad \text{for all } x_{i,0} \in \mathfrak{R}_+^n, \quad x_{0,j} \in \mathfrak{R}_+^n, \quad i, j \in Z_+.$$

Theorem 3. *The positive general model (1) is asymptotically stable if and only if there exists a strictly positive vector λ such that*

$$(11) \quad (A_0 + A_1 + A_2 - I_n)\lambda < 0.$$

Proof is given in [16, 18].

3. MAIN RESULT

Consider the general 2D model (1) with not necessarily nonnegative matrices A_0, A_1 and A_2 . We are looking for a gain matrix $K \in \mathfrak{R}^{m \times n}$ of the state-feedback

$$(12) \quad u_{ij} = Kx_{ij}$$

such that the closed-loop system

$$(13) \quad x_{i+1,j+1} = (A_0 + B_0K)x_{ij} + (A_1 + B_1K)x_{i+1,j} + (A_2 + B_2K)x_{i,j+1}$$

is positive and asymptotically stable.

Theorem 4. *The closed-loop system (13) is positive and asymptotically stable if and only if there exist a strictly positive vector*

$$(14) \quad \lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \quad (\lambda_k > 0, \quad k = 1, \dots, n)$$

and n real vectors

$$(15) \quad d_k \in \mathfrak{R}^m, \quad k = 1, \dots, n, \quad d = \sum_{k=1}^n d_k$$

such that

$$(16a) \quad a_{ij}^t + b_i^t d_j \geq 0 \quad \text{for } t = 0, 1, 2 \quad \text{and } 1 \leq i \leq n, \quad 1 \leq j \leq n$$

$$(16b) \quad (A_0 + A_1 + A_2 - I_n)\lambda + (B_0 + B_1 + B_2)d < 0,$$

where

$$(16c) \quad A_t = [a_{ij}^t] \in \mathfrak{R}^{n \times n}, \quad B_t = \begin{bmatrix} b_1^t \\ \vdots \\ b_n^t \end{bmatrix} \in \mathfrak{R}^{n \times m},$$

$$t = 0, 1, 2; \quad 1 \leq i \leq n, \quad 1 \leq j \leq n.$$

The gain matrix K has the form

$$(17) \quad K = \begin{bmatrix} d_1 \\ \lambda_1 \\ \dots \\ d_n \\ \lambda_n \end{bmatrix} \in \mathfrak{R}^{m \times n}.$$

Proof. First we shall show that the closed-loop system (13) is positive if and only if (16a) holds. Using (16c) and (17) for (i, j) entry of the matrix $(A_t + B_t K)$ we have

$$(A_t + B_t K)_{ij} = a_{ij}^t + b_i^t \frac{d_j}{\lambda_j} = (a_{ij}^t \lambda_j + b_i^t d_j) \frac{1}{\lambda_j} \geq 0$$

since (16a) holds. Hence

$$(18) \quad \bar{A}_t = A_t + B_t K \in \mathfrak{R}_+^{n \times n} \quad \text{for } t = 0, 1, 2$$

if and only if (16a) is satisfied.

Note that

$$(19) \quad (B_0 + B_1 + B_2)d = (B_0 + B_1 + B_2)K\lambda$$

since

$$K\lambda \begin{bmatrix} \frac{d_1}{\lambda_1}, \dots, \frac{d_n}{\lambda_n} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = \sum_{k=1}^n d_k = d.$$

If (16b) holds then using (19) we obtain

$$(20) \quad \begin{aligned} & (A_0 + A_1 + A_2 - I_n)\lambda + (B_0 + B_1 + B_2)d \\ &= (A_0 + B_0K + A_1 + B_1K + A_2 + B_2K - I_n)\lambda < 0. \end{aligned}$$

By Theorem 3 the positive closed-loop system (13) is asymptotically stable if and only if the conditions of Theorem 4 are satisfied. \blacksquare

Note that the problem of finding gain matrix K of the state-feedback (12) such that the closed-loop system is positive and asymptotically stable has been reduced to linear programming problem. Therefore, the problem can be solved by using the well-known linear programming methods [5, 7].

Using the matrices

$$(21) \quad D = [d_1 \dots d_n] \in \mathfrak{R}^{m \times n}, \quad \Lambda = \text{diag}[\lambda_1, \dots, \lambda_n]$$

we can write the gain matrix (17) in the form

$$(22) \quad K = D\Lambda^{-1}.$$

Substitution of (22) into (18) and postmultiplication by Λ yields

$$(23) \quad \hat{A}_t = A_t\Lambda + B_tD \in \mathfrak{R}_+^{n \times n} \quad \text{for } t = 0, 1, 2.$$

Note that the inequality (20) can be rewritten as

$$(24) \quad [(A_0 + A_1 + A_2)\Lambda + (B_0 + B_1 + B_2)D]\mathbf{1}_n < \Lambda\mathbf{1}_n$$

since for $\mathbf{1}_n = [1 \dots 1]^T \in \mathfrak{R}^n$

$$(25) \quad \Lambda \mathbf{1}_n = \lambda \quad \text{and} \quad D \mathbf{1}_n = d.$$

Substitution of (22) in (18) yields the relationship

$$(26) \quad \bar{A}_t = \hat{A}_t \Lambda^{-1}, \quad t = 0, 1, 2.$$

To solve the problem the following procedure can be used

Procedure

Step 1. Choose a diagonal matrix Λ with strictly positive diagonal entries and a real matrix D defined by (21) satisfying the conditions (23) and (24).

Step 2. Using the formula (22) find the gain matrix K .

Remark. If the 2D general model is unstable and there exist a diagonal matrix $\Lambda \in \mathfrak{R}_+^{n \times n}$ with strictly positive diagonal entries and a nonnegative matrix $D \in \mathfrak{R}_+^{n \times m}$ satisfying the conditions (23) and (24) then the model can be stabilized by positive gain matrix $K = D\Lambda^{-1} \in \mathfrak{R}_+^{m \times n}$.

Example 1. Let the unstable positive 2D general model (1) with the matrices

$$A_0 = \begin{bmatrix} 0.4 & 0.7 \\ 0.2 & 0.3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.3 & 0.5 \\ 0.4 & 0.9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.2 & 0.3 \\ 0.3 & 0.4 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0.6 \\ 0.3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.5 \\ 0.8 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix}$$

be given.

Find a gain matrix $K \in \mathfrak{R}^{1 \times 2}$ such that the closed-loop system is positive and asymptotically stable.

Using Procedure we obtain the following.

Step 1. We choose

$$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad D = [-1 \quad -2]$$

and we check the conditions (23)

$$\hat{A}_0 = A_0\Lambda + B_0D = \begin{bmatrix} 0.4 & 0.7 \\ 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0.6 \\ 0.3 \end{bmatrix} [-1 \ -2] = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0 \end{bmatrix}$$

$$\hat{A}_1 = A_1\Lambda + B_1D = \begin{bmatrix} 0.3 & 0.5 \\ 0.4 & 0.9 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.8 \end{bmatrix} [-1 \ -2] = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}$$

$$\hat{A}_2 = A_2\Lambda + B_2D = \begin{bmatrix} 0.2 & 0.3 \\ 0.3 & 0.4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix} [-1 \ -2] = \begin{bmatrix} 0.1 & 0 \\ 0.2 & 0 \end{bmatrix}$$

and the condition (24)

$$\begin{aligned} & [(A_0 + A_1 + A_2)\Lambda + (B_0 + B_1 + B_2)D]\mathbf{1}_n = \\ & = \left\{ \begin{bmatrix} 0.9 & 1.5 \\ 0.9 & 1.6 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1.4 \\ 1.5 \end{bmatrix} [-1 \ -2] \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ & = \begin{bmatrix} 0.6 \\ 0.5 \end{bmatrix} < \Lambda\mathbf{1}_n = \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \end{aligned}$$

Therefore, the conditions are satisfied.

Step 2. Using (22) we obtain the gain matrix of the form

$$K = D\Lambda^{-1} = [-0.5 \ -1].$$

The closed-loop system is positive since the matrices

$$\bar{A}_0 = A_0 + B_0K = \hat{A}_0\Lambda^{-1} = \begin{bmatrix} 0.1 & 0.1 \\ 0.05 & 0 \end{bmatrix},$$

$$\bar{A}_1 = A_1 + B_1K = \hat{A}_1\Lambda^{-1} = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$\bar{A}_2 = A_2 + B_2K = \hat{A}_2\Lambda^{-1} = \begin{bmatrix} 0.05 & 0 \\ 0.1 & 0 \end{bmatrix}$$

have nonnegative entries.

The closed-loop system is asymptotically stable since the condition (24) holds.

4. CONCLUDING REMARKS

The problem of finding a gain matrix of the state-feedback of 2D general model such that the closed-loop system is positive and asymptotically stable has been addressed. Necessary and sufficient conditions for the solvability of the problem have been established. It has been shown that the problem can be reduced to corresponding linear programming problem and can be solved using the well-known methods. A procedure for computation of the gain matrix has been proposed and illustrated by a numerical example. These considerations can be easily extended for 2D linear system with delay in state vector. An extension for 2D linear fractional systems is also possible.

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REFERENCES

- [1] N.K. Bose, Applied Multidimensional System Theory (Van Nostrand Reinhold Co, New York, 1982).
- [2] N.K. Bose, Multidimensional Systems Theory Progress, Directions and Open Problems (D. Reidel Publishing Co., 1985).
- [3] M. Busłowicz, *Stability of linear continuous-time fractional order systems with delays of the retarded type*, Bull. Pol. Acad. Sci. Techn. **56** (2008), 319–324.
- [4] M. Busłowicz, *Simple stability conditions for linear positive discrete-time systems with delays*, Bull. Pol. Acad. Sci. Techn. **56** (2008), 325–328.
- [5] H.A. Eiselt and C.L. Sandblom, Linear Programming and its Applications (Springer, Berlin, 2007).
- [6] L. Farina and S. Rinaldi, Positive Linear Systems; Theory and Applications (J. Wiley, New York, 2000).
- [7] M.C. Ferris, O.L. Mangasarian and S.J. Wright, Linear Programming with Matlab (SIAM, 2007).
- [8] E. Fornasini and G. Marchesini, *State-space realization theory of two-dimensional filters*, IEEE Trans, Autom. Contr. **AC-21** (1976), 481–491.
- [9] E. Fornasini and G. Marchesini, *Double indexed dynamical systems*, Math. Sys. Theory **12** (1978), 59–72.
- [10] K. Galkowski, *Elementary operation approach to state space realization of 2D systems*, IEEE Trans. On Circuit and Systems **44** (1997), 120–129.

- [11] K. Gałkowski, *State Space Realizations of Linear 2D Systems with Extensions to the General nD ($n > 2$) Case* (Springer-Verlag, London, 2001).
- [12] A. Hmamed, M. Ait Rami and M. Alfidi, *Controller synthesis for positive 2D systems described by the Roesser model*, submitted to IEEE Trans. on Circuits and Systems.
- [13] T. Kaczorek, *Two-dimensional Linear Systems* (Springer Verlag, Berlin, 1985).
- [14] T. Kaczorek, *Positive 1D and 2D Systems* (Springer-Verlag, London, 2001).
- [15] T. Kaczorek, Asymptotic stability of positive fractional 2D linear systems, *Bull. Polish Acad. Sci. Technical Sci.* **57** (3) (2009), 289–292.
- [16] T. Kaczorek, *Asymptotic stability of positive 1D and 2D linear systems*, (Recent Advances in Control and Automation, Acad. Publ. House EXIT 2008) 41–52.
- [17] T. Kaczorek, *LMI approach to stability of 2D positive systems*, *Multidim. Syst. Sign. Process.* **20** (1) (2009), 39–54.
- [18] T. Kaczorek, *Asymptotic stability of positive 2D linear systems with delays*, *Proc. of XII Scientific Conf. Computer Applications in Electr. Engin.* 2008.
- [19] T. Kaczorek, *Practical stability of positive fractional discrete-time systems*, *Bull. Pol. Acad. Sci. Techn.* **56** (2008), 313–318.
- [20] T. Kaczorek, *Choice of the forms of Lyapunov functions for positive 2D Roesser model*, *Intern. J. Applied Math. And Comp. Science* **17** (2007), 471–475.
- [21] T. Kaczorek, *Reachability and controllability of non-negative 2D Roesser type models*, *Bull. Acad. Pol. Sci. Techn.* **44** (1996), 405–410.
- [22] T. Kaczorek, *Reachability and minimum energy control of positive 2D systems with delays*, *Control and Cybernetics* **34** (2005), 411–423.
- [23] J. Klamka, *Controllability of Dynamical Systems* (Kluwer Academic Publ., Dordrecht, 1991).
- [24] J. Kurek, *The general state-space model for a two-dimensional linear digital systems*, *IEEE Trans. Autom. Contr.* **AC-30** (1985), 600–602.
- [25] R.P. Roesser, *A discrete state-space model for linear image processing*, *IEEE Trans. Autom. Contr.* **AC-20** (1975), 1–10.
- [26] M. Twardy, *An LMI approach to checking stability of 2D positive systems*, *Bull. Pol. Acad. Sci. Techn.* **55** (2007), 386–395.
- [27] M.E. Valcher, *On the internal stability and asymptotic behavior of 2D positive systems*, *IEEE Trans. On Circuits and Systems – I*, **44** (1997), 602–613.

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