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SECOND-ORDER VIABILITY RESULT IN BANACH SPACES

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Abstract

We show the existence result of viable solutions to the second-order differential inclusion

$$\ddot{x}(t) \in F(t, x(t), \dot{x}(t)),$$
$$x(0) = x_0, \quad \dot{x}(0) = y_0, \quad x(t) \in K \text{ on } [0, T],$$

where K is a closed subset of a separable Banach space E and $F(\cdot, \cdot, \cdot)$ is a closed multifunction, integrably bounded, measurable with respect to the first argument and Lipschitz continuous with respect to the third argument.

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1. Introduction

The aim of this paper is to establish the existence of local solutions of the second-order viability problem:

$$\text{(1.1)} \qquad \begin{cases} \ddot{x}(t) \in F(t, x(t), \dot{x}(t)) \ a.e.; \\ (x(0), \dot{x}(0)) = (x_0, y_0); \\ (x(t), \dot{x}(t)) \in K \times \Omega; \end{cases}$$

where K (resp. Ω) is a closed subset (resp. an open subset) of a separable Banach space E and $F : [0, 1] \times K \times \Omega \to 2^E$ is a measurable multifunction with respect to the first argument and Lipschitz continuous with respect to the third argument.

Second-order viability problem was first introduced by Cornet and Haddad [5]. The purpose was to study a problem of type:

$$\text{(1.2)} \qquad \begin{cases} \ddot{x}(t) \in Q(x(t), \dot{x}(t)) \quad a.e; \\ (x(0), \dot{x}(0)) = (x_0, y_0); \\ (x(t), \dot{x}(t)) \in Gr(T_K), \ K \subset \mathbb{R}^n; \end{cases}$$

where Q is a convex multifunction and $Gr(T_K)$ is the graph of the multifunction $x \mapsto T_K(x)$: the contingent cone at x. In order to obtain solutions of the problem (1.2), Cornet and Haddad imposed stronger conditions on the viability set K and the tangent vector y_0, namely $K = L \cap M$, y_0 belongs to $T_L(x_0) \cap TI_M(x_0)$ and $Gr(T_K)$ is locally compact. Here $TI_M(x_0)$ is the interior tangent cone introduced by Dubovitskij and Muljitin [6].

Another line of research appeared later, it considers the second order viability problems without convexity. Two kinds of problems are studied in this topic. In the first one, the space of state constraints is finite-dimensional (see [3, 9, 10]). In this work, the right hand-side is upper semi-continuous and cyclically monotone. Proof technics are based on Ascoli's theorem and the basic relation

$$\frac{d}{dt} V(x(t)) \quad = \quad \|\dot{x}(t)\|^2,$$

where V is a convex and lower semi-continuous function whose subdifferential ∂V contains the right-hand side. The non-convex case in Hilbert space has been studied by Morchadi and Sajid [11]. The authors proved the existence of solution for a second-order viability problem without convexity and compactness of the right-hand side of the inclusion. However, the viability set is convex. Their approach is based on the Baire category theorem. See also the work of Ibrahim and Alkulaibi [2].

First-order viability problem with the non-convex Carathéodory Lipschitzean right-hand side in Banach space has been studied recently by Duc Ha [7]. The author establishes a multi-valued version of Larrieu's work [8], assuming the following tangential condition:

$$\liminf_{h \mapsto 0^+} \frac{1}{h} e\left(x + \int_t^{t+h} F(s, x) ds, K \right) = 0,$$

where K is the viability set and $e(.,.)$ denotes the Hausdorff's excess.

In this paper we extend this result to the second-order case with the following tangential condition:

$$\liminf_{h\mapsto 0^+} \frac{1}{h^2} e\left(x + hy + \frac{h}{2}\int_t^{t+h} F(s,x,y)ds, K\right) = 0. \tag{1.3}$$

The case deserves mentioning: when F is compact values in finite-dimensional space and does not depend on time s. The condition (1.3) becomes

$$\liminf_{h\mapsto 0^+} \frac{1}{h^2} e\left(x + hy + \frac{h^2}{2} F(x,y), K\right) = 0,$$

which is equivalent to the following relation:

$$\forall v \in F(x,y) \quad \liminf_{h\mapsto 0^+} \frac{1}{h^2} d_K\left(x + hy + \frac{h^2}{2} v\right) = 0,$$

i.e., $F(x,y)$ is contained in the set second-order tangent of K at (x,y) introduced by Ben-Tal and defined by:

$$A_K(x,y) = \left\{ z \in E : \liminf_{h\mapsto 0^+} \frac{1}{\frac{h^2}{2}} d_K\left(x + hy + \frac{h^2}{2} z\right) = 0 \right\}.$$

This condition was used in [1, 9, 10].

2. PRELIMINARIES AND STATEMENT OF THE MAIN RESULT

Let E be a separable Banach space with the norm $\|\cdot\|$. For measurability purposes, E (resp. $\Omega \subset E$) is endowed with the σ-algebra $B(E)$ (resp. $B(\Omega)$) of Borel subsets for the strong topology and $[0,1]$ is endowed with Lebesgue measure and the σ-algebra of Lebesgue measurable subsets. For $x \in E$ and $r > 0$ let $B(x,r) := \{y \in E; \| y - x \| < r\}$ be the open ball centered at x with radius r and $\bar{B}(x,r)$ be its closure and put $B = B(0,1)$. For $x \in E$ and for nonempty sets A, B of E we denote $d_A(x)$ or $d(x,A) := \inf\{\| y - x \|; y \in A\}$, $e(A,B) := \sup\{d_B(x); x \in A\}$ and $H(A,B) := \max\{e(A,B), e(B,A)\}$. A multifunction is said to be measurable if its graph is measurable. For more details on measurability theory, we refer the reader to the book of Castaing-Valadier [4].

Let us recall the following Lemmas that will be used in the paper. For the proofs, we refer the reader to [12].

Lemma 2.1. *Let Ω be a nonempty set in E. Assume that $F : [a,b]\times\Omega \to 2^E$ is a multifunction with nonempty closed values satisfying:*

- *For every $x \in \Omega$, $F(\cdot, x)$ is measurable on $[a,b]$;*
- *For every $t \in [a,b]$, $F(t,\cdot)$ is (Hausdorff) continuous on Ω.*

Then for any measurable function $x(\cdot) : [a,b] \to \Omega$, the multifunction $F(\cdot, x(\cdot))$ is measurable on $[a,b]$.

Lemma 2.2. *Let $G : [a,b] \to 2^E$ be a measurable multifunction and $y(\cdot) : [a,b] \to E$ a measurable function. Then for any positive measurable function $r(\cdot) : [a,b] \to \mathbb{R}^+$, there exists a measurable selection $g(\cdot)$ of G such that for almost all $t \in [a,b]$*

$$\|g(t) - y(t)\| \le d\big(y(t), G(t)\big) + r(t).$$

Assume that the following hypotheses hold:

(H1) K is a nonempty closed subset in E and Ω is a nonempty open subset in E;

(H2) $F : [0,1] \times K \times \Omega \to 2^E$ is a set valued map with nonempty closed values satisfying

(i) For each $(x,y) \in K \times \Omega$, $t \mapsto F(t,x,y)$ is measurable;

(ii) There exists a function $m \in L^1([0,1], \mathbb{R}^+)$ such that for all $t \in [0,1]$ and for all $(x_1,y_1),(x_2,y_2) \in K\times\Omega$

$$H\Big(F(t,x_1,y_1), F(t,x_2,y_2)\Big) \le m(t)\|y_1 - y_2\|;$$

(iii) For each bounded subset S of $K \times \Omega$, there exists a function $g_S \in L^1([0,1],\mathbb{R}^+)$ such that for all $t \in [0,1]$ and for all $(x,y) \in S$

$$\|F(t,x,y)\| := \sup_{z\in F(t,x,y)} \|z\| \le g_S(t);$$

(H3) (**Tangential condition**) For every (t,x,y) fixed in $[0,1]\times K\times\Omega$,

$$\liminf_{h\mapsto 0^+} \frac{1}{h^2} e\left(x + hy + \frac{h}{2}\int_t^{t+h} F(s,x,y)ds, K\right) = 0.$$

For any $(x_0, y_0) \in K \times \Omega$, consider the problem:

$$
\begin{cases} \ddot{x}(t) \in F(t, x(t), \dot{x}(t)) \quad \text{a.e;} \\ x(0) = x_0, \dot{x}(0) = y_0; \\ x(t) \in K. \end{cases} \tag{2.4}
$$

Theorem 2.3. *If assumptions* (H1), (H2) and (H3) *are satisfied, then there exist $T > 0$ and an absolutely continuous function $x(\cdot) : [0, T] \to E$, for which $\dot{x}(\cdot) : [0, T] \to E$ is also absolutely continuous such that $x(\cdot)$ is a solution of* (2.4).

3. PROOF OF THE MAIN RESULT

Let $r > 0$ such that $\bar{B}(y_0, r) \subset \Omega$ and $g \in L^1([0, 1], \mathbb{R}^+)$ such that

$$
\| F(t, x, y) \| \leq g(t) \quad \forall (t, x, y) \in [0, 1] \times (K \cap B(x_0, r)) \times \bar{B}(y_0, r). \tag{3.1}
$$

Let $T_1 > 0$ and $T_2 > 0$ such that

$$
\int_0^{T_1} m(t)dt < 1 \ \text{ and } \ \int_0^{T_2} \Big(g(t) + r + \| y_0 \| + 1 \Big) dt < \frac{r}{2}. \tag{3.2}
$$

For $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that

$$
\left| \int_{t_1}^{t_2} g(\tau) d\tau \right| < \varepsilon \ \text{ whenever } \ |t_1 - t_2| < \eta(\varepsilon). \tag{3.3}
$$

Set

$$
T = \min \Big\{ T_1, T_2, 1 \Big\} \ \text{ and } \alpha = \min \left\{ T, \frac{1}{2}\eta(\frac{\varepsilon}{4}), \frac{\varepsilon}{4} \right\}. \tag{3.4}
$$

We will use the following approximation Lemma to prove the main result.

Lemma 3.1. *If assumptions* (H1)–(H3) *are satisfied, then for all $\varepsilon > 0$ and for all $y \in L^1([0, T], E)$, there exist $f \in L^1([0, T], E)$, $z : [0, T] \to E$ differentiable and a step function $\theta : [0, T] \to [0, T]$ such that*

- *$f(t) \in F\big(t, z(\theta(t)), \dot{z}(\theta(t))\big)$ for all $t \in [0, T]$;*

- $\big\|f(t)-y(t)\big\| \leq d\Big(y(t), F\big(t, z(\theta(t)), \dot{z}(\theta(t))\big)\Big) + \varepsilon$ *for all* $t \in [0,T]$;
- $\Big\|\dot{z}(t) - y_0 - \int_0^t f(\tau)d\tau\Big\| \leq \varepsilon$ *for all* $t \in [0,T]$.

Proof. By (H3), for $(0, x_0, y_0)$, we have

$$\liminf_{h \mapsto 0^+} \frac{1}{h^2} e\left(x_0 + hy_0 + \frac{h}{2}\int_0^h F(s, x_0, y_0)ds, K\right) = 0.$$

Hence, there exists $0 < h \leq \alpha$ such that

$$e\left(x_0 + hy_0 + \frac{h}{2}\int_0^h F(s, x_0, y_0)ds, K\right) < \frac{\alpha h^2}{4}.$$

Put

$$h_0 := \max\left\{h \in]0, \alpha] : e\left(x_0 + hy_0 + \frac{h}{2}\int_0^h F(s, x_0, y_0)ds, K\right) \leq \frac{\alpha h^2}{4}\right\}.$$

In view of Lemma 2.2, there exists a function $f_0 \in L^1([0, h_0], E)$ such that for all $t \in [0, h_0]$, one has

$$f_0(t) \in F(t, x_0, y_0) \quad \text{and} \quad \|f_0(t) - y(t)\| \leq d\big(y(t), F(t, x_0, y_0)\big) + \varepsilon.$$

Moreover, it is clear that

$$d_K\left(x_0 + h_0 y_0 + \frac{h_0}{2}\int_0^{h_0} f_0(s)ds\right) \leq \frac{\alpha h_0^2}{4}.$$

So, there exists $x_1 \in K$ such that

$$\begin{aligned} &\frac{2}{h_0^2}\left\|x_1 - \left(x_0 + h_0 y_0 + \frac{h_0}{2}\int_0^{h_0} f_0(s)ds\right)\right\| \\ &\leq \frac{2}{h_0^2} d_K\left(x_0 + h_0 y_0 + \frac{h_0}{2}\int_0^{h_0} f_0(s)ds\right) + \frac{\alpha}{4}, \end{aligned}$$

hence

$$\left\|\frac{x_1 - x_0 - h_0 y_0}{\frac{h_0^2}{2}} - \frac{1}{h_0}\int_0^{h_0} f_0(s)ds\right\| < \alpha.$$

Set
$$u_0 = \frac{x_1 - x_0 - h_0 y_0}{\frac{h_0^2}{2}},$$
then
$$x_1 = x_0 + h_0 y_0 + \frac{h_0^2}{2} u_0 \in K \text{ and } u_0 \in \frac{1}{h_0}\int_0^{h_0} f_0(s)ds + \alpha B. \tag{3.5}$$

Put $y_1 = y_0 + h_0 u_0$. Since $f_0(t) \in F(t, x_0, y_0)$ for all $t \in [0, h_0]$, by (3.1), (3.2) and (3.4), we have

$$\begin{aligned}\|x_1 - x_0\| &= \left\| h_0 y_0 + \frac{h_0^2}{2} u_0 \right\| \le h_0 \|y_0\| + \frac{h_0}{2}\int_0^{h_0} g(s)ds + \frac{h_0^2}{2}\alpha \\ &\le h_0\|y_0\| + \int_0^{h_0} g(s)ds + h_0 = \int_0^{h_0} \big(g(s) + \|y_0\| + 1\big) ds < \frac{r}{2}.\end{aligned}$$

Then $x_1 \in B(x_0, r)$. Also by (3.5), we have

$$\|y_1 - y_0\| = \|h_0 u_0\| \le \int_0^{h_0} g(s)ds + h_0\alpha \le \int_0^{h_0} \big(g(s) + 1\big) ds < \frac{r}{2}.$$

Then $y_1 \in B(y_0, r)$.

We reiterate this process for constructing sequences h_q, t_q, x_q, y_q, f_q and u_q satisfying for some rank $m \ge 1$ the following assertions:

(a) For all $q \in \{0, \dots, m-1\}$
$$h_q := \max\left\{ h \in]0, \alpha] : e\left(x_q + h y_q + \frac{h}{2}\int_{t_q}^{t_{q+1}} F(s, x_q, y_q) ds, K \right) \le \frac{\alpha h^2}{4} \right\}$$

(b) $t_0 = 0$, $t_{m-1} < T \le t_m$ with $t_q = \sum_{j=0}^{q-1} h_j$ for all $q \in \{1, \dots, m\}$;

(c) For all $q \in \{1, \dots, m\}$
$$x_q = \left(x_0 + \sum_{j=0}^{q-1} h_j y_j + \sum_{j=0}^{q-1} \frac{h_j^2}{2} u_j \right) \in K \cap B(x_0, r)$$

and

$$y_q = \left(y_0 + \sum_{j=0}^{q-1} h_j u_j \right) \in B(y_0, r);$$

(d) For all $q \in \{0, \ldots, m-1\}$, for every $t \in [t_q, t_{q+1}]$

$$u_q \in \frac{1}{h_q} \int_{t_q}^{t_{q+1}} f_q(s)ds + \alpha B, \;\; f_q(t) \in F(t, x_q, y_q)$$

and

$$\|f_q(t) - y(t)\| \, \leq \, d\Big(y(t), F(t, x_q, y_q)\Big) + \varepsilon;$$

(e) For all $q \in \{0, \ldots, m-1\}$

$$\left\| y_{q+1} - y_q - \int_{t_q}^{t_{q+1}} f_q(t)dt \right\| \, < \, h_q \alpha.$$

It is easy to see that for $q = 1$, the assertions (a)–(e) are fulfilled. Let now $q \geq 2$. Assume that (a)–(e) are satisfied for any $j = 1, \ldots, q$. If $T \leq t_{q+1}$, then we take $m = q + 1$ and so the process of iterations is stopped and we get (a)–(e) satisfied with $t_{m-1} < T \leq t_m$. In the other case: $t_{q+1} < T$, we define x_{q+1} and y_{q+1} as follows

$$x_{q+1} := x_q + h_q y_q + \frac{h_q^2}{2} u_q = x_0 + \sum_{j=0}^{q} h_j y_j + \sum_{j=0}^{q} \frac{h_j^2}{2} u_j \in K,$$

and

$$y_{q+1} := y_q + h_q u_q = y_0 + \sum_{j=0}^{q} h_j u_j.$$

By (3.1), (3.2) and (3.4), we have

$$\|x_{q+1} - x_0\| \leq \sum_{j=0}^{q} h_j \|y_j\| + \sum_{j=0}^{q} \frac{h_j^2}{2} \|u_j\| \leq \sum_{j=0}^{q} h_j \|y_j - y_0\| + h_j \|y_0\| + \|h_j u_j\|$$

$$\leq \sum_{j=0}^{q} h_j \big(r + \|y_0\|\big) + \sum_{j=0}^{q} \int_{t_j}^{t_{j+1}} \|f_j(t)\| dt + \alpha h_j$$

$$\leq \int_0^{t_{q+1}} \big(g(t) + r + \|y_0\| + 1\big) dt < r,$$

which ensures that $x_{q+1} \in K \cap B(x_0, r)$. Also we have

$$\|y_{q+1} - y_0\| \le \sum_{j=0}^{q} \|h_j u_j\| \le \sum_{j=0}^{q} \int_{t_j}^{t_{j+1}} \|f_j(t)\| dt + \alpha h_j \le \int_0^{t_{q+1}} \big(g(t) + 1\big) dt < r,$$

so that $y_{q+1} \in B(y_0, r)$. Thus the conditions (a)–(e) are satisfied for $q + 1$.

Now, we have to prove that this iterative process is finite, i.e., there exists a positive integer m such that $t_{m-1} < T \le t_m$. Suppose the contrary: $t_q \le T$ for all $q \ge 1$. Then the bounded increasing sequence $(t_q)_q$ converges to some $\bar{t}$ such that $\bar{t} \le T$. By (c) and (d), for $q > p$ we get

$$\begin{aligned}
\|x_q - x_p\| &\le \sum_{j=p}^{q-1} h_j \|y_j\| + \sum_{j=p}^{q-1} \frac{h_j^2}{2} \|u_j\| \\
&\le \sum_{j=p}^{q-1} h_j \big(\|y_j - y_0\| + \|y_0\|\big) + \sum_{j=p}^{q-1} \frac{h_j^2}{2} \frac{1}{h_j} \int_{t_j}^{t_{j+1}} \|f_j(s)\| ds + \alpha \sum_{j=p}^{q-1} \frac{h_j^2}{2} \\
&\le \big(r + \|y_0\| + \alpha\big) \sum_{j=p}^{q-1} h_j + \sum_{j=p}^{q-1} \int_{t_j}^{t_{j+1}} g(s) ds \\
&\le \int_{t_p}^{t_q} g(t) dt + (t_q - t_p)\big(\parallel y_0 \parallel + r + 1\big),
\end{aligned}$$

and

$$\begin{aligned}
\|y_q - y_p\| &\le \sum_{j=p}^{q-1} h_j \|u_j\| \le \sum_{j=p}^{q-1} h_j \frac{1}{h_j} \int_{t_j}^{t_{j+1}} \|f_j(s)\| ds + \alpha \sum_{j=p}^{q-1} h_j \\
&\le \sum_{j=p}^{q-1} \int_{t_j}^{t_{j+1}} g(s) ds + \alpha \sum_{j=p}^{q-1} h_j \le \int_{t_p}^{t_q} g(t) dt + t_q - t_p.
\end{aligned}$$

The last terms of the above tow inequalities converge to 0 when $p, q \to +\infty$, then $(x_q)_q$ and $(y_q)_q$ are Cauchy sequences, and hence they converge to some $\bar{x} \in K$ and $\bar{y} \in \bar{B}(y_0, r)$ respectively. As $(\bar{t}, \bar{x}, \bar{y}) \in [0, T] \times K \times \Omega$, by (H3), there exist $h \in]0, \alpha]$ and an integer $q_0 \ge 1$ such that for all $q \ge q_0$

$$
(3.6)\qquad \begin{cases} e\Big(\overline{x}+h\overline{y}+\frac{h}{2}\int_{\bar t}^{\bar t+h}F(s,\bar x,\bar y)ds,K\Big)\le \frac{h^2\alpha}{30};\\ \|x_q-\overline{x}\|\le \frac{h^2\alpha}{30};\\ \|y_q-\overline{y}\|\le \frac{h\alpha}{30};\\ \bar t-t_q<\min\{\eta(\frac{2h\alpha}{30}),h\};\\ \parallel y_q-\bar y\parallel \int_{\bar t}^{\bar t+h}m(t)dt\le \frac{2h\alpha}{30}. \end{cases}
$$

Let $q>q_0$ be given. For an arbitrary measurable selection ϕ_q of $F(t,x_q,y_q)$ on $[0,\bar t+h]$, there exists a measurable selection ϕ of $F(t,\bar x,\bar y)$ on $[0,\bar t+h]$ such that

$$
(3.7)\qquad \|\phi_q(t)-\phi(t)\|\le d\Big(\phi_q(t),F(t,\bar x,\bar y)\Big)+\frac{2\alpha}{30}\le m(t)\|y_q-\bar y\|+\frac{2\alpha}{30}.
$$

Relations (3.6) and (3.7) ensure

$$
\begin{aligned}
& d_K\left(x_q+hy_q+\frac{h}{2}\int_{t_q}^{t_q+h}\phi_q(s)ds\right)\\
\le\ & \|x_q-\overline{x}\|+h\|y_q-\overline{y}\|+\frac{h}{2}\int_{t_q}^{\bar t}\|\phi_q(s)\|ds+d_K\left(\overline{x}+h\overline{y}+\frac{h}{2}\int_{\bar t}^{\bar t+h}\phi(s)ds\right)\\
& +\frac{h}{2}\int_{\bar t}^{t_q+h}\|\phi_q(s)-\phi(s)\|ds+\frac{h}{2}\int_{t_q+h}^{\bar t+h}\|\phi(s)\|ds\\
\le\ & \|x_q-\overline{x}\|+h\|y_q-\overline{y}\|+\frac{h}{2}\int_{t_q}^{\bar t}g(s)ds+d_K\left(\overline{x}+h\overline{y}+\frac{h}{2}\int_{\bar t}^{\bar t+h}\phi(s)ds\right)\\
& +\frac{h}{2}\int_{\bar t}^{\bar t+h}m(s)\parallel y_q-\bar y\parallel ds+\frac{h^2\alpha}{30}+\frac{h}{2}\int_{t_q+h}^{\bar t+h}g(s)ds\\
\le\ & \frac{h^2\alpha}{30}+\frac{h^2\alpha}{30}+\frac{h^2\alpha}{30}+\frac{h^2\alpha}{30}+\frac{h^2\alpha}{30}+\frac{h^2\alpha}{30}+\frac{h^2\alpha}{30}<\frac{h^2\alpha}{4}.
\end{aligned}
$$

Since ϕ_q is an arbitrary measurable selection of $F(t, x_q, y_q)$ on $[0, \bar{t} + h]$, it follows that

$$e\left(x_q + hy_q + \frac{h}{2}\int_{t_q}^{t_q+h} F(t, x_q, y_q)ds, K\right) \leq \frac{h^2\alpha}{4}.$$

On the other hand, by (3.6), we have

$$t_{q+1} \leq \bar{t} < t_q + h \leq T, \text{ and hence } h > t_{q+1} - t_q = h_q.$$

Finally $h > h_q$, $0 < h \leq \alpha$ and

$$e\left(x_q + hy_q + \frac{h}{2}\int_{t_q}^{t_q+h} F(t, x_q, y_q)ds, K\right) \leq \frac{h^2\alpha}{4}.$$

This contradicts the maximality of h_q. Therefore, there exists an integer $m \geq 1$ such that $t_{m-1} < T \leq t_m$ and for which assertions (a)–(e) are fulfilled.

Now, we take $t_m = T$ and we define the function $\theta : [0, T] \to [0, T]$, $z : [0, T] \to E$ and $f \in L^1([0, T], E)$ by setting for all $t \in [t_q, t_{q+1}[$

$$\theta(t) = t_q,\ f(t) = f_q(t),\ z(t) = x_q + (t - t_q)y_q + \frac{(t - t_q)^2}{2}u_q.$$

Claim 3.2. *For all $q \in \{0, \dots, m\}$ we have*

$$\left\| y_q - y_0 - \int_0^{t_q} f(s)ds \right\| \leq \alpha t_q.$$

Proof. Obviously, for $q = 0$ the above assertion is fulfilled. By induction, assume that

$$\left\| y_j - y_0 - \int_0^{t_j} f(s)ds \right\| \leq \alpha t_j.$$

For any $j = 1, \dots, q-1$. By (d) we have

$$\begin{aligned}
&\left\| y_q - y_0 - \int_0^{t_q} f(s)ds \right\| = \\
&= \left\| y_{q-1} - y_0 - \int_0^{t_{q-1}} f(s)ds + h_{q-1}u_{q-1} - \int_{t_{q-1}}^{t_q} f(s)ds \right\| \\
&\leq \left\| y_{q-1} - y_0 - \int_0^{t_{q-1}} f(s)ds \right\| + \left\| h_{q-1}u_{q-1} - \int_{t_{q-1}}^{t_q} f(s)ds \right\| \\
&\leq \alpha t_{q-1} + \alpha h_{q-1} = \alpha t_{q-1} + \alpha t_q - \alpha t_{q-1} = \alpha t_q.
\end{aligned}$$

Now let $t \in [t_q, t_{q+1}]$, then by Claim 3.2, relations (3.1), (3.4) and (d), we have

$$\begin{aligned}
\left\| \dot{z}(t) - y_0 - \int_0^t f(s)ds \right\| &= \left\| y_q - y_0 - \int_0^{t_q} f(s)ds + (t - t_q)u_q - \int_{t_q}^t f(s)ds \right\| \\
&\leq \left\| y_q - y_0 - \int_0^{t_q} f(s)ds \right\| + \|h_q u_q\| + \int_{t_q}^{t_{q+1}} g(s)ds \\
&\leq \alpha t_q + 2\int_{t_q}^{t_{q+1}} g(s)ds + \alpha h_q \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

The proof of Lemma 3.1 is complete.

***Proof of Theorem* 2.3.** Let $(\varepsilon_n)_{n\geq 1}$ be a strictly decreasing sequence of positive scalars such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. In view of Lemma 3.1, we can define inductively sequences $(f_n(\cdot))_{n\geq 1} \subset L^1([0,T], E)$, $(z_n(\cdot))_{n\geq 1} \subset \mathcal{C}^1([0,T], E)$ and $(\theta_n(\cdot))_{n\geq 1} \subset S([0,T],[0,T])$, where $S([0,T],[0,T])$ denotes the space of step functions from $[0,T]$ into $[0,T]$ such that

(1) $f_n(t) \in F\big(t, z_n(\theta_n(t)), \dot{z}_n(\theta_n(t))\big)$ for all $t \in [0,T]$;

(2) $\big\| f_{n+1}(t) - f_n(t) \big\| \leq d\Big(f_n(t), F\big(t, z_{n+1}(\theta_{n+1}(t)), \dot{z}_{n+1}(\theta_{n+1}(t))\big)\Big) + \varepsilon_{n+1}$ for all $t \in [0,T]$;

(3) $\left\| \dot{z}_n(t) - y_0 - \int_0^t f_n(\tau)d\tau \right\| \leq \varepsilon_n$ for all $t \in [0,T]$.

By (1) and (2) we have

$$\begin{aligned}
&\big\|f_{n+1}(t)-f_n(t)\big\|\\
&\le H\Big(F\big(t,z_n(\theta_n(t)),\dot z_n(\theta_n(t))\big),F\big(t,z_{n+1}(\theta_{n+1}(t)),\dot z_{n+1}(\theta_{n+1}(t))\big)\Big)+\varepsilon_{n+1}\\
&\le m(t)\big\|\dot z_n(\theta_n(t))-\dot z_{n+1}(\theta_{n+1}(t))\big\|+\varepsilon_{n+1}\\
&\le m(t)\Big(\big\|\dot z_n(\theta_n(t))-\dot z_n(t)\big\|+\big\|\dot z_n(t)-\dot z_{n+1}(t)\big\|+\big\|\dot z_{n+1}(t)-\dot z_{n+1}(\theta_{n+1}(t))\big\|\Big)\\
&\quad+\varepsilon_{n+1}.
\end{aligned}$$

On the other hand, for $t\in[t_q,t_{q+1}[$ we have

$$\begin{aligned}
&\big\|\dot z_n(t)-\dot z_n(\theta_n(t))\big\|=\big\|\dot z_n(t)-\dot z_n(t_q)\big\|\\
&\le\ \left\|\dot z_n(t)-y_0-\int_0^t f_n(s)ds\right\|+\left\|y_q-y_0-\int_0^{t_q} f_n(s)ds\right\|+\int_{t_q}^t\|f_n(s)\|ds\\
&\le\ \varepsilon_n+\alpha t_q+\int_{t_q}^t g(s)ds\le\varepsilon_n+\frac{\varepsilon_n}{4}+\frac{\varepsilon_n}{4},
\end{aligned}$$

hence

$$\big\|\dot z_n(t)-\dot z_n(\theta_n(t))\big\|\ \le\ 2\varepsilon_n. \tag{3.8}$$

Thus, we get

$$\begin{aligned}
\big\|f_{n+1}(t)-f_n(t)\big\|\ &\le\ m(t)\Big(2\varepsilon_n+2\varepsilon_{n+1}+\|\dot z_n-\dot z_{n+1}\|_\infty\Big)+\varepsilon_{n+1}\\
&\le\ m(t)\Big(4\varepsilon_n+\|\dot z_n-\dot z_{n+1}\|_\infty\Big)+\varepsilon_{n+1}.
\end{aligned}\tag{3.9}$$

Relations (3.2) and (3.9) yield

$$\begin{aligned}
&\| \dot{z}_{n+1}(t) - \dot{z}_n(t) \| \\
&\quad \leq \left\| \dot{z}_{n+1}(t) - y_0 - \int_0^t f_{n+1}(s)ds \right\| + \left\| \dot{z}_n(t) - y_0 - \int_0^t f_n(s)ds \right\| \\
&\qquad + \int_0^t \| f_{n+1}(s) - f_n(s) \| \, ds \\
&\quad \leq \varepsilon_{n+1} + \varepsilon_n + \int_0^t m(s)\big(\| \dot{z}_n(\cdot) - \dot{z}_{n+1}(\cdot) \|_\infty + 4\varepsilon_n \big) ds + t\varepsilon_n \\
&\quad \leq 7\varepsilon_n + \| \dot{z}_n(\cdot) - \dot{z}_{n+1}(\cdot) \|_\infty \int_0^T m(s)ds.
\end{aligned}$$

Then

$$\| \dot{z}_n(\cdot) - \dot{z}_{n+1}(\cdot) \|_\infty \leq \frac{7\varepsilon_n}{1-L}, \tag{3.10}$$

where $L = \int_0^T m(s)ds$. Hence for, $n < m$, it follows that

$$\| \dot{z}_m(\cdot) - \dot{z}_n(\cdot) \|_\infty \leq \frac{7}{1-L} \sum_{i=n}^{m-1} \varepsilon_i.$$

Thus the sequence $(\dot{z}_n(\cdot))_{n\geq 1}$ converges uniformly on $[0,T]$ to a function $y(\cdot)$.

On the other hand, observe that $\dot{z}_n(\theta_n(t))$ converges uniformly to $y(t)$ on $[0,T]$. Indeed, by (3.8) since

$$\| \dot{z}_n(\theta_n(t)) - y(t) \| \leq \| \dot{z}_n(t) - \dot{z}_n(\theta_n(t)) \| + \| \dot{z}_n(t) - y(t) \|$$

then $(\dot{z}_n(\cdot))$ converges uniformly to $y(\cdot)$.
Thus

$$\dot{z}_n(\theta_n(\cdot)) \text{ converges uniformly to } y(\cdot) \text{ on } [0,T]. \tag{3.11}$$

By construction, we have $\dot{z}_n(\theta_n(t)) \in B(y_0, r)$ for every $t \in [0,T]$, then $y(t) \in \Omega$ for all $t \in [0,T]$.

Now we return to relation (3.9). By relation (3.10) we have

$$\begin{aligned}\big\|f_{n+1}(t)-f_n(t)\big\| &\le m(t)\Big(4\varepsilon_n+\parallel \dot z_n(\cdot)-\dot z_{n+1}(\cdot)\parallel_\infty\Big)+\varepsilon_n\\ &\le \Big(m(t)\Big(4+\frac{7}{1-L}\Big)+1\Big)\varepsilon_n.\end{aligned}$$

This implies (as above) that $(f_n(t))_{n\ge 1}$ is a Cauchy sequence and $(f_n(\cdot))_{n\ge 1}$ converges point-wisely to $f(\cdot)$. Further, since $\|f_n(t)\|\le g(t)$, by (3) and by Lebesgue dominated convergence theorem, we have

$$y(t)=\lim_{n\to\infty}\dot z_n(t)=\lim_{n\to\infty}\left(y_0+\int_0^t f_n(s)ds\right)=y_0+\int_0^t f(s)ds.$$

Hence $\dot y(t)=f(t)$. Since for $n<m$

$$\|\dot z_m(\cdot)-\dot z_n(\cdot)\|_\infty \le \frac{7}{1-L}\sum_{i=n}^{m-1}\varepsilon_i,$$

then we have for all $t\in[0,T]$

$$\|z_m(t)-z_n(t)\|\le\int_0^t\|\dot z_m(s)-\dot z_n(s)\|ds\le\frac{7}{1-L}\sum_{i=n}^{m-1}\varepsilon_i.$$

Thus the sequence $(z_n(\cdot))_{n\ge 1}$ converges uniformly on $[0,T]$ to a function $x(\cdot)$. Also the relation

$$z_n(t)=x_0+\int_0^t\dot z_n(s)ds$$

yields

$$x(t)=x_0+\int_0^t y(s)ds.$$

Therefore, $\dot x(t)=y(t)$ for all $t\in[0,T]$.

On the other hand, observe that $z_n(\theta_n(t))$ converges uniformly to $x(t)$ on $[0,T]$. Indeed, for $t\in[t_q,t_{q+1}[$ we have

$$\|z_n(t)-z_n(\theta_n(t))\|\le\int_{\theta_n(t)}^t\|\dot z_n(s)\|ds.$$

Since $\theta_n(t)$ converges to t and $\dot{z}_n(\cdot)$ is bounded, it follows that $\|z_n(t) - z_n(\theta_n(t))\|$ converges to 0 as $n \to \infty$. Since

$$\|z_n(\theta_n(t)) - x(t)\| \leq \|z_n(t) - z_n(\theta_n(t))\| + \|z_n(t) - x(t)\|$$

and $(z_n(\cdot))$ converges uniformly to $x(\cdot)$, then $z_n(\theta_n(t))$ converges uniformly to $x(t)$ on $[0,T]$. By construction, we have $z_n(\theta_n(t)) \in K$ for every $t \in [0,T]$ and K is closed, then $x(t) \in K$ for all $t \in [0,T]$.

Finally, observe that by (1),

$$d\Big(f(t), F(t,x(t),\dot{x}(t))\Big) \leq H\Big(F\big(t, z_n(\theta_n(t)), \dot{z}_n(\theta_n(t))\big), F\big(t, x(t), \dot{x}(t)\big)\Big)$$

$$+ \|f(t) - f_n(t)\| \leq \|f(t) - f_n(t)\| + m(t)\|\dot{z}_n(\theta_n(t)) - \dot{x}(t)\|.$$

Since $f_n(t)$ converges to $f(t)$ and by (3.11), the last term converges to 0. So that $\ddot{x}(t) = f(t) \in F(t,x(t),\dot{x}(t))$ a.e. in $[0,T]$. Hence the proof is complete.

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