

ON DISCONTINUOUS QUASI-VARIATIONAL INEQUALITIES

LIANG-JU CHU AND CHING-YANG LIN

*Department of Mathematics
National Taiwan Normal University
Taipei, Taiwan, Republic of China*

Abstract

In this paper, we derive a general theorem concerning the quasi-variational inequality problem : find $\bar{x} \in C$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and

$$\langle \bar{y}, z - \bar{x} \rangle \geq 0, \forall z \in S(\bar{x}),$$

where C, D are two closed convex subsets of a normed linear space X with dual X^* , and $T : X \longrightarrow 2^{X^*}$ and $S : C \longrightarrow 2^D$ are multifunctions. In fact, we extend the above to an existence result proposed by Ricceri [12] for the case where the multifunction T is required only to satisfy some general assumption without any continuity. Under a kind of Karamardian's condition, we give a partial affirmative answer to an unbounded quasi-variational inequality problem.

Keywords: variational inequality, quasi-variational inequality, Ricceri's conjecture, Karamardian condition, Hausdorff continuous multifunction, Kneser's minimax inequality.

2000 Mathematics Subject Classification: 47H04, 47H10, 49J35, 52A99.

1. INTRODUCTION AND PRELIMINARIES

Let X be a normed linear space, with dual X^* , C and D be two closed convex subsets of X , and $T : X \longrightarrow 2^{X^*}$ and $S : C \longrightarrow 2^D$ be two multifunctions. We shall deal with the following generalized *quasi-variational inequality problem*:

QVI(T, S, C, D): Find $\bar{x} \in C$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and

$$\langle \bar{y}, z - \bar{x} \rangle \geq 0, \quad \forall z \in S(\bar{x}).$$

This problem **QVI**(T, S, C, C) was introduced in 1982 by Chan and Pang [2] in a finite-dimensional setting ($X = X^* = R^n$). In the case of infinite-dimensional spaces, it was first studied by Shih and Tan [14] in 1985. In particular, if $S(x) \equiv C = D$, the problem **QVI**(T, S, C, D) reduces to the usual variational inequality:

VI(T, C): Find $\bar{x} \in C$ and $\bar{y} \in T(\bar{x})$ such that $\langle \bar{y}, z - \bar{x} \rangle \geq 0, \quad \forall z \in C$.

The importance of the variational inequality theory is well documented in the literature. In the last ten years, much of the study has been focused upon finding conditions to ensure the existence of a solution in the case where T need not be upper semicontinuous, since many applications in network equilibrium problems and control systems do not have such a continuity property. On the other hand, the usual Karamardian condition [13] is one way to control the difficulty of unbounded variational inequality problem. For a neighborhood V_0 of the origin in X , we shall say that T and S satisfy the *generalized V_0 -Karamardian condition* on (C, H, K) for some compact subsets H, K of C , with $H \subseteq K$, if for each $x \in (C + V_0) \setminus K$ and $y \in T(x)$, there is some $z \in S(x) \cap H$ satisfying $\langle y, z - x \rangle < 0$. When C is convex, $S(x) \equiv C$ and $H = K$, we may take $V_0 = \{0\}$, so that the generalized V_0 -Karamardian condition reduces to the usual Karamardian condition. Some variant Karamardian conditions are defined in [3]. In closed connection, Yau and Guo [15, Theorem 4.5] obtained the following

Theorem A. *Let $X = X^* = R^n$. Suppose that*

- (i) *$T(x)$ is nonempty, convex and compact for each $x \in C$;*
- (ii) *for each $z \in C$, the set*

$$\left\{ x \in C; \inf_{y \in T(x)} \langle y, x - z \rangle \leq 0 \right\}$$

is closed;

- (iii) *there exists a nonempty bounded subset K of C such that for each $x \in C \setminus K$ there exists $z \in K$ satisfying*

$$\inf_{y \in T(x)} \langle y, x - z \rangle > 0.$$

Then $VI(T, C)$ has a solution in K .

In 1996, Cubiotti [6, Theorem 3.2] showed that

Theorem B. *Let $X = X^* = R^n$, $C = D$, K a nonempty compact subset of C , and $B(r) = \{x \in X; \|x\| \leq r\}$ for $r > 0$. Suppose that the multifunctions T and S satisfy*

- (i) $T(x)$ is convex for each $x \in K$, with $x \in S(x)$;
- (ii) $T(x)$ is nonempty and compact for each $x \in C$;
- (iii) for each $z \in C - C$, the set

$$\left\{x \in C; \inf_{y \in T(x)} \langle y, z \rangle \leq 0\right\}$$

is closed;

- (iv) S is lower semicontinuous with a closed graph and $S(x)$ is convex for each $x \in C$.

If there exists an increasing sequence $\langle r_k \rangle$ of positive real numbers, with $C \cap B(r_1) \neq \emptyset$ and $\lim_{k \rightarrow \infty} r_k = +\infty$, such that for each k ,

- (v) $S(x) \cap B(r_k) \neq \emptyset$ for all $x \in C \cap B(r_k)$;
- (vi) for each $x \in C \cap B(r_k) \setminus K$ with $x \in S(x)$,

$$\sup_{z \in S(x) \cap B(r_k)} \inf_{y \in T(x)} \langle y, x - z \rangle > 0.$$

Then $QVI(T, S, C, C)$ has a solution in K .

In 1997, Lunsford [11, Theorem 3.4] proved

Theorem C. *Let X be a separable Banach space, $C = D$, and let K be a nonempty compact convex subset of C such that*

- (i) $T(x)$ is nonempty, convex and w^* -compact for each $x \in C$;
- (ii) the multifunction $L : C \longrightarrow 2^C$, defined by

$$L(x) = \left\{x \in C; \inf_{y \in T(x)} \langle y, x - z \rangle \leq 0\right\}$$

has a closed graph in $C \times C$;

- (iv) the multifunction $S^* : K \longrightarrow 2^K$, defined by $S^*(x) = S(x) \cap K$, is lower semicontinuous with a closed graph, and $S^*(x)$ is nonempty and convex for each $x \in K$.

- (vi) for each $x \in K$ with $\text{int}_{S(x)} S^*(x) \neq \emptyset$, and for all $x \in \partial_{S(x)} S^*(x)$, there is some $z \in \text{int}_{S(x)} S^*(x)$ such that $\langle y, x - z \rangle \geq 0$ for all $y \in T(x)$.

Then $QVI(T, S, C, C)$ has a solution.

In 1997, Cubiotti [7, Theorem 4.1] improved

Theorem D. Let $X = X^* = R^n$, $C = D$, and $B(r) = \{x \in X; \|x\| \leq r\}$ for $r > 0$. Suppose that the multifunctions T and S satisfy

- (i) $C_r := C \cap B(r) \neq \emptyset$;
- (ii) the multifunction $S^* : C_r \longrightarrow 2^{C_r}$, defined by $S^*(x) = S(x) \cap B(r)$, is continuous, and $S^*(x)$ is closed and convex for each $x \in C_r$;
- (iii) $T(x)$ is convex for each $x \in S(x)$;
- (iv) $T(x)$ is nonempty and compact for each $x \in C_r$;
- (v) $\text{aff}(S(x) \cap B(r)) = \text{aff}(C_r)$ for each $x \in C_r$, and the set

$$\left\{x \in C_r; \inf_{y \in T(x)} \langle y, x - z \rangle \leq 0\right\}$$

is closed for each $z \in C_r$;

- (vi) for all $x \in C$ with $x \in S(x)$ and $\|x\| = r$, and for all $y \in T(x)$, there exists some $z \in S(x)$ with $\|z\| < r$ such that $\langle y, x - z \rangle \geq 0$.

Then $QVI(T, S, C, C)$ has a solution \bar{x} satisfying $\|\bar{x}\| \leq r$.

The main purpose of the present paper is to deduce some generalized key results on $QVI(T, S, C, D)$ based on these very powerful results, together with some coercitive property. Indeed, we shall simplify and reformulate existence theorems of generalized quasi-variational inequalities on a non-compact region C . Beyond the realm of monotonicity nor continuity on T , the result derived here generalizes and unifies various earlier ones from classic optimization theory. We digress briefly now to list a little notation and review some definitions. Let X be a normed linear space, with dual X^* , and C be a convex subset of X . The *interior*, *relative interior*, *closure*, and *affine hull* of C will be denoted by $\text{int}C$, $\text{ri}C$, $\text{cl}C$, and $\text{aff}C$, respectively. Define $B(r) = \{x \in X; \|x\| \leq r\}$ and $d(x, C) = \inf_{y \in C} \|x - y\|$. A multifunction $T : C \longrightarrow 2^{X^*}$ is *upper semicontinuous* at x provided that for each open set V containing $T(x)$, there exists a neighborhood U of x in C such that $T(y)$ is contained in V for all $y \in U$. T is *lower semicontinuous* at x provided that for each open set V with $V \cap T(x) \neq \emptyset$, there exists a

neighborhood U of x such that $V \cap T(y) \neq \emptyset$ for all $y \in U$. We shall say that T is *upper (lower) semicontinuous* if it is upper (lower) semicontinuous at each point. We say that T is *Hausdorff upper semicontinuous* at x provided that for each $r > 0$, there is a neighborhood U of x in C such that $T(y) \subset T(x) + \text{int}B(r)$ for all $y \in U$. T is *Hausdorff lower semicontinuous* at x provided that for each $r > 0$, there is a neighborhood U of x in C such that $T(x) \subset T(y) + \text{int}B(r)$ for all $y \in U$. As before, we say that T is *Hausdorff upper (lower) semicontinuous* if it is Hausdorff upper (lower) semicontinuous at each point. We say that T is *Hausdorff continuous* if it is both Hausdorff upper and Hausdorff lower semicontinuous. It is known [9] that every upper semicontinuous multifunction is Hausdorff upper semicontinuous; conversely, every Hausdorff lower semicontinuous multifunction is lower semicontinuous. Moreover, T is Hausdorff upper semicontinuous at x if, and only if, for any sequence $\langle x_n \rangle$ converging to x , $\sup_{z \in T(x_n)} d(z, T(x))$ converges to 0. Such a property related to Berge's maximal theorem can be found in [9]; see also [1, pp. 118–123].

2. MAIN THEOREMS TO QVI(T, S, C, D)

Before proceeding with our main result, let us recall some key facts.

Proposition 2.1 [7, Proposition 2.1]. *If A is an open subset of R^n and B is a closed convex subset of R^n , with $A \cap B \neq \emptyset$, then $\text{aff}(A \cap B) = \text{aff}(B)$.*

Proposition 2.2 [7, Proposition 2.1]. *If C is a nonempty subset of a normed linear space X and $S : C \longrightarrow 2^X$ is a Hausdorff lower semicontinuous multifunction with nonempty values, then for each $r > 0$, the multifunction $S_r : C \longrightarrow 2^X$, defined by*

$$S_r(x) = \{z \in X; d(z, S(x)) < r\}, \quad \forall x \in C,$$

has open lower sections; that is, each $S_r^{-1}(z) = \{x : z \in S_r(x)\}$ is open for each $z \in X$.

Proposition 2.3 [7, Proposition 2.4]. *Let C be a closed subset of a normed linear space X , V be any affine set X , and $S : C \longrightarrow 2^V$ be a Hausdorff lower semicontinuous multifunction with nonempty closed convex values. If $y \in \text{int}_V S(\bar{x})$ for some $\bar{x} \in C$, then there exists a neighborhood U of \bar{x} in C such that $y \in \text{int}_V (\cap_{z \in U} S(z))$.*

Proposition 2.4. *If C is a nonempty closed subset of a normed linear space X , and $S : C \longrightarrow 2^X$ is a Hausdorff upper semicontinuous multifunction with nonempty values, then for each $r > 0$, the multifunction $clS_r : C \longrightarrow 2^X$, defined by*

$$clS_r(x) = \{z \in X; d(z, S(x)) \leq r\}, \quad \forall x \in C,$$

has a closed graph.

Proof. Let $G(clS_r)$ denote the graph of clS_r . Suppose that $(x_n, y_n) \in G(clS_r)$ and converges to (x, y) . Then we have $d(y_n, S(x_n)) \leq r$ for each n . This yields some $z_n \in S(x_n)$ such that $d(y_n, z_n) \leq r + \frac{1}{n}$. Notice that

$$\begin{aligned} d(y, S(x)) &\leq d(y, y_n) + d(y_n, z_n) + d(z_n, S(x)) \\ &\leq d(y, y_n) + r + \frac{1}{n} + \sup_{z \in S(x_n)} d(z, S(x)). \end{aligned}$$

Since S is Hausdorff upper semicontinuous, $\sup_{z \in S(x_n)} d(z, S(x))$ converges to 0. Therefore, taking the limits of the above inequality, we obtain

$$d(y, S(x)) \leq 0 + r + 0 + 0 = r.$$

This implies that $y \in clS_r(x)$. Equivalently, $(x, y) \in G(clS_r)$, and hence $G(clS_r)$ is closed.

From these, we are able to establish a basic existence theorem to the problem QVI(T, S, C, D) in R^n , which will be used to solve our main result of Ricceri's type [12].

Theorem 2.5. *Let C and D be closed convex subsets of R^n , and K be a nonempty compact subset of C . Suppose that the multifunctions $T : R^n \longrightarrow 2^{R^n}$ and $S : C \longrightarrow 2^D$ satisfy*

- (i) $S(x) \cap K \neq \emptyset, \quad \forall x \in C$;
- (ii) S is lower semicontinuous, the graph of S is closed, and $S(x)$ is convex for each $x \in C$;
- (iii) $T(x)$ is nonempty, convex and compact for each $x \in D$;

(iv) $aff(S(x)) = aff(D)$ for each $x \in C$, and the set

$$\left\{x \in D; \inf_{y \in T(x)} \langle y, x - z \rangle \leq 0\right\}$$

is closed for each $z \in D$;

(v) T and S satisfy the generalized V_0 -Karamardian condition on (D, K, K) for some neighborhood V_0 of the origin in R^n : for all $x \in (D + V_0) \setminus K$ and $y \in T(x)$, there exists some $z \in S(x) \cap K$ such that $\langle y, z - x \rangle < 0$.

Then $QVI(T, S, C, D)$ has a solution in K .

Proof. Observe that $C \cap D \neq \emptyset$, by the condition (i), so that we can replace C by $C \cap D$, if necessary. Since all the conditions (i) \sim (iii) are satisfied for the set $C \cap D$, we may assume without loss of generality that C is a subset of D . Define a multifunction $S_1 : D \longrightarrow 2^D$ by $S_1(x) = S(x)$ if $x \in C$, and $S_1(x) = D$ if $x \in D \setminus C$. Then S_1 is lower semicontinuous, with a closed graph, and $S_1(x)$ is convex for each $x \in D$. Indeed, let $x \in D$ and V be any open set satisfying $V \cap S_1(x) \neq \emptyset$. If $x \in C$, then $V \cap S_1(x) \neq \emptyset$. By the lower continuity of S , there is some neighborhood U of x such that $V \cap S(y) \neq \emptyset$ for all $y \in U \cap C$. It follows that $V \cap S_1(y) \neq \emptyset$ for all $y \in U \cap D$. If $x \in D \setminus C$, then $S_1(x) = D$. Taking any neighborhood U of x such that $U \cap C \neq \emptyset$, we have $V \cap S_1(y) \neq \emptyset$ for all $y \in U \cap D$. Now, let $r > 0$ satisfy $K \subset \text{int}B(r)$. Then, for any $x \in D \cap B(r)$, by (i), we can take one vector $y \in S(x) \cap K$. Then $y \in S_1(x) \cap \text{int}B(r)$, and hence

$$d(0, S_1(x)) = \inf_{z \in S_1(x)} \|z\| \leq \|y\| < r.$$

It follows from [5, Proposition 1] that the multifunction $S^* : D \cap B(r) \longrightarrow 2^{D \cap B(r)}$, defined by $S^*(x) = S_1(x) \cap B(r)$, is continuous, and $S^*(x)$ is nonempty, closed and convex for each $x \in D \cap B(r)$. By Proposition 2.2, together with (iv), we have $aff(S_1(x) \cap B(r)) = aff(D \cap B(r))$ for all $x \in D$. Also, for all $x \in D$, with $x \in S_1(x)$ and $\|x\| = r$, we have $x \notin K$ in view of $K \subset \text{int}B(r)$. Thus, by (v), for all $y \in T(x)$, there exists some $z \in S(x) \cap K$ such that $\langle y, z - x \rangle < 0$. Note that $z \in K \subset \text{int}B(r)$. It follows that $\|z\| < r$. Hence, by Theorem D, the problem $QVI(T, S_1, D, D)$ has a solution \bar{x} . That is, there exists some $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S_1(\bar{x})$ and $\langle \bar{y}, z - \bar{x} \rangle \geq 0$ for all $z \in S_1(\bar{x})$. Now, we show that $\bar{x} \in K$. Assume that $\bar{x} \notin K$. Then by (v),

there is some $z \in S(\bar{x}) \cap K$ such that $\langle \bar{y}, z - \bar{x} \rangle < 0$, a contradiction to the fact that \bar{x} is a solution of $\text{QVI}(T, S_1, D, D)$. It follows that $\bar{x} \in K \subset C$. Hence $\bar{x} \in S_1(\bar{x}) = S(\bar{x})$, and $\langle \bar{y}, z - \bar{x} \rangle \geq 0$ for all $z \in S(\bar{x})$. This shows that $\bar{x} \in K$ is a solution of $\text{QVI}(T, S, C, D)$.

In 1995, Ricceri [12] raised a problem concerning the existence of solutions to $\text{QVI}(T, S, C, C)$ as follows.

Ricceri's Conjecture 2.6. *Let C be a closed convex subset of a real Hausdorff topological vector space, with dual X^* , and $H \subseteq K$ be two compact subsets of C , where H is finite-dimensional. Suppose that the multifunctions $T : C \longrightarrow 2^{X^*}$ and $S : C \longrightarrow 2^C$ satisfy*

- (i) $S(x) \cap H \neq \emptyset, \forall x \in C$;
- (ii) S is lower semicontinuous with a closed graph, and $S(x)$ is closed and convex for each $x \in C$;
- (iii) $T(x)$ is nonempty, convex and w^* -compact for each $x \in C$;
- (iv) $\text{int}_{\text{aff}C} S(x) \neq \emptyset$ for each $x \in C$, and the set

$$\left\{ x \in C; \inf_{y \in T(x)} \langle y, z \rangle \leq 0 \right\}$$

is compactly closed for all $z \in C - C$;

- (v) *for all $x \in C \setminus K$ and $x \in S(x)$,*

$$\sup_{z \in S(x) \cap H} \inf_{y \in T(x)} \langle y, x - z \rangle > 0.$$

Then $\text{QVI}(T, S, C, C)$ has a solution in K .

In 1997, Cubioti [8, Theorem 3.1] partially solved this problem for the case where S is a Lipschitzian multifunction. Based on Theorem 2.5 equipped with Hausdorff continuity on S , we can establish a general existence result to $\text{QVI}(T, S, C, D)$ as follows.

Theorem 2.7. *Let C and D be closed convex subsets of a normed linear space X , with dual X^* , and $H \subseteq K$ be two compact subsets of C , where H is finite-dimensional. Suppose that the multifunctions $T : X \longrightarrow 2^{X^*}$ and $S : C \longrightarrow 2^D$ satisfy*

- (i) $S(x) \cap H \neq \emptyset, \forall x \in C$;
- (ii) S is Hausdorff continuous, and $S(x)$ is closed and convex for each $x \in C$;

- (iii) $T(x)$ is nonempty, convex and w^* -compact for each $x \in D$;
- (iv) $riS(x) \neq \emptyset$ for each $x \in K$, and the set

$$\left\{x \in D; \inf_{y \in T(x)} \langle y, x - z \rangle \leq 0\right\}$$

is closed for each $z \in D$;

- (v) T and S satisfy the generalized V_0 -Karamardian condition on (D, H, K) for some neighborhood V_0 of the origin in X : for all $x \in (D + V_0) \setminus K$ and $y \in T(x)$, there exists some $z \in S(x) \cap H$ such that $\langle y, z - x \rangle < 0$.

Then $QVI(T, S, C, D)$ has a solution in K .

Proof. Let Ω be the collection of all the finite-dimensional subspaces of X containing H . Equipped with the ordinary set inclusion \subseteq , the pair (Ω, \subseteq) becomes a partially ordered set. For each fixed $r > 0$, we define a multifunction $S_r : C \rightarrow 2^X$ by

$$S_r(x) = \{z \in X; d(z, S(x)) < r\}, \forall x \in C.$$

Since S is Hausdorff lower semicontinuous, by Proposition 2.2, each S_r has open lower sections. Since $S(x)$ is nonempty and convex, it is easy to see that each $S_r(x)$ is also nonempty and convex. For each $F \in \Omega$, we define a multifunction $S_F : C \cap F \rightarrow 2^{D \cap F}$ by $S_F(x) = clS_r(x) \cap D \cap F$ for each $x \in C \cap F$. Then the graph of S_F is closed, by Proposition 2.4. Also, each $S_F(x)$ is nonempty, closed and convex for each $x \in C \cap F$. Now, we show that S_F is lower semicontinuous in $C \cap F$. Since S_r has open lower sections, the multifunction $M : C \rightarrow 2^{D \cap F}$, defined by $M(x) = S_r(x) \cap D \cap F$, is lower semicontinuous in C . It follows that the multifunction $N : C \rightarrow 2^{D \cap F}$, defined by $N(x) = clM(x)$, is also lower semicontinuous in C . Notice that $S_F(x) = N(x)$ for all $x \in C \cap F$. Consequently, S_F is lower semicontinuous in $C \cap F$. Further, since $S_r(x)$ is open and $S_r(x) \cap (D \cap F) \neq \emptyset$, by applying Proposition 2.1 to $A = S_r(x)$ and $B = D \cap F$, we have $aff(S_r(x) \cap D \cap F) = aff(D \cap F)$ for all $x \in C$. In particular, for each $x \in C \cap F$, we have

$$aff(D \cap F) \supseteq aff(S_F(x)) \supseteq aff(S_r(x) \cap D \cap F) = aff(D \cap F).$$

It follows that $aff(S_F(x)) = aff(D \cap F)$ for all $x \in C \cap F$. Also, for each $z \in D \cap F$, the set

$$\left\{x \in D \cap F; \inf_{y \in T(x)} \langle y, x - z \rangle \leq 0\right\} = \left\{x \in D; \inf_{y \in T(x)} \langle y, x - z \rangle \leq 0\right\} \cap F$$

is closed by condition (iv). Notice that T and S_F satisfy the generalized V_0 -Karamardian condition on $(D \cap F, K \cap F, K \cap F)$. Indeed, for each $x \in (D \cap F) + V_0 \setminus (K \cap F)$ and $y \in T(x)$, we have $x \in (D + V_0) \setminus K$. By (v), there exists some $z \in S(x) \cap H \subseteq S_F(x) \cap (K \cap F)$ such that $\langle y, z - x \rangle < 0$. Thus, applying Theorem 2.5 to $\text{QVI}(T, S_F, C \cap F, D \cap F)$, we can obtain a solution x_F in $K \cap F$ for each $F \in \Omega$. That is, there exists some $y_F \in T(x_F)$ such that $x_F \in S_F(x_F)$ and

$$(2.1) \quad \langle y_F, z - x_F \rangle \geq 0, \quad \forall z \in S_F(x_F).$$

Since K is compact, the net $\langle x_F \rangle$ admits a cluster point \bar{x} in K . Let $V = \text{aff} S(\bar{x})$. Then by (iv), we have

$$(2.2) \quad \text{int}_V S(\bar{x}) = \text{ri} S(\bar{x}) \neq \emptyset.$$

Assume that there is some $\bar{y} \in \text{int}_V S(\bar{x})$ such that

$$(2.3) \quad \inf_{y \in T(\bar{x})} \langle y, \bar{x} - \bar{y} \rangle > 0.$$

Since S is Hausdorff lower semicontinuous at \bar{x} , by Proposition 2.3, there exists a neighborhood U of \bar{x} in C such that

$$(2.4) \quad \bar{y} \in \text{int}_V (\cap_{z \in U} S(z)).$$

Moreover, by (iv), there exists a neighborhood W of \bar{x} such that

$$(2.5) \quad W \subset U \cap \{x \in D; \inf_{y \in T(x)} \langle y, x - \bar{y} \rangle > 0\}.$$

Since \bar{x} is a cluster point of the net $\langle x_F \rangle$, we may have some F in Ω such that $x_F \in W$ and $\bar{y} \in F$. Thus, by (2.4), we obtain

$$\bar{y} \in \text{int}_V (\cap_{z \in W} S(z)) \subseteq S(x_F) \cap F \subseteq S_F(x_F).$$

It follows from (2.1) that

$$(2.6) \quad \langle y_F, \bar{y} - x_F \rangle \geq 0.$$

On the other hand, since $x_F \in W$, by (2.5), we have

$$\inf_{y \in T(x_F)} \langle y, x_F - \bar{y} \rangle > 0.$$

It follows that $\langle y_F, \bar{y} - x_F \rangle < 0$, which contradicts (2.6). Thus, our assumption (2.3) fails, and hence we conclude that

$$(2.7) \quad \inf_{y \in T(\bar{x})} \langle y, \bar{x} - z \rangle \leq 0, \quad \forall z \in \text{int}_V S(\bar{x}).$$

Recall that the existence \bar{x} in the above (2.7) is dependent on the parameter $r > 0$, and also $\bar{x} \in S_r(\bar{x})$. Now, we let $\langle r_n \rangle$ be a sequence of positive numbers converging to 0, and let $\langle x_n \rangle$ be the correspondent sequence in K , with $V_n = \text{aff} S(x_n)$, such that for each n , we have

$$(2.8) \quad \inf_{y \in T(x_n)} \langle y, x_n - z \rangle \leq 0, \quad \forall z \in \text{int}_{V_n} S(x_n).$$

Notice that since $x_n \in S_{r_n}(x_n)$, we have $d(x_n, S(x_n)) \leq r_n$ for each n . By the compactness of K , there exists a convergent subsequence of $\langle x_n \rangle$. Without loss of generality, we may assume that the sequence $\langle x_n \rangle$ converges to some vector \hat{x} of K . Let $V = \text{aff} S(\hat{x})$. Then by (iv), we have $\text{int}_V S(\hat{x}) = \text{ri} S(\hat{x}) \neq \emptyset$. Assume that there is some $\hat{y} \in \text{int}_V S(\hat{x})$ such that

$$(2.9) \quad \inf_{y \in T(\hat{x})} \langle y, \hat{x} - \hat{y} \rangle > 0.$$

Since S is Hausdorff lower semicontinuous at \hat{x} , by Proposition 2.3, there exists a neighborhood $U(\hat{x})$ of \hat{x} in C such that

$$(2.10) \quad \hat{y} \in \text{int}_V (\cap_{z \in U(\hat{x})} S(z)).$$

Moreover, by (iv), there exists a neighborhood $W(\hat{x})$ of \hat{x} such that

$$(2.11) \quad W(\hat{x}) \subset U(\hat{x}) \cap \left\{ x \in D; \inf_{y \in T(x)} \langle y, x - \hat{y} \rangle > 0 \right\}.$$

Let k be sufficiently large so that $x_k \in W(\hat{x})$. It follows from (2.11) that

$$(2.12) \quad \inf_{y \in T(x_n)} \langle y, x_n - \hat{y} \rangle > 0.$$

On the other hand, since $\hat{y} \in \text{int}_V(S(x_k))$, by (2.8), we have

$$\inf_{y \in T(x_n)} \langle y, x_n - \hat{y} \rangle \leq 0.$$

This is a contradiction to (2.12). Hence, the assumption (2.9) is not true, and therefore,

$$(2.13) \quad \inf_{y \in T(\hat{x})} \langle y, \hat{x} - z \rangle \leq 0, \quad \forall z \in \text{int}_A S(\hat{x}) = riS(\hat{x}).$$

From this, we conclude that

$$(2.14) \quad \sup_{z \in riS(\hat{x})} \inf_{y \in T(\hat{x})} \langle y, \hat{x} - z \rangle \leq 0.$$

Next, we show that $\hat{x} \in S(\hat{x})$. Notice that $\sup_{z \in S(x_n)} d(z, S(\hat{x}))$ converges to 0 as n tends to $+\infty$, since S is Hausdorff upper semicontinuous. Thus, we have

$$\begin{aligned} d(x_n, S(\hat{x})) &\leq d(x_n, S(x_n)) + \sup_{z \in S(x_n)} d(z, S(\hat{x})) \\ &\leq r_n + \sup_{z \in S(x_n)} d(z, S(\hat{x})) \longrightarrow 0 \text{ as } n \longrightarrow +\infty. \end{aligned}$$

Hence, by the continuity of the mapping $z \mapsto d(z, S(\hat{x}))$, we conclude that $d(\hat{x}, S(\hat{x})) \leq 0$, and hence $\hat{x} \in S(\hat{x})$. To complete the proof, we need to show that there is some $\hat{y} \in T(\hat{x})$ such that

$$(2.15) \quad \langle \hat{y}, z - \hat{x} \rangle \geq 0, \quad \forall z \in S(\hat{x}).$$

Since $T(\hat{x})$ is w^* -compact and convex, by Kneser's minimax theorem [10], there exists some $\hat{y} \in T(\hat{x})$ such that

$$\sup_{z \in riS(\hat{x})} \langle \hat{y}, z - \hat{x} \rangle = \inf_{y \in T(\hat{x})} \sup_{z \in riS(\hat{x})} \langle y, z - \hat{x} \rangle = \sup_{z \in riS(\hat{x})} \inf_{y \in T(\hat{x})} \langle y, z - \hat{x} \rangle \geq 0.$$

This implies that

$$\sup_{z \in S(\hat{x})} \langle \hat{y}, z - \hat{x} \rangle \geq 0.$$

The last inequality is equivalent to (2.15), and therefore, the proof is complete.

REFERENCES

- [1] J.P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, John Wiley & Sons, New York, 1984.
- [2] D. Chan and J.S. Pang, *The generalized quasi-variational inequality problem*, Math. Operations Research **7** (1982), 211–222.
- [3] L.J. Chu and C.Y. Lin, *Variational inequalities in noncompact nonconvex regions*, Disc. Math. Differential Inclusions, Control and Optimization **23** (2003), 5–19.
- [4] P. Cubiotti, *Finite-dimensional quasi-variational inequalities associated with discontinuous functions*, J. Optimization Theory and Applications **72** (1992), 577–582.
- [5] P. Cubiotti, *An existence theorem for generalized quasi-variational inequalities*, Set-Valued Analysis **1** (1993), 81–87.
- [6] P. Cubiotti, *An application of quasivariational inequalities to linear control systems*, J. Optim. Theory Appl. **89** (1) (1996), 101–113.
- [7] P. Cubiotti, *Generalized quasi-variational inequalities without continuities*, J. Optim. Theory Appl. **92** (3) (1997), 477–495.
- [8] P. Cubiotti, *Generalized quasi-variational inequalities in infinite-dimensional normed spaces*, J. Optim. Theory Appl. **92** (3) (1997), 457–475.
- [9] E. Klein and A.C. Thompson, *Theorem of Correspondences*, Wiley, New York, 1984.
- [10] H. Kneser, *Sur un théorème fondamental de la théorie des jeux*, Comptes Rendus de l'Académie des Sciences, Paris **234** (1952), 2418–2420.
- [11] M.L. Lunsford, *Generalized variational and quasivariational inequalities with discontinuous operators*, J. Math. Anal. Appl. **214** (1997), 245–263.
- [12] B. Ricceri, *Basic existence theorem for generalized variational and quasi-variational inequalities*, Variational Inequalities and Network Equilibrium Problems, Edited by F. Giannessi and A. Maugeri, Plenum Press, New York, 1995 (251–255).

- [13] R. Saigal, *Extension of the generalized complemetarity problem*, Math. Operations Research **1** (3) (1976), 260–266.
- [14] M.H. Shih and K.K. Tan, *Generalized quasi-variational inequalities in locally convex topological vector spaces*, J. Math. Anal. Appl. **108** (1985), 333–343.
- [15] J.C. Yao and J.S. Guo, *Variational and generalized variational inequalities with discontinuous mappings*, J. Math. Anal. Appl. **182** (1994), 371–392.

Received 1 August 2005