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EXISTENCE AND CONTROLLABILITY RESULTS FOR SEMILINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH NONLOCAL CONDITIONS

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Abstract

In this paper, we prove existence and controllability results for first and second order semilinear neutral functional differential inclusions with finite or infinite delay in Banach spaces, with nonlocal conditions. Our theory makes use of analytic semigroups and fractional powers of closed operators, integrated semigroups and cosine families.

Keywords and phrases: semilinear differential inclusions, nonlocal conditions, analytic semigroups, cosine functions, integrated semigroups, fixed point, nonlinear alternative, controllability.

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1. Introduction

In this paper, we shall be concerned with the existence and controllability for first and second order semilinear neutral functional differential inclusions in a real Banach space, with nonlocal conditions.

In Section 3, we study first order initial value problems for a semilinear neutral functional differential inclusion with nonlocal conditions of the form,

$$\frac{d}{dt}[y(t) - f(t, y_t)] \in Ay(t) + F(t, y_t), \quad \text{a.e. } t \in J = [0, T] \tag{1.1}$$

$$y(t) + h_t(y) = \phi(t), \quad t \in [-r, 0], \tag{1.2}$$

where $f : J \times \mathcal{D} \to E$, $F : J \times \mathcal{D} \longrightarrow \mathcal{P}(E)$ is a multivalued map, $h_t \in \mathcal{D}$, $\phi \in \mathcal{D}$, $\mathcal{D} = \{\psi : [-r, 0] \to E | \psi \text{ is continuous}\}$, A is the infinitesimal generator of a strongly continuous semigroup $S(t), t \geq 0$ and E is a separable real Banach space with the norm $\|\cdot\|$.

For any continuous function y defined on the interval $[-r, T]$ and any $t \in J$, we denote by y_t the element of $\mathcal{D}$ defined by

$$y_t(\theta) = y(t + \theta), \ \ \theta \in [-r, 0].$$

For $\psi \in \mathcal{D}$ the norm of ψ is defined by

$$\|\psi\|_{\mathcal{D}} = \sup\{\|\psi(\theta)\| : \theta \in [-r, 0]\}.$$

The nonlocal condition $h_t(y)$ may be given by

$$h_t(y) = \sum_{i=1}^{p} c_i y(t_i + t), \quad t \in [-r, 0]$$

where $c_i, i = 1, \dots, p$, are given constants and $0 < t_1 < \dots < t_p \leq T$. At time $t = 0$, we have

$$h_0(y) = \sum_{i=1}^{p} c_i y(t_i).$$

In Section 4, we consider a general form of the problem (1.1)–(1.2) where $A : D(A) \subset E \to E$ is a nondensely defined closed linear operator.

In Section 5, we study second order initial value problems for a semilinear neutral functional differential inclusion with nonlocal conditions of the form

$$\frac{d}{dt}[y'(t) - f(t, y_t)] \in Ay(t) + F(t, y_t), \ \ t \in J := [0, T], \tag{1.3}$$

$$y(t) + h_t(y) = \phi(t), \ \ t \in [-r, 0], \quad y'(0) + h_1(y) = \eta, \tag{1.4}$$

where A is the infinitesimal generator of a family of cosine operators $\{C(t) : t \geq 0\}$, $\eta \in E$ and f, F, ϕ, h_t are as in the problem (1.1)–(1.2) and $h_1 : C(J, E) \to E$ is continuous.

Nonlocal conditions for evolution equations were initiated by Byszewski. We refer the reader to [7] and the references cited therein for a motivation regarding nonlocal initial conditions. The nonlocal condition can be applied in physics and is more natural than the classical initial condition $y(0) = y_0$.

IVPs (1.1)–(1.2) and (1.3)–(1.4) were studied in the literature under growth conditions on F. Here, by using the ideas in [2] we obtain new results if instead of growth conditions we assume the existence of a maximal solution to an appropriate problem.

Our existence theory is based on fixed point methods, in particular the Leray-Schauder Alternative for single valued and Kakutani maps, Kakutani's fixed point theorem and on a selection theorem for lower semicontinuous maps.

In Section 6, we study controllability results for the problems (1.1)–(1.2) and (1.3)–(1.4) by using the Leray-Schauder Alternative for Kakutani maps. We refer to [5] for recent controllability results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that are used throughout this paper.

Let (X, d) be a metric space. We use the notations:
$\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $\mathcal{P}_c(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$, $\mathcal{P}_{c,cp}(X) = \mathcal{P}_c(X) \cap \mathcal{P}_{cp}(X)$ etc. A multivalued map $G : X \to \mathcal{P}(X)$ is *convex* (*closed*) *valued* if $G(x)$ is convex (closed) for all $x \in X$. G is *bounded on bounded sets* if $G(\mathcal{B}) = \cup_{x \in \mathcal{B}} G(x)$ is bounded in X for all $\mathcal{B} \in \mathcal{P}_b(X)$ (i.e., $\sup_{x \in \mathcal{B}}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called *upper semi-continuous* (*u.s.c.*) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set $\mathcal{U}$ of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{V}$ of x_0 such that $G(\mathcal{V}) \subseteq \mathcal{U}$.

G is said to be *completely continuous* if $G(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in \mathcal{P}_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a *fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.

A multivalued map $N : J \to \mathcal{P}_{cl}(E)$ is said to be *measurable*, if for every $y \in E$, the function $t \longmapsto d(y, N(t)) = \inf\{\|y-z\| : z \in N(t)\}$ is measurable. For more details on multivalued maps see the books of Aubin and Cellina [4], Deimling [10], Górniewicz [12] and Hu and Papageorgiou [17].

Throughout this paper, E will be a separable Banach space provided with norm $\|\cdot\|$ and $A : D(A) \to E$ will be the infinitesimal generator of an analytic semigroup, $S(t),\ t \geq 0$, of bounded linear operators on E. For the theory of strongly continuous semigroup, we refer the reader to Pazy [21]. If $S(t),\ t \geq 0$, is a uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha \leq 1$, as closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in E, and the expression

$$\|x\|_\alpha = \|(-A)^\alpha x\|, \quad x \in D(-A)^\alpha$$

defines a norm on $D(-A)^\alpha$. Hereafter we denote by E_α the Banach space $D(-A)^\alpha$ normed with $\|\cdot\|_\alpha$. Then for each $0 < \alpha \leq 1$, E_α is a Banach space, and $E_\alpha \hookrightarrow E_\beta$ for $0 < \beta \leq \alpha \leq 1$ and the imbedding is compact whenever the resolvent operator of A is compact. Also for every $0 < \alpha \leq 1$ there exists $C_\alpha > 0$ such that

$$\|(-A)^\alpha S(t)\| \leq \frac{C_\alpha}{t^\alpha}, \quad 0 < t \leq T.$$

We say that a family $\{C(t) \mid t \in \mathbb{R}\}$ of operators in $B(E)$ is a *strongly continuous cosine family* if

(i) $C(0) = I$,

(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$, for all $s, t \in \mathbb{R}$,

(iii) the map $t \mapsto C(t)(x)$ is strongly continuous, for each $x \in E$.

The strongly continuous sine family $\{S(t) \mid t \in \mathbb{R}\}$, associated with the given strongly continuous cosine family $\{C(t) \mid t \in \mathbb{R}\}$, is defined by

$$S(t)(x) = \int_0^t C(s)(x)\, ds, \quad x \in E,\ t \in \mathbb{R}. \tag{2.1}$$

The infinitesimal generator $A : E \to E$ of a cosine family $\{C(t) \mid t \in \mathbb{R}\}$ is defined by

$$A(x) = \frac{d^2}{dt^2} C(t)(x)\Big|_{t=0}.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books of Goldstein [13], Heikkila and Lakshmikantham [15] and Fattorini [11] and the papers [22] and [23].

Proposition 2.1 [22]. *Let $C(t), t \in \mathbb{R}$ be a strongly continuous cosine family in E. Then:*

(i) *there exist constants $M_1 \geq 1$ and $\omega \geq 0$ such that $\|C(t)\| \leq M_1 e^{\omega|t|}$ for all $t \in \mathbb{R}$;*

(ii) $\|S(t_1) - S(t_2)\| \leq M_1 \left| \int_{t_2}^{t_1} e^{\omega|s|} ds \right|$ *for all $t_1, t_2 \in \mathbb{R}$.*

Definition 2.2. A multi-valued map $F : J \times \mathcal{D} \to \mathcal{P}_{c,cp}(E)$ is said to be L^1-Carathéodory if

(i) $t \mapsto F(t, x)$ is measurable for each $x \in \mathcal{D}$,

(ii) $x \mapsto F(t, x)$ is upper semi-continuous for almost all $t \in J$, and

(iii) for each real number $\rho > 0$, there exists a function $h_\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, u)\| := \sup\{\|v\| : v \in F(t, u)\} \leq h_\rho(t), \quad a.e.\ t \in J$$

for all $u \in \mathcal{D}$ with $\|u\|_{\mathcal{D}} \leq \rho$.

We need also the following result, see [16].

Lemma 2.3. *Let $v(\cdot), w(\cdot) : [0, T] \to [0, \infty)$ be continuous functions. If $w(\cdot)$ is nondecreasing and there are constants $\theta > 0, 0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + \theta \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}}\, ds, \quad t \in [0, T],$$

then

$$v(t) \le e^{\theta^n \Gamma(\alpha)^n t^{n\alpha}/\Gamma(n\alpha)} \sum_{j=0}^{n-1} \left(\frac{\theta T^\alpha}{\alpha}\right)^j w(t),$$

for every $t \in [0,T]$ *and every* $n \in \mathbb{N}$ *such that* $n\alpha > 1$, *and* $\Gamma(\cdot)$ *is the Gamma function.*

3. First order semilinear neutral functional differential inclusions with nonlocal conditions

Let us start by defining what we mean by a solution to the problem (1.1)–(1.2).

Definition 3.1. A function $y \in C([-r,T],E)$ is said to be a mild solution of (1.1)–(1.2) if $y(t) + h_t(y) = \phi(t)$ on $[-r,0]$, and exists $v \in L^1(J,E)$ such that $v(t) \in F(t,y_t)$ a.e on J and

$$\begin{aligned} y(t) &= S(t)[\phi(0) - h_0(y) - f(0,\phi)] + f(t,y_t) + \int_0^t AS(t-s)f(s,y_s)\,ds \\ &\quad + \int_0^t S(t-s)v(s)\,ds,\ t \in J. \end{aligned}$$

For the multivalued map F and for each $y \in C(J,E)$, we define $S_{F,y}$ by

$$S_{F,y} = \{v \in L^1(J,E) : v(t) \in F(t,y(t)) \ \text{ for a.e. } \ t \in J\}.$$

Our first existence result for the IVP (1.1)–(1.2) is the following.

Theorem 3.2. *Assume that:*

(3.2.1) $A : D(A) \subset E \to E$ *is the infinitesimal generator of an analytic semigroup* $S(t), t \ge 0$, *of bounded linear operators on* E. *Assume that* $0 \in \rho(A)$, $S(t)$ *is compact for* $t > 0$, *and there exist constants* $M \ge 1$ *and* $C_{1-\beta}$ *such that*

$$\|S(t)\|_{B(E)} \le M \quad \text{and} \quad \|(-A)^{1-\beta}S(t)\| \le \frac{C_{1-\beta}}{t^{1-\beta}}, \quad forall\ t > 0;$$

(3.2.2) (i) *the map* $H : C([-r,T],E) \to C([-r,T],E)$, *given by* $H(y)(t) = f(t,y_t)$ *for* $t \in [0,T]$, *is continuous and completely continuous;*

(ii) *f is E_β-valued, and there exist constants $c_1, c_2 \geq 0$ such that $c_1\|(-A)^{-\beta}\| < 1$ and*

$$\|(-A)^\beta f(t,x)\| \leq c_1\|x\|_{\mathcal{D}} + c_2, \quad (t,x) \in J \times \mathcal{D};$$

(3.2.3) *given $\epsilon > 0$, then for any bounded subset D of $C([-r,T],E)$ there exists a $\delta > 0$ with $\|(S(h)-I)h_0(y)\| < \epsilon$ for all $y \in D$ and $h \in [0,\delta]$ and $\|h_t(y)-h_s(y)\| < \epsilon$ for all $y \in D$ and $t,s \in [-r,0]$ with $|t-s| < \delta$;*

(3.2.4) *for each $t \in [-r,0]$ the function h_t is continuous and completely continuous and there exists $Q > 0$ such that $\|h_t(u)\| \leq Q, u \in C([-r,b],E)$ and $t \in [-r,0]$;*

(3.2.5) *$F : J \times E \to \mathcal{P}_{c,cp}(E)$ is an L^1-Carathéodory multivalued map;*

(3.2.6) *there exist an L^1-Carathéodory function $g : J \times [0,\infty) \to [0,\infty)$ such that*

$$\|F(t,u)\| := \sup\{\|v\| : v \in F(t,u)\} \leq g(t, \|u\|_{\mathcal{D}})$$

for almost all $t \in J$ and all $u \in \mathcal{D}$;

(3.2.7) *$g(t,x)$ is nondecreasing in x for a.e. $t \in J$;*

(3.2.8) *the problem*

$$v'(t) = bK_2 g(t, v(t)), \quad a.e.\ t \in J,$$
$$v(0) = bK_0,$$

where

$$K_0 = \Lambda(1 - c_1\|(-A)^{-\beta}\|)^{-1},$$

$$K_1 = C_{1-\beta}c_1(1 - c_1\|(-A)^{-\beta}\|)^{-1},$$

$$K_2 = M(1 - c_1\|(-A)^{-\beta}\|)^{-1},$$

$$b = e^{K_1^n(\Gamma(\beta))^n T^{n\beta}/\Gamma(n\beta)} \sum_{j=0}^{n-1} \left(\frac{K_1 T^\beta}{\beta}\right)^j,$$

$$\Lambda = M\|\phi\|_{\mathcal{D}} \left\{1 + c_1\|(-A)^{-\beta}\|\right\} + MQ + c_2\|(-A)^{-\beta}\|\{M+1\} + \frac{C_{1-\beta}c_2T^\beta}{\beta},$$

and n is the first integer such that $n\beta > 1$, has a maximal solution $r(t)$.

Then the IVP (1.1)–(1.2) *has at least one mild solution on* $[-r,T]$.

Proof. Transform the problem (1.1)–(1.2) into a fixed point problem. Consider the operator $N : C([-r,T],E) \longrightarrow \mathcal{P}(C([-r,T],E))$ defined by:

$$N(y) =$$

$$= \left\{ h \in C([-r,T],E) : h(t) = \begin{cases} \phi(t) - h_t(y), & t \in [-r,0], \\ S(t)[\phi(0) - h_0(y) - f(0,\phi(0))] & \\ +f(t,y_t) + \int_0^t AS(t-s)f(s,y_s)ds & \\ + \int_0^t S(t-s)v(s)ds, & t \in J, \end{cases} \right\}$$

where $v \in S_{F,y}$.

We shall show that N has a fixed point. The proof is given in several steps.

Step 1. $N(y)$ is convex for each $y \in C([-r,T],E)$.
This is obvious since $S_{F,y}$ is convex (because F has convex values).

Step 2. N maps bounded sets into bounded sets in $C([-r,T],E)$.
Let $B_q := \{y \in C([-r,T],E) : \|y\| = \sup_{t\in[-r,T]} \|y(t)\| \le q\}$ be a bounded set in $C([-r,T],E)$ and $y \in B_q$. Then for each $h \in N(y)$ there exists $v \in S_{F,y}$ such that

$$\begin{aligned} h(t) &= S(t)[\phi(0) - h_0(y) - f(0,\phi(0))] + f(t,y_t) \\ &\quad + \int_0^t AS(t-s)f(s,y_s)ds + \int_0^t S(t-s)v(s)ds, \;\; t \in J. \end{aligned}$$

Thus for each $t \in J$ we get (see [21, Theorem 2.6.8 (d) and Theorem 2.6.13 (b)]

$$\|h(t)\| \le$$

$$\le M\|\phi\|_{\mathcal{D}} + MQ + M\|(-A)^{-\beta}\|(c_1\|\phi\|_{\mathcal{D}} + c_2) + \|(-A)^{-\beta}\|[c_1\|y_t\|_{\mathcal{D}} + c_2]$$

$$+ \int_0^t \|(-A)^{1-\beta}S(t-s)\|\|(-A)^{\beta}f(s,y_s)\|ds + M\int_0^t \|v(s)\|\, ds$$

$$\leq M\|\phi\|_{\mathcal{D}} + MQ + M\|(-A)^{-\beta}\|(c_1\|\phi\|_{\mathcal{D}} + c_2) + \|(-A)^{-\beta}\|[c_1\|y_t\|_{\mathcal{D}} + c_2]$$

$$+ C_{1-\beta}c_1 \int_0^t \frac{\|y_s\|_{\mathcal{D}}}{(t-s)^{1-\beta}} ds + \frac{C_{1-\beta}c_2T^\beta}{\beta} + M\int_0^t h_\rho(s)\, ds$$

$$\leq M\|\phi\|_{\mathcal{D}} + MQ + M\|(-A)^{-\beta}\|(c_1\|\phi\|_{\mathcal{D}} + c_2) + \|(-A)^{-\beta}\|[c_1 q + c_2]$$

$$+ \frac{C_{1-\beta}T^\beta}{\beta}[c_1 q + c_2] + M\|h_\rho\|_{L^1};$$

here h_q is chosen as in Definition 2.2. Then for each $h \in N(B_q)$ we have

$$\|h\| \leq M\|\phi\|_{\mathcal{D}} + MQ + M\|(-A)^{-\beta}\|(c_1\|\phi\|_{\mathcal{D}} + c_2) + \|(-A)^{-\beta}\|[c_1 q + c_2]$$

$$+ \frac{C_{1-\beta}T^\beta}{\beta}[c_1 q + c_2] + M\|h_\rho\|_{L^1} := \ell.$$

Step 3. N maps bounded sets into equicontinuous sets of $C([-r,T],E)$. We consider B_q as in Step 2 and let $h \in N(y)$ for $y \in B_q$. Let $\epsilon > 0$ be given. Now let $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$. We consider two cases $\tau_1 > \epsilon$ and $\tau_1 \leq \epsilon$.

Case 1. If $\tau_1 > \epsilon$, then

$$\|h(\tau_2) - h(\tau_1)\| \leq$$

$$\leq \|\, [S(\tau_2) - S(\tau_1)][\phi(0) - h_0(y) - f(0,\phi(0)]\, \|$$

$$+ \|f(\tau_2, y_{\tau_2}) - f(\tau_1, y_{\tau_1})\|$$

$$+ \int_0^{\tau_1-\epsilon} \|(-A)^{1-\beta}[S(\tau_2 - s) - S(\tau_1 - s)](-A)^\beta f(s,y_s)\| ds$$

$$+ \int_{\tau_1-\epsilon}^{\tau_1} \|(-A)^{1-\beta}[S(\tau_2 - s) - S(\tau_1 - s)](-A)^\beta f(s,y_s)\| ds$$

$$+ \int_{\tau_1}^{\tau_2} \|(-A)^{1-\beta}S(\tau_2 - s)(-A)^\beta f(s,y_s)\| ds$$

$$+ \int_0^{\tau_1-\epsilon} \|S(\tau_2 - s) - S(\tau_1 - s)\| \|v(s)\| ds$$

$$
\begin{aligned}
&+\int_{\tau_1-\epsilon}^{\tau_1}\|S(\tau_2-s)-S(\tau_1-s)\|\|v(s)\|ds\\
&+\int_{\tau_1}^{\tau_2}\|S(\tau_2-s)\|\|v(s)\|ds\\
\leq\ &\|\,[S(\tau_2)-S(\tau_1)][\phi(0)-f(0,\phi(0)]\,\|\\
&+M\|S(\tau_2-\tau_1+\epsilon)-S(\epsilon)\|_{B(E)}\|h_0(B_q)\|\\
&+\|f(\tau_2,y_{\tau_2})-f(\tau_1,y_{\tau_1})\|\\
&+M\int_0^{\tau_1-\epsilon}\|(-A)^{1-\beta}[S(\tau_2-\tau_1+\epsilon)-S(\epsilon)](-A)^{\beta}f(s,y_s)\|ds\\
&+M\|(-A)^{-\beta}\|C_{1-\beta}(c_1q+c_2)\left[\frac{(\tau_2-\tau_1+\epsilon)^{\beta}}{\beta}-\frac{(\tau_2-\tau_1)^{\beta}}{\beta}+\frac{\epsilon^{\beta}}{\beta}\right]\\
&+M\|(-A)^{-\beta}\|C_{1-\beta}(c_1q+c_2)\frac{(\tau_2-\tau_1)^{\beta}}{\beta}\\
&+M\|S(\tau_2-\tau_1+\epsilon)-S(\epsilon)\|_{B(E)}\int_0^{\tau_1-\epsilon}h_q(s)ds\\
&+2M\int_{\tau_1-\epsilon}^{\tau_1}h_q(s)ds\\
&+M\int_{\tau_1}^{\tau_2}h_q(s)ds\\
\leq\ &\|\,[S(\tau_2)-S(\tau_1)][\phi(0)-f(0,\phi(0)]\,\|\\
&+M\|S(\tau_2-\tau_1+\epsilon)-S(\epsilon)\|_{B(E)}\|h_0(B_q)\|\\
&+\|f(\tau_2,y_{\tau_2})-f(\tau_1,y_{\tau_1})\|\\
&+M\|(-A)^{1-\beta}\|\|S(\tau_2-\tau_1+\epsilon)-S(\epsilon)\|_{B(E)}\int_0^{\tau_1-\epsilon}[c_1q+c_2]ds\\
&+M\|(-A)^{-\beta}\|C_{1-\beta}(c_1q+c_2)\left[\frac{(\tau_2-\tau_1+\epsilon)^{\beta}}{\beta}-\frac{(\tau_2-\tau_1)^{\beta}}{\beta}+\frac{\epsilon^{\beta}}{\beta}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ M\|(-A)^{-\beta}\|C_{1-\beta}(c_1 q + c_2)\frac{(\tau_2-\tau_1)^\beta}{\beta}\\
&+ M\|S(\tau_2-\tau_1+\epsilon) - S(\epsilon)\|_{B(E)}\int_0^{\tau_1-\epsilon} h_q(s)ds\\
&+ 2M\int_{\tau_1-\epsilon}^{\tau_1} h_q(s)ds\\
&+ M\int_{\tau_1}^{\tau_2} h_q(s)ds
\end{aligned}
$$

where we have used the semigroup identities

$$
S(\tau_2-s) = S(\tau_2-\tau_1+\epsilon)S(\tau_1-s-\epsilon), \quad S(\tau_1-s) = S(\tau_1-s-\epsilon)S(\epsilon),
$$

$$
S(\tau_2) = S(\tau_2-\tau_1+\epsilon)S(\tau_1-\epsilon), \quad S(\tau_1) = S(\tau_1-\epsilon)S(\epsilon).
$$

Case 2. Let $\tau_1 \leq \epsilon$. For $\tau_2 - \tau_1 < \epsilon$ we get

$$
\begin{aligned}
|h(\tau_2) - h(\tau_1)| \leq\ & \|\,[S(\tau_2) - S(\tau_1)][\phi(0) - f(0,\phi(0)]\,\|\\
&+ M\|S(\tau_2-\tau_1)h_0(y) - h_0(y)\|\\
&+ \|f(\tau_2, y_{\tau_2}) - f(\tau_1, y_{\tau_1})\|\\
&+ \int_0^{\tau_2} \|(-A)^{1-\beta}S(\tau_2-s)(-A)^\beta f(s,y_s)\|ds\\
&+ \int_0^{\tau_1} \|(-A)^{1-\beta}S(\tau_1-s)(-A)^\beta f(s,y_s)\|ds\\
&+ \int_0^{\tau_2} \|S(\tau_2-s)\|h_q(s)ds\\
&+ \int_0^{\tau_1} \|S(\tau_1-s)\|h_q(s)ds\\
\leq\ & \|\,[S(\tau_2) - S(\tau_1)][\phi(0) - f(0,\phi(0)]\,\|\\
&+ M\|S(\tau_2-\tau_1)h_0(y) - h_0(y)\|
\end{aligned}
$$

$$
\begin{aligned}
&+\|f(\tau_2, y_{\tau_2}) - f(\tau_1, y_{\tau_1})\| \\
&+\|(-A)^{-\beta}\|C_{1-\beta}(c_1q+c_2)\frac{(2\epsilon)^\beta}{\beta} \\
&+\|(-A)^{-\beta}\|C_{1-\beta}(c_1q+c_2)\frac{\epsilon^\beta}{\beta} \\
&+M\int_0^{2\epsilon} h_q(s)ds \\
&+M\int_0^{\epsilon} h_q(s)ds.
\end{aligned}
$$

Now (3.2.2), (3.2.3) and the fact that $s \to (-A)^{1-\beta}S(s)$ is continuous in the uniform operator topology on $(0,T]$ implies the equicontinuity.

The equicontinuity for the case $\tau_1 < \tau_2 \leq 0$ follows from the uniform continuity of ϕ on the interval $[-r, 0]$, and for the case $\tau_1 \leq 0 \leq \tau_2$ by combining the previous cases.

Let $0 < t \leq T$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_q$ and $v \in S_{F,y}$ we define

$$
\begin{aligned}
r_\epsilon(t) &= \int_0^{t-\epsilon} AS(t-s)f(s,y_s)ds + \int_0^{t-\epsilon} S(t-s)v(s)ds \\
&= \int_0^{t-\epsilon} (-A)^{1-\beta}S(t-s)(-A)^\beta f(s,y_s)ds \\
&\quad + S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)v(s)ds \\
&= S(\epsilon)\int_0^{t-\epsilon} (-A)^{1-\beta}S(t-s-\epsilon)(-A)^\beta f(s,y_s)ds \\
&\quad + S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)v(s)ds.
\end{aligned}
$$

Note

$$
\left\{\int_0^{t-\epsilon} (-A)^{1-\beta}S(t-s-\epsilon)(-A)^\beta f(s,y_s)ds : y \in B_q \text{ and } v \in S_{F,y}\right\}
$$

is a bounded set since

$$\Big\|\int_0^{t-\epsilon}(-A)^{1-\beta}S(t-s-\epsilon)(-A)^{\beta}f(s,y_s)ds\Big\|$$

$$\leq MC_{1-\beta}\|(-A)^{-\beta}\|(c_1q+c_2)\int_0^{t-\epsilon}\frac{ds}{(t-s-\epsilon)^{1-\beta}}.$$

Also, note

$$\left\{\int_0^{t-\epsilon}S(t-s-\epsilon)v(s)ds : y\in B_q \text{ and } v\in S_{F,y}\right\}$$

is a bounded set since $\big\|\int_0^{t-\epsilon}S(t-s-\epsilon)v(s)ds\big\| \leq M\int_0^{t-\epsilon}h_q(s)ds$ and now since $S(t)$ is a compact operator for $t>0$, the set $Y_\varepsilon(t)=\{r_\varepsilon(t) : y\in B_q \text{ and } v\in S_{F,y}\}$ is relatively compact in E for every ε, $0<\varepsilon<t$. Moreover, for $r=r_0$ we have

$$\|r(t)-r_\varepsilon(t)\|\leq MC_{1-\beta}\|(-A)^{-\beta}\|(c_1q+c_2)\int_0^{t-\epsilon}\frac{ds}{(t-s-\epsilon)^{1-\beta}}$$

$$+M\int_{t-\varepsilon}^{t}h_q(s)ds.$$

Therefore, the set $Y(t)=\{r(t) : y\in B_q \text{ and } v\in S_{F,y}\}$ is totally bounded. Hence $Y(t)$ is relatively compact in E.

As a consequence of Steps 2, 3 and the Arzelá-Ascoli theorem we can conclude that $N : C([-r,T],E)\longrightarrow \mathcal{P}(C([-r,T],E))$ is completely continuous.

Step 4. N has a closed graph.
Let $y_n\longrightarrow y_*$, $h_n\in N(y_n)$, and $h_n\longrightarrow h_*$. We shall prove that $h_*\in N(y_*)$. Now $h_n\in N(y_n)$ means that there exists $v_n\in S_{F,y_n}$ such that for each $t\in J$

$$h_n(t)=S(t)[\phi(0)-h_0(y_n)-f(0,\phi(0))]+f(t,y_{nt})+\int_0^t AS(t-s)f(s,y_{ns})\,ds$$

$$+\int_0^t S(t-s)v_n(s)ds.$$

We must prove that there exists $v_*\in S_{F,y_*}$ such that for each $t\in J$

$$h_*(t)=S(t)[\phi(0)-h_0(y_*)-f(0,\phi(0))]+f(t,y_{*t})+\int_0^t AS(t-s)f(s,y_{*s})\,ds$$

$$+\int_0^t S(t-s)v_*(s)ds.$$

Now since $s \to AT(t-s)$ is continuous in the uniform operator topology on $[0,t)$ we have that

$$\Big\|\Big(h_n - S(t)[\phi(0) - h_0(y_n) - f(0,\phi(0))] - f(t,y_{nt}) - \int_0^t AS(t-s)f(s,y_{ns})\,ds\Big)$$
$$-\Big(h_* - S(t)[\phi(0) - h_0(y_*) - f(0,\phi(0))] - f(t,y_{*t}) - \int_0^t AS(t-s)f(s,y_{*s})\,ds\Big)\Big\| \to 0,$$

as $n \to \infty$.

Consider the linear continuous operator

$$\Gamma : L^1(J,E) \longrightarrow C(J,E)$$

$$v \longmapsto \Gamma(v)(t) = \int_0^t S(t-s)v(s)ds.$$

It follows that $\Gamma \circ S_F$ is a closed graph operator ([20]).

Also from the defintion of Γ we have that

$$h_n(t) - S(t)[\phi(0) - h_0(y_n) - f(0,\phi(0))] - f(t,y_{nt}) - \int_0^t AS(t-s)f(s,y_{ns})\,ds$$
$$\in \Gamma(S_{F,y_n}).$$

Since $y_n \longrightarrow y_*$, it follows that

$$h_*(t) - S(t)[\phi(0) - h_0(y_*) - f(0,\phi(0))] - f(t,y_{*t}) - \int_0^t AS(t-s)f(s,y_{*s})\,ds$$
$$= \int_0^t S(t-s)v_*(s)ds$$

for some $v_* \in S_{F,y_*}$.

Step 5. Now it remains to show that the set

$$\mathcal{M} := \{y \in C([-r,T],E) : \lambda y \in N(y), \quad \text{for some} \quad \lambda > 1\}$$

is bounded.

Let $y \in \mathcal{M}$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus there exists $v \in S_{F,y}$ such that

$$\begin{aligned} y(t) &= \lambda^{-1}S(t)[\phi(0)-h_0(y)-f(0,\phi(0))]+\lambda^{-1}f(t,y_t) \\ &\quad + \lambda^{-1}\int_0^t AS(t-s)f(s,y_s)ds + \lambda^{-1}\int_0^t S(t-s)v(s)ds, \;\; t\in J. \end{aligned}$$

Then

$$\begin{aligned} &\|y(t)\| \leq \\ &\leq M\|\phi\|_{\mathcal{D}} + MQ + M\|(-A)^{-\beta}\|[c_1\|\phi\|_{\mathcal{D}}+c_2] + \|(-A)^{-\beta}\|[c_1\|y_t\|_{\mathcal{D}}+c_2] \\ &\quad + \int_0^t \|(-A)^{1-\beta}S(t-s)\|\|(-A)^{\beta}f(s,y_s)\|\,ds + M\int_0^t g(s,\|y_s\|_{\mathcal{D}})ds \\ &\leq M\|\phi\|_{\mathcal{D}} + MQ + M\|(-A)^{-\beta}\|[c_1\|\phi\|_{\mathcal{D}}+c_2] + \|(-A)^{-\beta}\|[c_1\|y_t\|_{\mathcal{D}}+c_2] \\ &\quad + C_{1-\beta}c_1\int_0^t \frac{\|y_s\|_{\mathcal{D}}}{(t-s)^{1-\beta}}\,ds + \frac{C_{1-\beta}c_2T^{\beta}}{\beta} + M\int_0^t g(s,\|y_s\|_{\mathcal{D}})ds \\ &\leq \Lambda + c_1\|(-A)^{-\beta}\|\|y_t\|_{\mathcal{D}} \\ &\quad + C_{1-\beta}c_1\int_0^t \frac{\|y_s\|_{\mathcal{D}}}{(t-s)^{1-\beta}}\,ds + M\int_0^t g(s,\|y_s\|_{\mathcal{D}})ds, \quad t\in J. \end{aligned}$$

Put $w(t)=\max\{\|y(s)\| : -r\leq s\leq t\}$, $t\in J$. Then $\|y_t\|_{\mathcal{D}}\leq w(t)$ for all $t\in J$ and there is a point $t^*\in[-r,t]$ such that $w(t)=y(t^*)$. Hence we have

$$\begin{aligned} w(t) &= \|y(t^*)\| \\ &\leq \Lambda + c_1\|(-A)^{-\beta}\|\|y_{t^*}\|_{\mathcal{D}} + C_{1-\beta}c_1\int_0^{t^*}\frac{\|y_s\|_{\mathcal{D}}}{(t-s)^{1-\beta}}\,ds \\ &\quad + M\int_0^{t^*} g(s,\|y_s\|_{\mathcal{D}})ds \\ &\leq \Lambda + c_1\|(-A)^{-\beta}\|w(t) + C_{1-\beta}c_1\int_0^t \frac{w(s)}{(t-s)^{1-\beta}}\,ds \\ &\quad + M\int_0^t g(s,w(s))\,ds, \end{aligned}$$

or

$$w(t) \leq \frac{1}{1 - c_1\|(-A)^{-\beta}\|}\left\{\Lambda + C_{1-\beta}c_1\int_0^t \frac{w(s)}{(t-s)^{1-\beta}}\,ds + M\int_0^t g(s,w(s))\,ds\right\}$$

$$\leq K_0 + K_1\int_0^t \frac{w(s)}{(t-s)^{1-\beta}}\,ds + K_2\int_0^t g(s,w(s))\,ds, \quad t \in J.$$

From Lemma 2.3 we have

$$w(t) \leq b\left(K_0 + K_2\int_0^t g(s,w(s))\,ds\right),$$

where

$$b = e^{K_1^n(\Gamma(\beta))^n T^{n\beta}/\Gamma(n\beta)}\sum_{j=0}^{n-1}\left(\frac{K_1T^\beta}{\beta}\right)^j.$$

Let

$$m(t) = b\left(K_0 + K_2\int_0^t g(s,w(s))\,ds\right), \quad t \in J.$$

Then we have $w(t) \leq m(t)$ for all $t \in J$. Differentiating with respect to t, we obtain

$$m'(t) = bK_2g(t,w(t)), \text{ a.e. } t \in J, \; m(0) = bK_0.$$

Using the nondecreasing character of g we get

$$m'(t) \leq bK_2g(t,m(t)), \; t \in J.$$

This implies that ([19] Theorem 1.10.2) $m(t) \leq r(t)$ for $t \in J$, and hence $w(t) \leq b_0 = \sup_{t\in J} r(t)$. Thus

$$\sup\{\|y(t)\| : -r \leq t \leq T\} \leq b_0' := \max\{\|\phi\|_{\mathcal{D}}, b_0\},$$

where b_0 depends only on T and on the function r. This shows that $\mathcal{M}$ is bounded.

As a consequence of the Leray-Schauder Alternative for Kakutani maps [14] we deduce that N has a fixed point which is a solution of (1.1)–(1.2). ■

Theorem 3.3. *Assume that* (3.2.1)–(3.2.5) *hold. In addition, suppose that the following condition is satisfied:*

(3.3.1) *there exists a continuous non-decreasing function* $\psi : [0, \infty) \longrightarrow (0, \infty)$, $p \in L^1(J, \mathbb{R}_+)$ *such that*

$$\begin{aligned}\|F(t,u)\| &:= \sup\{\|v\| : v \in F(t,u)\} \\ &\leq p(t)\psi(\|u\|_{\mathcal{D}}) \quad \textit{for each} \quad (t,u) \in J \times \mathcal{D}\end{aligned}$$

and there exists a constant $M_* > 0$ *with*

$$\frac{\left(1 - K_1 \dfrac{T^\beta}{\beta}\right) M_*}{K_0 + K_2 \psi(M_*) \displaystyle\int_0^T p(s)\, ds} > 1,$$

where K_0, K_1, K_2 *are defined in Theorem* 3.2, *with* $1 - K_1 \frac{T^\beta}{\beta} > 0$.

Then the IVP (1.1)–(1.2) *has at least one mild solution on* $[-r, T]$.

Proof. Define N as in the proof of Theorem 3.2. As in Theorem 3.2 we can prove that N is completely continuous.

We show there exists an open set $U \subseteq C(J, E)$ with $y \notin \lambda N(y)$ for $\lambda \in (0,1)$ and $y \in \partial U$. Let $\lambda \in (0,1)$ and let $y \in \lambda N(y)$. Then we have

$$\begin{aligned}
&\|y(t)\| \leq \\
&\leq M\|\phi\|_{\mathcal{D}} + MQ + M\|(-A)^{-\beta}\|[c_1\|\phi\|_{\mathcal{D}} + c_2] + \|(-A)^{-\beta}\|[c_1\|y_t\|_{\mathcal{D}} + c_2] \\
&\quad + \int_0^t \|(-A)^{1-\beta} S(t-s)\| \|(-A)^\beta f(s, y_s)\|\, ds \\
&\quad + M \int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}}) ds \\
&\leq M\|\phi\|_{\mathcal{D}} + MQ + M\|(-A)^{-\beta}\|[c_1\|\phi\|_{\mathcal{D}} + c_2] + \|(-A)^{-\beta}\|[c_1\|y_t\|_{\mathcal{D}} + c_2] \\
&\quad + C_{1-\beta} c_1 \int_0^t \frac{\|y_s\|_{\mathcal{D}}}{(t-s)^{1-\beta}} + \frac{C_{1-\beta} c_2 T^\beta}{\beta} + M \int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}}) ds
\end{aligned}$$

$$\leq \Lambda + c_1\|(-A)^{-\beta}\|\|y_t\|_{\mathcal{D}}$$

$$+ C_{1-\beta}c_1\int_0^t \frac{\|y_s\|_{\mathcal{D}}}{(t-s)^{1-\beta}}\,ds + M\int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}})ds, \quad t\in J.$$

Put $w(t) = \max\{\|y(s)\| : -r \leq s \leq t\}$, $t \in J$. Then $\|y_t\|_{\mathcal{D}} \leq w(t)$ for all $t \in J$ and there is a point $t^* \in [-r, t]$ such that $w(t) = y(t^*)$. Hence we have

$$\begin{aligned}
w(t) &= \|y(t^*)\| \\
&\leq \Lambda + c_1\|(-A)^{-\beta}\|\|y_{t^*}\|_{\mathcal{D}} + C_{1-\beta}c_1\int_0^{t^*}\frac{\|y_s\|_{\mathcal{D}}}{(t-s)^{1-\beta}}\,ds \\
&\quad + M\int_0^{t^*} p(s)\psi(\|y_s\|_{\mathcal{D}})ds \\
&\leq \Lambda + c_1\|(-A)^{-\beta}\|w(t) + C_{1-\beta}c_1\int_0^t\frac{w(s)}{(t-s)^{1-\beta}}\,ds \\
&\quad + M\int_0^t p(s)\psi(w(s))\,ds,
\end{aligned}$$

or

$$w(t) \leq K_0 + K_1\int_0^t \frac{w(s)}{(t-s)^{1-\beta}}\,ds + K_2\int_0^t p(s)\psi(w(s))\,ds, \quad t\in J.$$

Then we have

$$\begin{aligned}
\|w\| &\leq K_0 + K_1\|w\|\int_0^t\frac{1}{(t-s)^{1-\beta}}\,ds + K_2\psi(\|w\|)\int_0^t p(s)\,ds \\
&\leq K_0 + K_1\|w\|\frac{T^\beta}{\beta} + K_2\psi(\|w\|)\int_0^t p(s)\,ds.
\end{aligned}$$

Consequently

$$\frac{\left(1 - K_1\dfrac{T^\beta}{\beta}\right)\|w\|}{K_0 + K_2\psi(\|w\|)\displaystyle\int_0^T p(s)\,ds} \leq 1.$$

Then by (3.3.1), there exists M_* such that $\|w\| \neq M_*$.

Set

$$U = \{y \in C(J, E) : \|y\| < M_*\}.$$

From the choice of U there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for $\lambda \in (0, 1)$. As a consequence of the Leray-Schauder Alternative for Kakutani maps [14] we deduce that N has a fixed point and therefore the problem (1.1)–(1.2) has a solution on $[-r, T]$. ■

Next, we study the case where F is not necessarily convex valued. Our approach here is based on the Leray-Schauder Alternative for single valued maps combined with a selection theorem due to Bressan and Colombo [6] for lower semicontinuous multivalued operators with decomposable values.

Theorem 3.4 *Suppose that:*

(3.4.1) $F : J \times \mathcal{D} \longrightarrow \mathcal{P}(E)$ *is a nonempty, compact-valued, multivalued map such that:*

(a) $(t, u) \mapsto F(t, u)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable;*

(b) $u \mapsto F(t, u)$ *is lower semi-continuous for a.e.* $t \in J$;

(3.4.2) *for each* $\rho > 0$, *there exists a function* $\varphi_\rho \in L^1(J, \mathbb{R}^+)$ *such that*

$$\|F(t, u)\| = \sup\{\|v\| : v \in F(t, u)\} \leq \varphi_\rho(t) \text{ for a.e. } t \in J$$

and for $u \in E$ *with* $\|u\|_{\mathcal{D}} \leq \rho$.

In addition, suppose (3.2.1)–(3.2.4), (3.2.6)–(3.2.8) *are satisfied. Then the initial value problem* (1.1)–(1.2) *has at least one solution on* $[-r, T]$.

Proof. Assumptions (3.4.1) and (3.4.2) imply that F is of lower semicontinuous type. Then there exists ([6]) a continuous function $p : C(J, E) \to L^1(J, E)$ such that $p(y) \in \mathcal{F}(y)$ for all $y \in C(J, E)$, where $\mathcal{F}$ is the Nemitsky operator defined by

$$\mathcal{F}(y) = \{w \in L^1(J, E) : w(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}.$$

Consider the problem

$$\frac{d}{dt}[y(t) - f(t, y_t)] - Ay(t) = p(y)(t), \quad t \in J, \tag{3.1}$$

$$y(t) + h_t(y) = \phi(t), \quad t \in [-r, 0]. \tag{3.2}$$

It is obvious that if $y \in C([-r,T],E)$ is a solution to the problem (3.1)–(3.2), then y is a solution to the problem (1.1)–(1.2).

Transform the problem (3.1)–(3.2) into a fixed point problem considering the operator $N : C([-r,T],E) \to C([-r,T],E)$ defined by:

$$N(y)(t) := \begin{cases} \phi(t) - h_t(y), & \text{if } t \in [-r,0] \\ S(t)[\phi(0) - h_0(y) - f(0,\phi(0))] + f(t,y_t) & \\ + \displaystyle\int_0^t AS(t-s)f(s,y_s)ds & \\ + \displaystyle\int_0^t S(t-s)p(y)(s)ds, & t \in J. \end{cases}$$

We prove that $N : C([-r,T],E) \longrightarrow C([-r,T],E)$ is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \longrightarrow y$ in $C([-r,T],E)$. Then there is an integer q such that $\|y_n\| \le q$ for all $n \in \mathbb{N}$ and $\|y\| \le q$, so $y_n \in B_q$ and $y \in B_q$. We have then by the dominated convergence theorem

$$\begin{aligned} \|N(y_n) - N(y)\| \le\ & M\|h_0(y_n) - h_0(y)\| \\ & + \|f(t,y_{nt}) - f(t,y_t)\| \\ & + \left\| \Big[\int_0^t AS(t-s)|f(s,y_{ns}) - f(s,y_s)|ds \Big] \right\| \\ & + M \left\| \Big[\int_0^t |p(y_n) - p(y)|ds \Big] \right\| \longrightarrow 0. \end{aligned}$$

Thus N is continuous. Also the argument in Theorem 3.2 guarantees that N is completely continuous and that there is no $y \in \partial U$ (U as defined in Theorem 3.3), such that $y = \lambda N(y)$ for some $\lambda \in (0,1)$.

As a consequence of the Leray-Schauder Alternative for single valued maps we deduce that N has a fixed point y which is a mild solution to the problem (3.1)–(3.2). Then y is a mild solution to the problem (1.1)–(1.2). ∎

We state without proof the analogous of Theorem 3.3 for the lower semicontinuous case.

Theorem 3.5. *Assume that the conditions* (3.2.1)–(3.2.4), (3.4.1), (3.4.2) *and* (3.3.1) *are satisfied. Then the initial value problem* (1.1)–(1.2) *has at least one solution on* $[-r, T]$.

4. SEMILINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH NONDENSE DOMAIN AND NONLOCAL CONDITIONS

Recently, in [1] the authors have considered the following general class of nonlinear partial neutral functional differential equations with infinite delay

$$\frac{d}{dt}[x(t) - f(t, x_t)] = A[x(t) - f(t, x_t)] + F(t, x_t), \quad t \geq 0 \tag{4.1}$$

$$x_0 = \phi \in \mathcal{F} \tag{4.2}$$

where the operator A is nondensely defined, $f, F : [0, \infty) \times \mathcal{F} \to E$ and $\mathcal{F}$ is the phase space of functions mapping $(-\infty, 0]$ into E. There are many examples where evolution equations are nondensely defined. For example, when we look at a one-dimensional heat equation with Dirichlet conditions on $[0, 1]$ and consider $A = \frac{\partial^2}{\partial x^2}$ in $C([0, 1], \mathbb{R})$ in order to measure the solutions in the sup-norm, then the domain,

$$D(A) = \{\phi \in C^2([0, 1], \mathbb{R}) : \phi(0) = \phi(1) = 0\},$$

is not dense in $C([0, 1], \mathbb{R})$ with the sup-norm. See [9] for more examples and remarks concerning nondensely defined operators.

In this section, we consider the following first order semilinear neutral functional differential inclusion with nonlocal conditions

$$\frac{d}{dt}[y(t) - f(t, x_t)] \in A[y(t) - f(t, x_t)] + F(t, y_t), \text{ a.e. } t \in J, \tag{4.3}$$

$$y(t) + h_t(y) = \phi(t), \quad t \in [-r, 0], \tag{4.4}$$

where f, F, h_t, ϕ are as in the problem (1.1)–(1.2) and A is nondensely defined. We give an existence result by assuming the existence of a maximal solution to an appropriate problem. The basic tool for this study is the theory of integrated semigroups.

Definition 4.1 ([3]). Let E be a Banach space. An integrated semigroup is a family of operators $(S(t))_{t\geq 0}$ of bounded linear operators $S(t)$ on E with the following properties:

(i) $S(0) = 0$;

(ii) $t \to S(t)$ is strongly continuous;

(iii) $S(s)S(t) = \int_0^s (S(t+r) - S(r))dr$, for all $t, s \geq 0$.

Definition 4.2. An integrated semigroup $(S(t))_{t\geq 0}$ is called exponentially bounded, if there exists a constant $M \geq 0$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\| \leq Me^{\omega t}, \qquad \text{for } t \geq 0.$$

Moreover, $(S(t))_{t\geq 0}$ is called nondegenerate, if $S(t)x = 0$, for all $t \geq 0$, implies $x = 0$.

Definition 4.3. An operator A is called a generator of an integrated semigroup, if there exists $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ (the resolvent set of A), and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of linear bounded operators such that $S(0) = 0$ and $(\lambda I - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S(t)dt$ for all $\lambda > \omega$.

If A is the generator of an integrated semigroup $(S(t))_{t\geq 0}$ which is locally Lipschitz, then from [3], $S(\cdot)x$ is continuously differentiable if and only if $x \in \overline{D(A)}$. In particular, $S'(t)x := \frac{d}{dt}S(t)x$ defines a bounded operator on the set $E_1 := \{x \in E : t \to S(t)x \text{ is continously differentiable on } [0, \infty)\}$ and $(S'(t))_{t\geq 0}$ is a C_0 semigroup on $\overline{D(A)}$. Here and hereafter, we assume that A satisfies the Hille-Yosida condition, that is, there exists $M \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$, $\sup\{(\lambda I - \omega)^n \|(\lambda I - A)^{-n}\| : \lambda > \omega,\ n \in \mathbb{N}\} \leq M$, where $\rho(A)$ is the resolvent operator set of A and I is the identity operator.

Let $(S(t))_{t\geq 0}$, be the integrated semigroup generated by A. We note that, since A satisfies the Hille-Yosida condition, $\|S'(t)\|_{B(E)} \leq Me^{\omega t}$, $t \geq 0$, where M and ω are from the Hille-Yosida condition (see [18]).

In the sequel, we give some results for the existence of solutions to the following problem:

$$y'(t) = Ay(t) + g(t), \quad t \geq 0, \tag{4.5}$$

$$y(0) = y_0 \in E, \tag{4.6}$$

where A satisfies the Hille-Yosida condition, without being densely defined.

Theorem 4.4 [18]. *Let $g : [0, b] \to E$ be a continuous function. Then for $y_0 \in \overline{D(A)}$, there exists a unique continuous function $y : [0, b] \to E$ such that*

(i) $\int_0^t y(s)ds \in D(A)$ *for* $t \in [0, b]$,

(ii) $y(t) = y_0 + A\int_0^t y(s)ds + \int_0^t g(s)ds, \quad t \in [0, b]$,

(iii) $\|y(t)\| \le Me^{\omega t}\left(\|y_0\| + \int_0^t e^{-\omega s}\|g(s)\|ds\right), \quad t \in [0, b]$.

Moreover, y satisfies the following variation of constant formula:

$$y(t) = S'(t)y_0 + \frac{d}{dt}\int_0^t S(t-s)g(s)ds, \quad t \ge 0. \tag{4.7}$$

Let $B_\lambda = \lambda R(\lambda, A) := \lambda(\lambda I - A)^{-1}$. Then ([18]) for all $x \in \overline{D(A)}, B_\lambda x \to x$ as $\lambda \to \infty$. Also from the Hille-Yosida condition (with $n = 1$) it easy to see that $\lim\limits_{\lambda\to\infty} \|B_\lambda x\| \le M\|x\|$, since

$$\|B_\lambda\| = \|\lambda(\lambda I - A)^{-1}\| \le \frac{M\lambda}{\lambda - \omega}.$$

Thus $\lim\limits_{\lambda\to\infty} \|B_\lambda\| \le M$. Also if y satisfies (4.7), then

$$y(t) = S'(t)y_0 + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda g(s)ds, \quad t \ge 0. \tag{4.8}$$

We are now in a position to define what we mean by an integral solution of the IVP (4.3)–(4.4).

Definition 4.5. We say that $y : J \to E$ is an integral solution of (4.3)–(4.4) if

(i) $y \in C([-r, T], E)$,

(ii) $\displaystyle\int_0^t [y(s) - f(s, y_s)]ds \in D(A)$ for $t \in J$,

(iii) there exists a function $v \in L^1(J, E)$, such that $v(t) \in F(t, y(t))$ a.e. in J and

$$y(t) = S'(t)[\phi(0) - h_0(y) - f(0, \phi(0))] + f(t, y_t) + \frac{d}{dt}\int_0^t S(t-s)v(s)ds$$

and $y(t) = \phi(t) - h_t(y), \quad t \in [-r, 0]$.

Theorem 4.6 *Assume that* (3.2.2) (i), (3.2.4)–(3.2.7) *hold and in addition, suppose that the following conditions are satisfied:*

(4.6.1) *A satisfies the Hille-Yosida condition;*

(4.6.2) *the operator $S'(t)$ is compact in $\overline{D(A)}$ whenever $t > 0$.*

(4.6.3) $\phi(0) - h_0(y) - f(0, \phi(0)) \in \overline{D(A)}$;

(4.6.4) *there exist constants $0 < c_1 < 1, c_2 \geq 0$ such that*

$$\|f(t, x)\| \leq c_1\|x\|_{\mathcal{D}} + c_2, \quad (t, x) \in J \times \mathcal{D};$$

(4.6.5) *given $\epsilon > 0$, then for any bounded subset D of $C([-r, T], E)$ there exists a $\delta > 0$ with $\|(S'(h) - I)h_0(y)\| < \epsilon$ for all $y \in D$ and $h \in [0, \delta]$ and $\|h_t(y) - h_s(y)\| < \epsilon$ for all $y \in \mathcal{D}$ and $t, s \in [-r, 0]$ with $|t-s| < \delta$;*

(4.6.6) *the problem*

$$\begin{aligned} v'(t) &= \frac{M^*}{1-c_1}e^{-\omega t}g(t, v(t)), \quad a.e.\ t \in J, \\ v(0) &= \frac{M^*}{1-c_1}\left[(1+c_1)\|\phi\|_{\mathcal{D}} + Q + c_2 + \frac{c_2}{M^*}\right], \quad M^* = \max\{e^{\omega T}, 1\}, \end{aligned}$$

has a maximal solution $r(t)$.

Then the IVP (4.3)–(4.4) *has at least one integral solution on $[-r, T]$.*

Proof. Transform the problem (4.3)–(4.4) into a fixed point problem. Consider the operator $\overline{N} : C([-r, T], E) \to \mathcal{P}(C([-r, T], E))$ defined by

$$\overline{N}(y) := \begin{cases} \phi(t) - h_t(y), & \text{if } t \in [-r, 0], \\ S'(t)[\phi(0) - h_0(y) - f(0, \phi(0))] + f(t, y_t) & \\ +\dfrac{d}{dt}\displaystyle\int_0^t S(t-s)v(s)ds, & \text{if } t \in J, \end{cases}$$

where $v \in S_{F,y}$.

We shall show that $\overline{N}$ has a fixed point. The proof is given in several steps.

Step 1. $\overline{N}$ is convex for each $y \in C([-r, T], E)$.
This is obvious, since F has convex values.

Step 2. $\overline{N}$ maps bounded sets into bounded sets in $C([-r, T], E)$.
Let $B_q = \{y \in C([-r, T], E) : \|y\| := \sup_{t\in[-r,T]} \|y(t)\| \leq q\}$ be a bounded set in $C([-r, T], E)$ and $y \in B_q$. Then for $h \in N(y)$ there exists $v \in S_{F,y}$ such that

$$h(t) = S'(t)[\phi(0) - h_0(y) - f(0, \phi(0))] + f(t, y_t) + \frac{d}{dt}\int_0^t S(t-s)v(s)ds, \quad t \in J.$$

Thus for each $t \in J$ we get

$$\begin{aligned} \|y(t)\| &\leq Me^{\omega t}[(1 + c_1)\|\phi\|_{\mathcal{D}} + Q + c_2] + c_1\|y_t\|_{\mathcal{D}} + c_2 \\ &\quad + Me^{\omega t}\int_0^t e^{-\omega s}\|v(s)\|ds \\ &\leq M^*[(1 + c_1)\|\phi\|_{\mathcal{D}} + Q + c_2] + c_1 q + c_2 + M^*\int_0^t e^{-\omega s}h_q(s)ds; \end{aligned}$$

here h_q is chosen as in Definition 2.2 and $M^* = e^{\omega T}$ if $\omega > 0$ or $M^* = 1$ if $\omega \leq 0$. Then for each $h \in \overline{N}(B_q)$ we have

$$\|h\| \leq M^*[(1 + c_1)\|\phi\|_{\mathcal{D}} + Q + c_2] + c_1 q + c_2 + M^*\int_0^T e^{-\omega s}h_q(s)ds := \ell.$$

Step 3. $\overline{N}$ sends bounded sets into equicontinuous sets in $C([-r, T],\ E)$.
We consider B_q as in Step 2 and let $h \in \overline{N}(y)$ for $y \in B_q$. Let $\epsilon > 0$ be given. Now let $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$. We consider two cases $\tau_1 > \epsilon$ and $\tau_1 \leq \epsilon$.

Case 1. It $\tau_1 > \epsilon$ then

$$\begin{aligned}
\|h(\tau_2) - h(\tau_1)\| \leq\ & \| [S'(\tau_2) - S'(\tau_1)][\phi(0) - h_0(y) - f(0, \phi(0))] \| \\
& + \|f(\tau_2, y_{\tau_2}) - f(\tau_2, y_{\tau_2})\| \\
& + \left\| \lim_{\lambda \to \infty} \int_0^{\tau_1 - \epsilon} [S'(\tau_2 - s) - S'(\tau_1 - s)] B_\lambda v(s) ds \right\| \\
& + \left\| \lim_{\lambda \to \infty} \int_{\tau_1 - \epsilon}^{\tau_1} [S'(\tau_2 - s) - S'(\tau_1 - s)] B_\lambda v(s) ds \right\| \\
& + \left\| \lim_{\lambda \to \infty} \int_{\tau_1}^{\tau_2} S'(\tau_2 - s) B_\lambda v(s) ds \right\| \\
\leq\ & \| [S'(\tau_2) - S'(\tau_1)][\phi(0) - f(0, \phi(0))] \| \\
& + M^* \|S'(\tau_2 - \tau_1 + \epsilon) - S'(\epsilon)\|_{B(e)} \|h_0(B_q)\| \\
& + \|f(\tau_2, y_{\tau_2}) - f(\tau_2, y_{\tau_2})\| \\
& + M^* \|S'(\tau_2 - \tau_1 + \epsilon) - S'(\epsilon)\|_{B(E)} \int_0^{\tau_1 - \epsilon} e^{-\omega s} h_q(s) ds \\
& + 2M^* \int_{\tau_1 - \epsilon}^{\tau_1} e^{-\omega s} h_q(s) ds \\
& + M^* \int_{\tau_1}^{\tau_2} e^{-\omega s} h_q(s) ds.
\end{aligned}$$

Case 2. Let $\tau_1 \leq \epsilon$. For $\tau_2 - \tau_1 < \epsilon$ we get

$$\begin{aligned}
\|h(\tau_2) - h(\tau_1)\| \leq\ & \| [S'(\tau_2) - S'(\tau_1)][\phi(0) - f(0, \phi(0))] \| \\
& + M \|S'(\tau_2 - \tau_1) h_0(y) - h_0(y)\| \\
& + \|f(\tau_2, y_{\tau_2}) - f(\tau_2, y_{\tau_2})\| \\
& + M^* \int_0^{2\epsilon} e^{-\omega s} h_q(s) ds \\
& + M^* \int_0^{\epsilon} e^{-\omega s} h_q(s) ds.
\end{aligned}$$

Note that equicontinuity follows since (i). $S'(t), t \geq 0$ is a strongly continuous semigroup, (ii). (4.6.5) and (iii). $S'(t)$ is compact for $t > 0$ (so $S'(t)$ is continuous in the uniform operator topology for $t > 0$).

Let $0 < t \leq T$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_q$ and $v \in S_{F,y}$ we define

$$\begin{aligned} r_\epsilon(t) &= \lim_{\lambda \to \infty} \int_0^{t-\epsilon} S'(t-s) B_\lambda v(s) ds \\ &= S'(\epsilon) \lim_{\lambda \to \infty} \int_0^{t-\epsilon} S'(t-s-\epsilon) B_\lambda v(s) ds. \end{aligned}$$

Note

$$\left\{ \lim_{\lambda \to \infty} \int_0^{t-\epsilon} S'(t-s-\epsilon) B_\lambda v(s) ds : y \in B_q \text{ and } v \in S_{F,y} \right\}$$

is a bounded set since

$$\left\| \lim_{\lambda \to \infty} \int_0^{t-\epsilon} S'(t-s-\epsilon) B_\lambda v(s) ds \right\| \leq M^* \int_0^{t-\epsilon} e^{-\omega s} h_q(s) ds$$

and now since $S'(t)$ is a compact operator for $t > 0$, the set $Y_\epsilon(t) = \{r_\epsilon(t) : y \in B_q \text{ and } v \in S_{F,y}\}$ is relatively compact in E for every ε, $0 < \varepsilon < t$. Moreover, for $r = r_0$ we have

$$\|r(t) - r_\epsilon(t)\| \leq M \int_{t-\epsilon}^{t} e^{-\omega s} h_q(s) ds.$$

Therefore, the set $Y(t) = \{r(t) : y \in B_q \text{ and } v \in S_{F,y}\}$ is totally bounded. Hence $Y(t)$ is relatively compact in E.

As a consequence of Steps 2, 3 and the Arzelá-Ascoli theorem we can conclude that $\overline{N} : C([-r,T], E) \longrightarrow \mathcal{P}(C([-r,T], E))$ is completely continuous.

Step 4. $\overline{N}$ has a closed graph.
Let $y_n \longrightarrow y_*$, $h_n \in \overline{N}(y_n)$ and $h_n \longrightarrow h_*$. We shall prove that $h_* \in \overline{N}(y_*)$. Now $h_n \in \overline{N}(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that

$$\begin{aligned} h_n(t) = S'(t)[\phi(0) - h_0(y_n) - f(0, \phi(0))] + f(t, y_{nt}) \\ + \lim_{\lambda \to \infty} \int_0^t S'(t-s) B_\lambda v_n(s) ds, \quad t \in J. \end{aligned}$$

We must prove that there exists $v_* \in S_{F,y_*}$ such that

$$h_*(t) = S'(t)[\phi(0) - h_0(y_*) - f(0,\phi(0))] + f(t,y_{*t}) \\ + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda v_*(s)ds, \ \ t \in J.$$

Consider the linear continuous operator $\Gamma : L^1(J,E) \longrightarrow C(J,E)$ defined by

$$(\Gamma v)(t) = \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda v(s)ds.$$

We have

$$\|(h_n(t) - S'(t)[\phi(0) - h_0(y_n) - f(0,\phi(0))] + f(t,y_{nt})) \\ -(h_*(t) - S'(t)[\phi(0) - h_0(y_*) - f(0,\phi(0))] + f(t,y_{*t}))\| \longrightarrow 0$$

as $n \longrightarrow \infty$. It follows that $\Gamma\circ S_F$ is a closed graph operator ([20]). Moreover, we have

$$h_n(t) - S'(t)[\phi(0) - h_0(y_n) - f(0,\phi(0))] + f(t,y_{nt}) \in \Gamma(S_{F,y_n}).$$

Since $y_n \longrightarrow y_*$, it follows that

$$h_*(t) = S'(t)[\phi(0) - h_0(y_*) - f(0,\phi(0))] + f(t,y_{*t}) \\ + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda v_*(s)ds, \ \ t \in J.$$

for some $v_* \in S_{F,y_*}$.

Step 5. The set

$$\mathcal{M} := \{y \in C([-r,T],E) : \lambda y \in \overline{N}(y), \text{ for some } \lambda > 1\}$$

is bounded.

Let $y \in \mathcal{M}$ be such that $\lambda y \in \overline{N}(y)$ for some $\lambda > 1$. Then

$$y(t) = \lambda^{-1}S'(t)[\phi(0) - h_0(y) - f(0,\phi(0))] + \lambda^{-1}f(t,y_t) \\ + \lambda^{-1}\frac{d}{dt}\int_0^t S(t-s)v(s)ds, \ \ t \in J.$$

Thus

$$\|y(t)\| \le M^*[(1+c_1)\|\phi\|_{\mathcal{D}} + Q + c_2] + c_1\|y_t\|_{\mathcal{D}} + c_2$$
$$+ M^* \int_0^t e^{-\omega s} g(s, \|y_s\|_{\mathcal{D}}) ds, \;\; t \in J.$$

We consider the function μ defined by

$$\mu(t) := \sup\{\|y(s)\| : -r \le s \le t\}, \quad t \in [0,T].$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = \|y(t^*)\|$. If $t^* \in [0,T]$, then by the previous inequality, we have for $t \in [0,T]$,

$$(1-c_1)\mu(t) \le M^*[(1+c_1)\|\phi\|_{\mathcal{D}} + Q + c_2] + c_2 + M^* \int_0^t e^{-\omega s} g(s, \mu(s)) ds,$$

or

$$\mu(t) \le \frac{M^*}{1-c_1}\left[(1+c_1)\|\phi\|_{\mathcal{D}} + Q + c_2 + \frac{c_2}{M^*} + \int_0^t e^{-\omega s} g(s,\mu(s)) ds\right], \; t \in J.$$

If $t^* \in [-r,0]$ then $\mu(t) \le \|\phi\|_{\mathcal{D}} + Q$ and the inequality holds. Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$v(0) = \frac{M^*}{1-c_1}\left[(1+c_1)\|\phi\|_{\mathcal{D}} + Q + c_2 + \frac{c_2}{M^*}\right]$$

and

$$\begin{aligned} v'(t) &= \frac{M^*}{1-c_1} e^{-\omega t} g(t, \mu(t)) \\ &\le \frac{M^*}{1-c_1} e^{-\omega t} g(t, v(t)), \;\; t \in [0,T]. \end{aligned}$$

This implies that ([19] Theorem 1.10.2) $v(t) \le r(t)$ for $t \in J$, and hence $\|y(t)\| \le b' = \sup_{t\in[-r,T]} r(t)$, $t \in J_0$ where b' depends only on T and on the function r. This shows that $\mathcal{M}$ is bounded.

As a consequence of the Leray-Schauder Alternative for Kakutani maps [14] we deduce that $\overline{N}$ has a fixed point which is a solution of (4.3)–(4.4). ■

Theorem 4.7. *Assume that* (3.2.2) (i), (3.2.4), (3.2.5), (4.6.1)–(4.6.5) *hold. In addition, suppose that the following condition is satisfied:*

(4.7.1) *there exists a continuous non-decreasing function* $\psi : [0,\infty) \longrightarrow (0,\infty)$, $p \in L^1(J,\mathbb{R}_+)$ *such that*

$$\|F(t,u)\| := \sup\{\|v\| : v \in F(t,u)\} \le p(t)\psi(\|u\|_{\mathcal{D}})$$
$$\text{for each } \ (t,u) \in J \times \mathcal{D}$$

and there exists a constant $M_* > 0$ *with*

$$\frac{(1-c_1)M_*}{M^*(1+c_1)\|\phi\|_{\mathcal{D}} + M^*(c_2+Q) + c_2 + M^*\psi(M_*)\displaystyle\int_0^T e^{-\omega s}p(s)\,ds} > 1.$$

Then the IVP (4.3)–(4.4) *has at least one integral solution on* $[-r,T]$.

Proof. Define N as in the proof of Theorem 4.6. As in Theorem 4.6 we can prove that N is completely continuous.

We show there exists an open set $U \subseteq C(J,E)$ with $y \notin \lambda N(y)$ for $\lambda \in (0,1)$ and $y \in \partial U$. Let $\lambda \in (0,1)$ and let $y \in \lambda N(y)$. Then we have

$$\|y(t)\| \le M^*[(1+c_1)\|\phi\|_{\mathcal{D}} + Q + c_2] + c_1\|y_t\|_{\mathcal{D}} + c_2$$
$$+ M^*\int_0^t e^{-\omega s}p(s)\psi(\|y_s\|_{\mathcal{D}})ds, \ t \in J.$$

We consider the function μ defined by

$$\mu(t) := \sup\{\|y(s)\| : -r \le s \le t\}, \quad t \in [0,T].$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = \|y(t^*)\|$. If $t^* \in [0,T]$, then by the previous inequality, we have for $t \in [0,T]$,

$$(1-c_1)\mu(t) \le M^*[(1+c_1)\|\phi\|_{\mathcal{D}} + Q + c_2] + c_2 + M^*\int_0^t e^{-\omega s}p(s)\psi(\mu(s))ds,$$

or

$$\mu(t) \le \frac{M^*}{1-c_1}\left[(1+c_1)\|\phi\|_{\mathcal{D}} + Q + c_2 + \frac{c_2}{M^*} + \int_0^t e^{-\omega s}p(s)\psi(\mu(s))ds\right], \ t \in J.$$

If $t^* \in [-r,0]$, then $\mu(t) \le \|\phi\|_{\mathcal{D}} + Q$ and the inequality holds.

Consequently

$$\frac{(1-c_1)\|y\|}{M^*(1+c_1)\|\phi\|_{\mathcal{D}} + M^*(c_2+Q) + c_2 + M^*\psi(\|y\|)\displaystyle\int_0^T e^{-\omega s}p(s)\,ds} \leq 1.$$

Then by (4.7.1), there exists M_* such that $\|y\| \neq M_*$. Set

$$U = \{y \in C(J,E) : \|y\| < M_*\}.$$

From the choice of U there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for $\lambda \in (0,1)$. As a consequence of the Leray-Schauder Alternative for Kakutani maps [14] we deduce that N has a fixed point and therefore the problem (4.3)–(4.4) has a solution on $[-r,T]$. ■

We state also without proof two results concerning the lower semicontinuous case for nondensely defined operators.

Theorem 4.8. *Assume that the conditions* (3.2.2) (i), (3.2.4), (3.4.1), (3.4.2) *and* (4.6.1)–(4.6.6) *are satisfied. Then the problem* (4.3)–(4.4) *has at least one integral solution on* $[-r,T]$.

Theorem 4.9. *Assume that the conditions* (3.2.2) (i), (3.2.4), (3.4.1), (3.4.2), (4.6.1)–(4.6.5) *and* (4.7.1) *are satisfied. Then the problem* (4.3)–(4.4) *has at least one integral solution on* $[-r,T]$.

5. Second order semilinear neutral functional differential inclusions with nonlocal conditions

In this section, we study the problem (1.3)–(1.4).

Definition 5.1. A function $y \in C([-r,T],E)$ is said to be a mild solution of (1.3)–(1.4) if $y(t) + h_t(y) = \phi(t),\ t \in [-r,0], y'(0) + h_1(y) = \eta$ and there exists $v \in L^1(J,E)$ such that $v(t) \in F(t,y(t))$ a.e. on J and

$$\begin{aligned} y(t) &= C(t)[\phi(0) - h_0(y)] + S(t)[\eta - h_1(y) - f(0,\phi(0))] \\ &\quad + \int_0^t C(t-s)f(s,y_s)ds + \int_0^t S(t-s)v(s)ds,\ \ t \in J. \end{aligned}$$

Theorem 5.2. *Assume* (3.2.4)–(3.2.7), (4.6.4) *and the conditions*

(5.2.1) *the function* $h_1 : C(J,E) \to E$ *is continuous and completely continuous and there exists a constant* $Q_1 > 0$ *such that* $\|h_1(y)\| \leq Q_1$, *for all* $y \in C(J,E)$;

(5.2.2) $A : D(A) \subset E \to E$ *is the infinitesimal generator of a strongly continuous cosine family* $\{C(t) : t \in J\}$, *and there exist constants* $N_1 \geq 1$, *and* $N_2 \geq 1$ *such that* $\|C(t)\|_{B(E)} \leq N_1$, $\|S(t)\|_{B(E)} \leq N_2$ *for all* $t \in \mathbb{R}$;

(5.2.3) *for each bounded* $B \subseteq C([-r,b],E)$, *and* $t \in J$ *the set*

$$\Big\{C(t)[\phi(0) - h_0(y)] + S(t)[\eta - h_1(y) - f(0,\phi(0))] + \int_0^t C(t-s)f(s,y_s)ds + \int_0^t S(t-s)v(s)ds, v \in S_{F,B}\Big\}$$

is relatively compact in E, *where* $y \in B$ *and* $S_{F,B} = \cup\{S_{F,y} : y \in B\}$;

(5.2.4) *the problem*

$$\begin{aligned} v'(t) &= N_1c_1v(t) + N_2g(t,v(t)), \quad a.e.\ t \in J,\\ v(0) &= C_1, \end{aligned}$$

where

$$C_1 = N_1[\|\phi\|_{\mathcal{D}} + Q] + N_2[\|\eta\| + Q_1 + c_1\|\phi\|_{\mathcal{D}} + c_2] + N_1c_2T,$$

has a maximal solution $r(t)$;

(5.2.5) *given* $\epsilon > 0$, *then for any bounded subset* D *of* $C([-r,b],E)$ *there exists a* $\delta > 0$ *with* $\|[C(\tau_2) - C(\tau_1)]h_0(y)]\| < \epsilon$ *for all* $y \in D$ *and* $\tau_1, \tau_2 \in [0,\delta]$ *and* $\|h_t(y) - h_s(y)\| < \epsilon$ *for all* $y \in D$ *and* $t, s \in [-r,0]$ *with* $|t-s| < \delta$;

(5.2.6) *given* $\epsilon > 0$, *then for any bounded subset* D *of* $C([-r,b],E)$ *there exists a* $\delta > 0$ *with* $\|[S(\tau_2) - S(\tau_1)]h_1(y)]\| < \epsilon$ *for all* $y \in D$ *and* $\tau_1, \tau_2 \in [0,\delta]$;

(5.2.7) *for every* $q > 0$ *the set* $f(I \times B_q(0))$ *is relatively compact in* E, *where* $B_q(0)$ *denotes the closed ball with center at* 0 *and radius* $q > 0$,

are satisfied. Then the problem (1.3)–(1.4) *has at least one mild solution on* $[-r,T]$.

Proof. We transform the problem (1.3)–(1.4) into a fixed point problem. Consider the multivalued map $\overline{N} : C([-r,T],E) \longrightarrow \mathcal{P}(C([-r,T],E))$ defined by

$$\overline{N}(y) := \left\{ h \in C([-r,T],E) : h(t) = \begin{cases} \phi(t) - h_t(y), & \text{if } t \in [-r,0] \\ C(t)[\phi(0) - h_0(y)] \\ + S(t)[\eta - h_1(y) - f(0,\phi(0))] \\ + \displaystyle\int_0^t C(t-s) f(s,y_s)ds \\ + \displaystyle\int_0^t S(t-s)v(s)ds, & \text{if } t \in [0,T] \end{cases} \right\}$$

where $v \in S_{F,y}$. We shall show that $\overline{N}$ has a fixed point. The proof will be given in several steps.

Step 1. $\overline{N}(y)$ is convex for each $y \in C([-r,T],E)$.
This is obvious, since F has convex values.

Step 2. N maps bounded sets into bounded sets in $C([-r,b],E)$.
Let $B_q := \{y \in C([-r,T],E) : \|y\| = \sup_{t\in[-r,T]} \|y(t)\| \leq q\}$ be a bounded set in $C([-r,T],E)$ and $y \in B_q$. Then for each $h \in \overline{N}(y)$ there exists $v \in S_{F,y}$ such that

$$\begin{aligned} h(t) &= C(t)[\phi(0) - h_0(y)] + S(t)[\eta - h_1(y) - f(0,\phi(0))] \\ &\quad + \int_0^t C(t-s) f(s,y_s)ds + \int_0^t S(t-s)v(s)ds, \ \ t \in J. \end{aligned}$$

Thus for each $t \in J$ we get

$$\begin{aligned} \|h(t)\| &\leq N_1[\|\phi\|_{\mathcal{D}} + Q] + N_2[\|\eta\| + Q_1 + c_1\|\phi\|_{\mathcal{D}} + c_2] \\ &\quad + N_1 \int_0^t [c_1\|y_s\| + c_2]ds + N_2 \int_0^t \|v(s)\|ds \\ &\leq N_1[\|\phi\|_{\mathcal{D}} + Q] + N_2[\|\eta\| + Q_1 + c_1\|\phi\|_{\mathcal{D}} + c_2] \\ &\quad + N_1 \int_0^t [c_1\|y_s\| + c_2]ds + N_2\|h_q\|_{L^1}; \end{aligned}$$

here h_q is chosen as in Definition 2.2. Then for each $h \in \overline{N}(B_q)$ we have

$$\|h\| \le N_1[\|\phi\|_{\mathcal{D}}+Q]+N_2[\|\eta\|+Q_1+c_1\|\phi\|_{\mathcal{D}}+c_2]+N_1T[c_1q+c_2]+N_2\|h_q\|_{L^1}:=\ell.$$

Step 3. N maps bounded sets into equicontinuous sets of $C([-r,T],E)$. We consider B_q as in Step 2 and we fix $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$. For $y \in B_q$, we have using Proposition 2.1

$$\begin{aligned}
\|h(\tau_2)-h(\tau_1)\| \le\ & \|\,[C(\tau_2)-C(\tau_1)]\phi(0)\| + \|\,[C(\tau_2)-C(\tau_1)]h_0(y)\| \\
& + \|\,[S(\tau_2)-S(\tau_1)][\phi(0)-f(0,\phi(0)]\,\| \\
& + \|\,[S(\tau_2)-S(\tau_1)]h_1(y)\| \\
& + \int_0^{\tau_1} \|\,[C(\tau_2-s)-C(\tau_1-s)]f(s,y_s)\|ds \\
& + \int_{\tau_1}^{\tau_2} \|C(\tau_2-s)\|\|f(s,y_s)\|ds \\
& + \int_0^{\tau_1} \|\,[S(\tau_2-s)-S(\tau_1-s)]v(s)\|ds \\
& + \int_{\tau_1}^{\tau_2} \|S(\tau_2-s)\|\,\|v(s)\|ds \\
\le\ & \|\,[C(\tau_2)-C(\tau_1)]\phi(0)\| + \|\,[C(\tau_2)-C(\tau_1)]h_0(y)\| \\
& + \|\,[S(\tau_2)-S(\tau_1)][\phi(0)-f(0,\phi(0)]\,\| \\
& + \|\,[S(\tau_2)-S(\tau_1)]h_1(y)\| \\
& + \int_0^{\tau_1} \|\,[C(\tau_2-s)-C(\tau_1-s)]f(s,y_s)\|ds \\
& + N_1\int_{\tau_1}^{\tau_2}[c_1q+c_2]ds + \int_0^{\tau_1}\int_{\tau_1-s}^{\tau_2-s} e^{\omega x}dxv(s)ds \\
& + N_2\int_{\tau_1}^{\tau_2} h_q(s)ds
\end{aligned}$$

$$\leq \| [C(\tau_2) - C(\tau_1)]\phi(0)\| + \| [C(\tau_2) - C(\tau_1)]h_0(y)\|$$
$$+ \| [S(\tau_2) - S(\tau_1)] [\phi(0) - h_1(y) - f(0, \phi(0)] \|$$
$$+ \int_0^{\tau_1} \| [C(\tau_2 - s) - C(\tau_1 - s)]f(s, y_s)\|ds$$
$$+ N_1(\tau_2 - \tau_1)[c_1 q + c_2]$$
$$+ e^{\omega b}(\tau_2 - \tau_1) \int_0^{\tau_1} h_q(s)ds + N_2 \int_{\tau_1}^{\tau_2} h_q(s)ds.$$

As a consequence of Steps 2, 3, (5.2.3), (5.2.5), (5.2.6) the strong continuity of $C(t), t \in J$ and the compactness of f, (note for a given $\epsilon > 0$ we can choose a $\delta > 0$ such that

$$\|C(t)f(s, z) - C(t')f(s, z)\| < \epsilon, \quad t, t', s \in J, z \in B_q(0),$$

with $|t - t'| \leq \delta$) and the Arzelá-Ascoli theorem we can conclude that N is completely continuous.

Step 4. N has a closed graph.
Let $y_n \longrightarrow y_*,\ h_n \in N(y_n)$, and $h_n \longrightarrow h_*$. We shall prove that $h_* \in N(y_*)$. Now $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that

$$h_n(t) = C(t)[\phi(0) - h_0(y)] + S(t)[\eta - h_1(y) - f(0, \phi)] + \int_0^t C(t - s)f(s, y_{ns})ds$$
$$+ \int_0^t S(t - s)v_n(s)ds, \quad t \in J.$$

We must prove that there exists $v_* \in S_{F,y_*}$ such that

$$h_*(t) = C(t)[\phi(0) - h_0(y)] + S(t)[\eta - h_1(y) - f(0, \phi)] + \int_0^t C(t - s)f(s, y_{*s})ds$$
$$+ \int_0^t S(t - s)v_*(s)ds, \quad t \in J.$$

Since f is continuous we have that

$$\Big\|\Big(h_n - C(t)[\phi(0) - h_0(y)] - S(t)[\eta - h_1(y) - f(0,\phi)] - \int_0^t C(t-s)f(s,y_{ns})ds\Big) -$$

$$\Big(h_* - C(t)[\phi(0) - h_0(y)] - S(t)[\eta - h_1(y) - f(0,\phi)] - \int_0^t C(t-s)f(s,y_{*s})ds\Big)\Big\| \to 0,$$

as $n \longrightarrow \infty$.

Consider the linear continuous operator

$$\Gamma : L^1(J,E) \longrightarrow C(J,E)$$

$$v \longmapsto \Gamma(v)(t) = \int_0^t S(t-s)v(s)ds.$$

It follows that $\Gamma \circ S_F$ is a closed graph operator ([20]).

Moreover, we have that

$$h_n(t) - C(t)[\phi(0) - h_0(y)] - S(t)[\eta - h_1(y) - f(0,\phi)]$$

$$- \int_0^t C(t-s)f(s,y_{ns})ds \in \Gamma(S_{F,y_n}).$$

Since $y_n \longrightarrow y^*$, it follows that

$$h_*(t) - C(t)[\phi(0) - h_0(y)] - S(t)[\eta - h_1(y) - f(0,\phi)]$$

$$- \int_0^t C(t-s)f(s,y_{*s})ds = \int_0^t S(t-s)v_*(s)ds$$

for some $v_* \in S_{F,y^*}$.

Step 5. Now it remains to show that the set

$$\mathcal{M} := \{y \in C([-r,T],E) : \lambda y \in N(y), \quad \text{for some} \quad \lambda > 1\}$$

is bounded.

Let $\lambda > 1$ and $\lambda y \in N(y)$. Then for $t \in J$

$$\begin{aligned} y(t) &= \lambda^{-1}C(t)[\phi(0) - h_0(y)] + \lambda^{-1}S(t)[\eta - h_1(y) - f(0,\phi)] \\ &\quad + \lambda^{-1}\int_0^t C(t-s)f(s,y_s)ds + \lambda^{-1}\int_0^t S(t-s)v(s)ds, \quad t \in J. \end{aligned}$$

This implies by our assumptions that for each $t \in J$ we have

$$\begin{aligned} \|y(t)\| &\leq N_1[\|\phi\|_{\mathcal{D}} + Q] + N_2[\|\eta\| + Q_1 + c_1\|\phi\|_{\mathcal{D}} + c_2] \\ &\quad + N_1\int_0^t (c_1\|y_s\|_{\mathcal{D}} + c_2)ds + N_2\int_0^t g(s, \|y_s\|_{\mathcal{D}})ds. \end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\{\|y(s)\| : -r \leq s \leq t\}, \quad 0 \leq t \leq T.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = \|y(t^*)\|$. If $t^* \in J$, by the previous inequality we have for $t \in J$

$$\begin{aligned} &\mu(t) \leq \\ &\leq N_1[\,\|\phi\|_{\mathcal{D}} + Q] + N_2[\,\|\eta\| + Q_1 + c_1\|\phi\|_{\mathcal{D}} + c_2] + N_1\int_0^{t^*} (c_1\mu(s) + c_2)ds \\ &\quad + N_2\int_0^{t^*} g(s, \mu(s))ds \\ &\leq N_1[\,\|\phi\|_{\mathcal{D}} + Q] + N_2[\,\|\eta\| + Q_1 + c_1\|\phi\|_{\mathcal{D}} + c_2] + N_1c_1\int_0^t \mu(s)ds + N_1c_2T \\ &\quad + N_2\int_0^t g(s, \mu(s))ds \\ &\leq C_1 + N_1c_1\int_0^t \mu(s)ds + N_2\int_0^t g(s, \mu(s))ds. \end{aligned}$$

If $t^* \in J_0$, then $\mu(t) \leq \|\phi\|_{\mathcal{D}} + Q$ and the previous inequality holds.

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$v(0) = C_1$$

and

$$\begin{aligned} v'(t) &= N_1 c_1 \mu(t) + N_2 g(t, \mu(t)) \\ &\leq N_1 c_1 v(t) + N_2 g(t, v(t)), \quad t \in [0, T]. \end{aligned}$$

This implies that ([19] Theorem 1.10.2) $v(t) \leq r(t)$ for $t \in J$, and hence $\|y(t)\| \leq b' = \sup_{t \in [-r,T]} r(t),\ t \in J_0$ where b' depends only on T and on the function r. This shows that $\mathcal{M}$ is bounded.

As a consequence of the Leray-Schauder Alternative for Kakutani maps [14] we deduce that $\overline{N}$ has a fixed point which is a solution of (4.3)–(4.4). ■

Theorem 5.3. *Assume that* (3.2.4), (3.2.5), (4.6.4), (5.2.1)–(5.2.3), (5.2.5)–(5.2.7) *hold. In addition, suppose that the following condition is satisfied:*

(5.3.1) *there exist a continuous non-decreasing function* $\psi : [0, \infty) \longrightarrow (0, \infty)$, $p \in L^1(J, \mathbb{R}_+)$ *such that*

$$\|F(t, u)\| := \sup\{\|v\| : v \in F(t, u)\} \leq p(t)\psi(\|u\|_{\mathcal{D}})$$
$$\textit{for each} \quad (t, u) \in J \times \mathcal{D}$$

and there exists a constant $M_* > 0$ *with*

$$\frac{(1 - Tc_1 N_1) M_*}{C_1 + N_2 \psi(M_*) \displaystyle\int_0^T p(s)\, ds} > 1,$$

where C_1 *is defined in Theorem* 5.2 *and* $1 - Tc_1 N_1 > 0$.

Then the IVP (1.3)–(1.4) *has at least one mild solution on* $[-r, T]$.

Proof. Define N as in the proof of Theorem 5.2. As in Theorem 5.2 we can prove that N is completely continuous.

We show there exists an open set $U \subseteq C(J, E)$ with $y \notin \lambda N(y)$ for $\lambda \in (0, 1)$ and $y \in \partial U$. Let $\lambda \in (0, 1)$ and let $y \in \lambda N(y)$. Then we have

$$\begin{aligned} \|y(t)\| \leq\ & N_1[\|\phi\|_{\mathcal{D}} + Q] + N_2[\|\eta\| + Q_1 + c_1\|\phi\|_{\mathcal{D}} + c_2] \\ & + N_1 \int_0^t (c_1 \|y_s\|_{\mathcal{D}} + c_2) ds + N_2 \int_0^t p(s)\psi(\|y_s\|_{\mathcal{D}}) ds. \end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\{\|y(s)\| : -r \le s \le t\}, \quad 0 \le t \le T.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = \|y(t^*)\|$. If $t^* \in J$, by the previous inequality we have for $t \in J$

$$\begin{aligned}\mu(t) &\le N_1[\,\|\phi\|_{\mathcal{D}} + Q] + N_2[|\eta| + Q_1 + c_1\|\phi\|_{\mathcal{D}} + c_2] \\ &\quad + N_1 c_1 \int_0^t \mu(s)ds + N_1 c_2 T + N_2 \int_0^t p(s)\psi(\mu(s))ds \\ &\le C_1 + N_1 c_1 \int_0^t \mu(s)ds + N_2 \int_0^t p(s)\psi(\mu(s))ds.\end{aligned}$$

If $t^* \in J_0$, then $\mu(t) \le \|\phi\|_{\mathcal{D}} + Q$ and the previous inequality holds.

Consequently,

$$\frac{(1 - Tc_1N_1)\|y\|}{C_1 + N_2\psi(\|y\|)\displaystyle\int_0^T p(s)\,ds} \le 1.$$

Then by (5.3.1), there exists M_* such that $\|y\| \ne M_*$. Set

$$U = \{y \in C(J, E) : \|y\| < M_*\}.$$

From the choice of U there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for $\lambda \in (0, 1)$. As a consequence of the Leray-Schauder Alternative for Kakutani maps [14] we deduce that N has a fixed point and therefore the problem (1.3)–(1.4) has a solution on $[-r, T]$. ∎

For the lower semicontinuous case we state without proof the following results.

Theorem 5.4. *Assume that the conditions* (3.2.4), (3.2.6), (3.2.7), (3.4.1), (3.4.2), (4.6.4), (5.2.1)–(5.2.7) *are satisfied. Then the problem* (1.3)–(1.4) *has at least one mild solution on* $[-r, T]$.

Theorem 5.5. *Assume that the conditions* (3.2.4), (3.4.1), (3.4.2), (4.6.4), (5.2.1)–(5.2.3), (5.2.5) *and* (5.3.1) *are satisfied. Then the problem* (1.3)–(1.4) *has at least one mild solution on* $[-r, T]$.

6. Controllability results

In this section, we study controllability for first and second order semilinear neutral functional differential inclusions with nonlocal conditions. We consider first the problem

$$\frac{d}{dt}[y(t) - f(t, y_t)] \in Ay(t) + F(t, y_t) + \mathcal{B}u(t), \quad t \in J := [0, b], \tag{6.1}$$

$$y(t) + h_t(y) = \phi(t), \ t \in [-r, 0] \tag{6.2}$$

where f, A, F, ϕ are as in the problem (1.1)–(1.2) and the control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions. U is a Banach space. Finally, $\mathcal{B}$ is a bounded linear operator from U to E.

Definition 6.1. A function $y \in C([-r, T], E)$ is said to be a mild solution of (6.1)–(6.2) if $y(t) + h_t(y) = \phi(t), t \in [-r, 0]$ and there exists $v \in L^1(J, E)$ such that $v(t) \in F(t, y(t))$ a.e. on J, and

$$\begin{aligned} y(t) \ &= S(t)[\phi(0) - h_0(y) - f(0, \phi)] + f(t, y_t) + \int_0^t AS(t-s) f(s, y_s)\, ds \\ &\quad + \int_0^t S(t-s)[\mathcal{B}u(s) + v(s)]\, ds, \ t \in J. \end{aligned}$$

Definition 6.2. The system (6.1)–(6.2) is said to be nonlocally controllable on the interval J, if for every $\phi \in \mathcal{D}$ and $y_1 \in E$ there exists a control $u \in L^2(J, U)$, such that the mild solution $y(t)$ of (6.1)–(6.2) satisfies $y(T) + h_T(y) = y_1$.

Theorem 6.3. *Assume that the conditions* (3.2.1)–(3.2.5) *hold. In addition, assume the following conditions are satisfied:*

(6.3.1) *the linear operator* $W : L^2(J, U) \to E$, *defined by*

$$Wu = \int_0^T S(T-s)\mathcal{B}u(s)\, ds,$$

has a bounded invertible operator $W^{-1} : E \to L^2(J, U)$ *and there exist positive constants* M_1, M_2 *such that* $\|\mathcal{B}\| \le M_1$ *and* $\|W^{-1}\| \le M_2$;

(6.3.2) *there exists a continuous non-decreasing function* $\psi : [0,\infty) \longrightarrow (0,\infty)$, $p \in L^1(J,\mathbb{R}_+)$ *such that*

$$\|F(t,u)\| := \sup\{\|v\| : v \in F(t,u)\} \leq p(t)\psi(\|u\|_{\mathcal{D}})$$
$$\text{for each } (t,u) \in J \times \mathcal{D}$$

and there exists a constant $M_* > 0$ *with*

$$\frac{\left[1-(K_2+K_4)\dfrac{T^\beta}{\beta}\right]M_*}{K_1+(K_3+K_5)\psi(M_*)\displaystyle\int_0^T p(s)ds} > 1,$$

where

$$\begin{aligned}
\Lambda = {} & M\|\phi\|_{\mathcal{D}}\Big\{1+c_1\|(-A)^{-\beta}\|\Big\}+MQ+c_2\|(-A)^{-\beta}\|\{M+1\}+\frac{C_{1-\beta}c_2T^\beta}{\beta}\\
& +TMM_1M_2\Big[\|y_1\|+M\|\phi\|_{\mathcal{D}}+M\|(-A)^{-\beta}\|\,[c_1\|\phi\|_{\mathcal{D}}+c_2]\\
& +c_2\|(-A)^{-\beta}\|+\frac{C_{1-\beta}c_2T^\beta}{\beta}\Big],\\
K_0 = {} & 1-c_1\|(-A)^{-\beta}\|(1+TMM_1M_2)>0, \qquad K_1=\frac{\Lambda}{K_0},\\
K_2 = {} & \frac{TMM_1M_2C_{1-\beta}c_1}{K_0},\ \ K_3=\frac{TM^2M_1M_2}{K_0},\ \ K_4=\frac{M}{K_0},\ \ K_5=\frac{C_{1-\beta}c_1}{K_0},
\end{aligned}$$

and

$$(K_2+K_4)\frac{T^\beta}{\beta} < 1.$$

Then the problem (6.1)–(6.2) *is nonlocally controllable on* $[-r,T]$.

Proof. Using hypothesis (6.3.1) for an arbitrary function $y(\cdot)$ define the control

$$\begin{aligned}
u_y(t) = {} & W^{-1}\Big[y_1-S(T)(\phi(0)-h_0(y)-f(0,\phi))-f(T,y_T)\\
& -\int_0^T AS(T-s)f(s,y_s)ds-\int_0^T S(T-s)v(s)ds\Big](t),
\end{aligned}$$

where $v \in S_{F,y}$. Then we must show that when using this control, the operator $N : C([-r,T],E) \longrightarrow \mathcal{P}(C([-r,T],E))$ defined by:

$$N(y) = \left\{ h \in C([-r,T],E) : h(t) = \begin{cases} \phi(t) - h_y(y), & t \in [-r,0], \\ S(t)[\phi(0) - h_0(y) - f(0,\phi(0))] & \\ +f(t,y_t) + \int_0^t AS(t-s)f(s,y_s)ds & \\ + \int_0^t S(t-s)[\mathcal{B}u_y(s) + v(s)]ds, & t \in J, \end{cases} \right\}$$

where $v \in S_{F,y}$, has a fixed point. As in Theorem 3.2 we can prove that N is a completely continuous multivalued map, u.s.c. with convex values.

We now show there exists an open set $U \subseteq C(J,E)$ with $u \notin \lambda N(u)$ for $\lambda \in (0,1)$ and $u \in \partial U$.

Let $\lambda \in (0,1)$ and let $u \in \lambda N u$. Then there exists $v \in S_{F,u}$ such that

$$\begin{aligned} u(t) &= \lambda S(t)[\phi(0) - h_0(y) - f(0,\phi(0))] + \lambda f(t,u_t) \\ &\quad + \lambda \int_0^t AS(t-s)f(s,u_s)ds \\ &\quad + \lambda \int_0^t S(t-s)BW^{-1}\Big[y_1 - S(T)(\phi(0) - h_0(y) - f(0,\phi)) - f(T,u_T) \\ &\quad - \int_0^T AS(T-s)f(s,u_s)ds - \int_0^T S(T-s)v(s)ds\Big](\eta)d\eta \\ &\quad + \lambda \int_0^t S(t-s)v(s)ds, \quad t \in J. \end{aligned}$$

Then

$$\begin{aligned} &\|u(t)\| \leq \\ &\leq M[\,\|\phi\|_{\mathcal{D}} + Q] + M\|(-A)^{-\beta}\|\,[c_1\|\phi\|_{\mathcal{D}} + c_2] + \|(-A)^{-\beta}\|\,[c_1\|u_t\|_{\mathcal{D}} + c_2] \\ &\quad + \int_0^t \|(-A)^{1-\beta}S(t-s)\|\,\|(-A)^{\beta}f(s,u_s)\|\,ds \\ &\quad + TMM_1M_2\Big[\|y_1\| + M\|\phi\|_{\mathcal{D}} + M\|(-A)^{-\beta}\|\,[c_1\|\phi\|_{\mathcal{D}} + c_2] \end{aligned}$$

$$
\begin{aligned}
&+\|(-A)^{-\beta}\|\,[c_1\|u_T\|_{\mathcal{D}}+c_2]+\int_0^T\|(-A)^{1-\beta}S(t-s)\|\,\|(-A)^{\beta}f(s,u_s)\|\,ds\\
&+M\int_0^T p(s)\psi(\|u_s\|_{\mathcal{D}})ds\Big]+M\int_0^t p(s)\psi(\|u_s\|_{\mathcal{D}})ds\\
\le{}& M[\|\phi\|_{\mathcal{D}}+Q]+M\|(-A)^{-\beta}\|\,[c_1\|\phi\|_{\mathcal{D}}+c_2]+\|(-A)^{-\beta}\|\,[c_1\|u_t\|_{\mathcal{D}}+c_2]\\
&+C_{1-\beta}c_1\int_0^t\frac{\|u_s\|_{\mathcal{D}}}{(t-s)^{1-\beta}}\,ds+\frac{C_{1-\beta}c_2T^{\beta}}{\beta}\\
&+TMM_1M_2\Big[\|y_1\|+M\|\phi\|_{\mathcal{D}}+M\|(-A)^{-\beta}\|\,[c_1\|\phi\|_{\mathcal{D}}+c_2]\\
&+\|(-A)^{-\beta}\|\,[c_1\|u_T\|_{\mathcal{D}}+c_2]+C_{1-\beta}c_1\int_0^T\frac{\|u_s\|_{\mathcal{D}}}{(t-s)^{1-\beta}}\,ds+\frac{C_{1-\beta}c_2T^{\beta}}{\beta}\\
&+M\int_0^T p(s)\psi(\|u_s\|_{\mathcal{D}})ds\Big]+M\int_0^t p(s)\psi(\|u_s\|_{\mathcal{D}})ds\\
\le{}&\Lambda+c_1\|(-A)^{-\beta}\|\|u_t\|_{\mathcal{D}}+C_{1-\beta}c_1\int_0^t\frac{\|u_s\|_{\mathcal{D}}}{(t-s)^{1-\beta}}\,ds\\
&+TMM_1M_2c_1\|(-A)^{-\beta}\|\|u_T\|+TMM_1M_2C_{1-\beta}c_1\int_0^T\frac{\|u_s\|_{\mathcal{D}}}{(t-s)^{1-\beta}}\,ds\\
&+TM^2M_1M_2\int_0^T p(s)\psi(\|u_s\|_{\mathcal{D}})ds+M\int_0^t p(s)\psi(\|u_s\|_{\mathcal{D}})ds,\quad t\in J.
\end{aligned}
$$

Put $w(t)=\max\{\|u(s)\|:-r\le s\le t\}$, $t\in J$. Then $\|u_t\|_{\mathcal{D}}\le w(t)$ for all $t\in J$ and there is a point $t^*\in[-r,t]$ such that $w(t)=\|u(t^*)\|$. Hence we have

$$
\begin{aligned}
w(t)&=\|u(t^*)\|\\
&\le\Lambda+c_1\|(-A)^{-\beta}\|\|u_{t^*}\|_{\mathcal{D}}+C_{1-\beta}c_1\int_0^{t^*}\frac{\|u_s\|_{\mathcal{D}}}{(t^*-s)^{1-\beta}}\,ds\\
&\quad+TMM_1M_2c_1\|(-A)^{-\beta}\|\|u_T\|_{\mathcal{D}}+TMM_1M_2C_{1-\beta}c_1\int_0^T\frac{\|u_s\|}{(T-s)^{1-\beta}}\,ds\\
&\quad+TM^2M_1M_2\int_0^T p(s)\psi(\|u_s\|_{\mathcal{D}})ds+M\int_0^{t^*} p(s)\psi(\|u_s\|_{\mathcal{D}})ds
\end{aligned}
$$

$$\begin{aligned}
&\le \Lambda + c_1\|(-A)^{-\beta}\|w(t) + C_{1-\beta}c_1\int_0^t \frac{w(s)}{(t-s)^{1-\beta}}\,ds \\
&\quad + TMM_1M_2c_1\|(-A)^{-\beta}\|w(t) + TMM_1M_2C_{1-\beta}c_1\int_0^T \frac{w(s)}{(T-s)^{1-\beta}}\,ds \\
&\quad + TM^2M_1M_2\int_0^T p(s)\psi(w(s))ds + M\int_0^t p(s)\psi(w(s))\,ds,
\end{aligned}$$

or

$$\begin{aligned}
w(t) \;\le\; & K_1 + K_2\int_0^T \frac{w(s)}{(T-s)^{1-\beta}}\,ds + K_3\int_0^T p(s)\psi(w(s))\,ds \\
& + K_4\int_0^t \frac{w(s)}{(t-s)^{1-\beta}}\,ds + K_5\int_0^t p(s)\psi(w(s))\,ds, \quad t\in J.
\end{aligned}$$

If $t^* \in [-r,0]$, then $w(t) \le \|\phi\|_{\mathcal{D}} + Q$ and the previous inequality holds.
Consequently,

$$\begin{aligned}
\|u\| \;\le\; & K_1 + K_2\|u\|\int_0^T \frac{ds}{(T-s)^{1-\beta}} + K_3\psi(\|u\|)\int_0^T p(s)ds \\
& + K_4\|u\|\int_0^T \frac{ds}{(T-s)^{1-\beta}} + K_5\psi(\|u\|)\int_0^T p(s)ds \\
\le\; & K_1 + (K_2+K_4)\frac{T^\beta}{\beta}\|u\|_\infty + (K_3+K_5)\psi(\|u\|)\int_0^T p(s)ds,
\end{aligned}$$

and therefore

$$\frac{\left[1-(K_2+K_4)\dfrac{T^\beta}{\beta}\right]\|u\|}{K_1 + (K_3+K_5)\psi(\|u\|)\displaystyle\int_0^T p(s)ds} \le 1.$$

Then by (3.2.5), there exists M_* such that $\|u\| \ne M_*$. Set

$$U = \{u \in C([-r,T],E) : \|u\| < M_*\}.$$

From the choice of U there is no $u \in \partial U$ such that $u \in \lambda N(u)$ for $\lambda \in (0,1)$. As a consequence of the Leray-Schauder Alternative for Kakutani maps [14] we deduce that N has a fixed point and therefore the problem (6.1)–(6.2) is nonlocally controllable. ■

For the lower semicontinuous case we state without proof the following result.

Theorem 6.4. *Assume that the conditions* (3.2.1)–(3.2.4) (3.4.1), (3.4.2), (6.3.1) *and* (6.3.2) *are satisfied. Then the problem* (6.1)–(6.2) *is nonlocally controllable on* $[-r, T]$.

In the case where A is non densely defined we state without proof the following result:

Theorem 6.5. *Assume that the conditions* (3.2.2) (i), (3.2.4), (3.2.5), (4.6.1)–(4.6.5) *hold. In addition, assume the following conditions are satisfied:*

(6.5.1) *the linear operator* $W : L^2(J, U) \to E$, *defined by*

$$Wu = \lim_{\lambda \to \infty} \int_0^b S'(T-s) B_\lambda (\mathcal{B}u)(s)\, ds,$$

has a bounded invertible operator $W^{-1} : E \to L^2(J, U)$ *and there exist positive constants* M_1, M_2 *such that* $\|\mathcal{B}\| \leq M_1$ *and* $\|W^{-1}\| \leq M_2$;

(6.5.2) *there exists a continuous non-decreasing function* $\psi : [0, \infty) \longrightarrow (0, \infty)$, $p \in L^1(J, \mathbb{R}_+)$ *such that*

$$\|F(t,u)\| := \sup\{\|v\| : v \in F(t,u)\} \leq p(t)\psi(\|u\|_{\mathcal{D}})$$

$$\textit{for each } (t,u) \in J \times E$$

and there exists a constant $M'_* > 0$ *with*

$$\frac{(1 - c_1 - C_2)M'_*}{C_1 + C_3\psi(M'_*) \int_0^T e^{-\omega s} p(s)ds + M^*\psi(M'_*) \int_0^t e^{-\omega s} p(s)ds} > 1,$$

where

$$\begin{aligned}
C_1 &= M^*[(1+c_1)\|\phi\|_{\mathcal{D}} + Q + c_2] + c_2 \\
&\quad + M^*MM_1M_2T\Big[\|y_1\| + M^*(1+c_1)\|\phi\|_{\mathcal{D}} + Q + M^*c_2 + c_2\Big], \\
C_2 &= M^*MM_1M_2Tc_1, \\
C_3 &= M^{*2}MM_1M_2T
\end{aligned}$$

and

$$c_1 + C_2 < 1.$$

Then the problem (6.1)–(6.2) *is nonlocally controllable on* $[-r,T]$.

Consider now the second order functional differential inclusion of the form

$$\frac{d}{dt}[y'(t) - f(t,y_t)] \in Ay(t) + F(t,y_t) + (\mathcal{B}u)(t), \quad t \in J := [0,T], \tag{6.3}$$

$$y(t) + h_t(y) = \phi(t),\ t \in [-r,0],\ \ y'(0) + h_1(y) = \eta, \tag{6.4}$$

where $f, F, A, \mathcal{B}, h_t, \phi$ are as in the problem (6.1)–(6.2) and $\eta \in E$.

Definition 6.6. A function $y \in C([-r,T],E)$ is called a mild solution to the problem (6.3)–(6.4) if $y(t) + h_t(y) = \phi(t), t \in [-r,0]$, $y'(0) + h_1(y) = \eta$ and there exists $v \in L^1(J,E)$ such that $v(t) \in F(t,y_t)$ a.e. on J and

$$\begin{aligned}
y(t) &= C(t)[\phi(0) - h_0(y)] + S(t)[\eta - h_1(y) - f(0,\phi)] + \int_0^t C(t-s)f(s,y_s)\,ds \\
&\quad + \int_0^t S(t-s)[(\mathcal{B}u(s) + v(s)]\,ds,\ t \in J.
\end{aligned}$$

Definition 6.7. The system (6.3)–(6.4) is said to be nonlocally controllable on the interval $[-r,T]$, if for every continuous initial function $\phi \in \mathcal{D}$ and every $\eta, y_1 \in E$ there exists a control $u \in L^2(J,U)$, such that the mild solution $y(t)$ of (6.3)–(6.4) satisfies $y(T) + h_T(y) = y_1$.

Theorem 6.8. *Assume that* (3.2.4), (3.2.5), (4.6.4), (5.2.1), (5.2.2), (5.2.5)–(5.2.7) *hold and in addition, we suppose that the following conditions are satisfied:*

(6.8.1) *the linear operator* $W : L^2(J,U) \to E$, *defined by*

$$Wu = \int_0^T S(T-s)\mathcal{B}u(s)\,ds,$$

has a bounded invertible operator $W^{-1} : E \to L^2(J,U)$ *and there exist positive constants* M_1, M_2 *such that* $\|\mathcal{B}\| \le M_1$ *and* $\|W^{-1}\| \le M_2$;

(6.8.2) *for each bounded* $B \subseteq C([-r,T],E)$, *and* $t \in J$ *the set*

$$\Big\{C(t)[\phi(0) - h_0(y)] \;+\; S(t)[\eta - h_1(y) - f(0,\phi(0)] + \int_0^t S(t-s)[(\mathcal{B}u_y)(s) + v(s)]ds, v \in S_{F,B}\Big\}$$

is relatively compact in E, *where* $S_{F,B} = \cup\{S_{F,y} : y \in B\}$, *and*

$$u_y(t) \;=\; W^{-1}\Big[y_1 - C(T)[\phi(0) - h_0(y)] - S(T)[\eta - h_1(y) - f(0,\phi)] - \int_0^T C(T-s)f(s,y_s)ds - \int_0^T S(T-s)v(s)ds\Big](t);$$

(6.8.3) *there exists a continuous non-decreasing function* $\psi : [0,\infty) \longrightarrow (0,\infty)$, $p \in L^1(J,\mathbb{R}_+)$ *such that*

$$\|F(t,u)\| := \sup\{\|v\| : v \in F(t,u)\} \le p(t)\psi(\|u\|_{\mathcal{D}})$$

$$\textit{for each } (t,u) \in J \times \mathcal{D}$$

and there exists a constant $M_* > 0$ *with*

$$\frac{[1 - T(c_1N_1 + \Lambda_2)]M_*}{\Lambda_1 + (\Lambda_3 + N_2)\psi(M_*)\displaystyle\int_0^T p(s)ds} > 1,$$

where

$$\begin{aligned}\Lambda_1 \;&=\; N_1[\,\|\phi\|_{\mathcal{D}}+Q]+N_2(|\eta|+Q_1+c_1\|\phi\|_{\mathcal{D}}+c_2)+N_1Tc_2\\ &\quad+TN_2M_1M_2\Big[\|y_1\|+N_1[\|\phi\|_{\mathcal{D}}+Q]\\ &\quad+N_2(|\eta|+Q_1+c_1\|\phi\|_{\mathcal{D}}+c_2)+N_1Tc_2\Big],\\ \Lambda_2 \;&=\; TN_2M_1M_2N_1,\\ \Lambda_3 \;&=\; TN_2^2M_1M_2,\end{aligned}$$

and

$$T(c_1N_1+\Lambda_2)<1.$$

Then the problem (6.3)–(6.4) *is nonlocally controllable on* $[-r,T]$.

Proof. Using hypothesis (6.8.2) for an arbirtary function $y(\cdot)$ define the control

$$\begin{aligned}u_y(t) \;&=\; W^{-1}\Big[y_1-C(T)[\phi(0)-h_0(y)]-S(T)[\eta-h_1(y)-f(0,\phi)]\\ &\quad-\int_0^T C(T-s)f(s,y_s)ds-\int_0^T S(T-s)v(s)ds\Big](t),\end{aligned}$$

where $v\in S_{F,y}$. We shall now show that when using this control, the operator $N: C(J_1,E)\longrightarrow \mathcal{P}(C(J_1,E)),\ J_1:=[-r,T]$, defined by:

$$N(y):=\left\{h\in C(J_1,E): h(t)=\begin{cases}\phi(t)-h_t(y), & \text{if } t\in[-r,0]\\ C(t)[\phi(0)-h_0(y)] & \\ +S(t)[\eta-h_1(y)-f(0,\phi)] & \\ +\displaystyle\int_0^t C(t-s)f(s,y_s)\,ds & \\ +\displaystyle\int_0^t S(t-s)[(\mathcal{B}u_y)(s)+v(s)]ds, & \text{if } t\in J\end{cases}\right\}$$

where $v\in S_{F,y}$, has a fixed point. This fixed point is then a solution of the system (6.3)–(6.4).

Clearly, $y_1\in(Ny)(T)$.

The argument in Theorem 5.2 guarantees that N is completely continuous. Also as in Theorem 5.2 one can prove that N has bounded, closed, convex values and is upper semicontinuous.

We now show there exists an open set $U \subseteq C(J,E)$ with $y \notin \lambda Ny$ for $\lambda \in (0,1)$ and $y \in \partial U$.

Let $\lambda \in (0,1)$ and let $y \in \lambda N(y)$. Then there exists $v \in S_{F,y}$ such that

$$\begin{aligned} y(t) \,=\, & \lambda C(t)[\phi(0)-h_0(y)] - \lambda S(t)[\eta - h_1(y) - f(0,\phi)] \\ & + \lambda \int_0^t C(t-s) f(s,y_s) ds \\ & + \lambda \int_0^t S(t-s) BW^{-1}\Big[y_1 - C(T)[\phi(0)-h_0(y)] \\ & - S(T)[\eta - h_1(y) - f(0,b)] \\ & - \int_0^T C(T-s) f(s,y_s)\, ds - \int_0^T S(T-s) v(s)\, ds\Big](\eta) d\eta \\ & + \lambda \int_0^t S(t-s) g(s) ds, \quad t \in J. \end{aligned}$$

This implies by our assumptions that for each $t \in J$ we have

$$\begin{aligned} & \|y(t)\| \leq \\ \leq\; & N_1[\|\phi\|_{\mathcal{D}} + Q] + N_2(\|\eta\| + Q_1 + c_1\|\phi\|_{\mathcal{D}} + c_2) + N_1 \int_0^t (c_1\|y_s\|_{\mathcal{D}} + c_2) ds \\ & + TN_2M_1M_2\Big[\|y_1\| + N_1\{\|\phi\|_{\mathcal{D}} + Q\} + N_2(\|\eta\| + Q_1 + c_1\|\phi\|_{\mathcal{D}} + c_2) \\ & + N_1 \int_0^T (c_1\|y_T\|_{\mathcal{D}} + c_2) ds + N_2 \int_0^T p(s)\psi(\|y\|_{\mathcal{D}}) ds\Big] \\ & + N_2 \int_0^t p(s)\psi(\|y\|_{\mathcal{D}}) ds \\ \leq\; & \Lambda_1 + c_1 N_1 \int_0^t \|y_s\|_{\mathcal{D}} ds + \Lambda_2 \int_0^T \|y_T\|_{\mathcal{D}} ds \\ & + \Lambda_3 \int_0^T p(s)\psi(\|y\|_{\mathcal{D}}) ds + N_2 \int_0^t p(s)\psi(\|y\|_{\mathcal{D}}) ds. \end{aligned}$$

We consider the function μ defined by

$$\mu(t) := \sup\{\|y(s)\| : -r \le s \le t\}, \quad t \in [0,T].$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = \|y(t^*)\|$. If $t^* \in [0,T]$, then by the previous inequality, we have for $t \in [0,T]$,

$$\begin{aligned}\mu(t) \le\ & \Lambda_1 + c_1 N_1 \int_0^t \mu(s)ds + \Lambda_2 \int_0^T \mu(s)ds \\ & + \Lambda_3 \int_0^T p(s)\psi(\mu(s))ds + N_2 \int_0^t p(s)\psi(\mu(s))ds.\end{aligned}$$

If $t^* \in [-r,0]$, then $\mu(t) \le \|\phi\|_{\mathcal{D}} + Q$ and the inequality holds.

Consequently,

$$\|y\| \le \Lambda_1 + T(c_1 N_1 + \Lambda_2)\|y\| + (\Lambda_3 + N_2)\psi(\|y\|) \int_0^T p(s)ds,$$

and therefore

$$\frac{[1 - T(c_1 N_1 + \Lambda_2)]\|y\|}{\Lambda_1 + (\Lambda_3 + N_2)\psi(\|y\|) \int_0^T p(s)ds} \le 1.$$

Then by (6.8.4), there exists M_* such that $\|y\| \ne M_*$.

Set

$$U = \{y \in C(J,E) : \|y\| < M_*\}.$$

From the choice of U there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for $\lambda \in (0,1)$. As a consequence of the Leray-Schauder Alternative for Kakutani maps [14] we deduce that N has a fixed point and therefore the problem (6.3)–(6.4) is controllable on $[-r,T]$. ∎

For the lower semicontinuous case we state without proof the following result.

Theorem 6.9. *Assume that the conditions* (3.2.4), (3.4.1), (3.4.2), (4.6.4), (5.2.1), (5.2.2), (5.2.5)–(5.2.7), (6.8.1)–(6.8.3) *are satisfied. Then the problem* (6.3)–(6.4) *is nonlocally controllable on* $[-r,T]$.

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