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# PROJECTION METHOD WITH RESIDUAL SELECTION FOR LINEAR FEASIBILITY PROBLEMS

Robert Dylewski

*Faculty of Mathematics, Computer Science and Econometrics*
*University of Zielona Góra, Poland*
*Szafrana 4a, 65–516 Zielona Góra, Poland*

**e-mail:** r.dylewski@wmie.uz.zgora.pl

**Abstract**

We propose a new projection method for linear feasibility problems. The method is based on the so called residual selection model. We present numerical results for some test problems.

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## 1. Introduction

In this paper, we consider the linear feasibility problem:

Given a system of linear inequalities

$$G^\top x \le b, \tag{1}$$

where $G$ is a matrix of size $n \times m$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

Find a solution $x^* \in M_0 = \{x : G^\top x \le b\}$ or detect that $M_0 = \emptyset$.

We use the following notation:

$x_k$ – $k$th element of a sequence $(x_k)$,
$x^\top y$ – the standard scalar product of vectors $x, y \in \mathbb{R}^n$,
$\|x\|$ – the Euclidean norm of a vector $x \in \mathbb{R}^n$,
$P_D x$ – the metric projection of a point $x \in \mathbb{R}^n$ onto a closed and convex subset $D \subset \mathbb{R}^n$,

$A^+ = (A^\top A)^{-1}A^\top$ – the Moore-Penrose pseudoinverse of a full column rank matrix $A$.

We study the projection method for the problem (1) of the form

$$x_1 \in \mathbb{R}^n \text{ – arbitrary} \qquad (2)$$
$$x_{k+1} = x_k + \lambda_k t_k,$$

where

$$t_k = P_{\{x : G^\top_{L_k} x \le b_{L_k}\}} x_k - x_k,$$

and $\lambda_k \in (0, 2)$.

We denote by $G_{L_k}$ the submatrix of $G$ which consists of the columns $L_k \subset J = \{1, 2, \ldots, m\}$ and by $b_{L_k}$ the subvector of $b$ which consists of the coordinates $L_k \subset J$.

In the method (2) we have a problem: how to choose $L_k \subset J$ such that $x_k^+ = P_{\{x : G^\top_{L_k} x \le b_{L_k}\}} x_k$ approximates a solution $x^* \in M_0$ essentially better than $x_k$ and such that $x_k^+$ can easily be evaluated.

Suppose that $G_{L_k}$ has a full column rank. Then, the equation system $G^\top_{L_k} x = b_{L_k}$ has a solution and

$$x'_k = P_{\{x : G^\top_{L_k} x = b_{L_k}\}} x_k = x_k - G_{L_k} \left( G^\top_{L_k} G_{L_k} \right)^{-1} \left( G^\top_{L_k} x_k - b_{L_k} \right). \qquad (3)$$

Of course, $x'_k$ is not necessarily equal to $x_k^+$. Nevertheless, it can be shown that

$$x'_k = x_k^+ \iff y := \left( G^\top_{L_k} G_{L_k} \right)^{-1} \left( G^\top_{L_k} x_k - b_{L_k} \right) \ge 0. \qquad (4)$$

If $G^\top_{L_k} x_k \ge b_{L_k}$ and $\left( G^\top_{L_k} G_{L_k} \right)^{-1} \ge 0$, then $y \ge 0$. Selections of $L_k \subset J$ with such properties were employed for convex feasibility problems or for convex minimization problems in [1, 2, 3, 4]. We call such a selection an *obtuse cone selection* since the columns of a full column rank matrix $A$ generate an obtuse cone if and only if $\left( A^\top A \right)^{-1} \ge 0$. In this paper, we study selections of $L_k \subset J$ such that $y \ge 0$ without assuming that $G^\top_{L_k} x_k \ge b_{L_k}$, for the linear feasibility problems. Such selections were employed for convex minimization problems in [5, 6].

## 2. RESIDUAL SELECTION MODEL

Let be given a system of linear inequalities (1) and an approximation $\overline{x} \in \mathbb{R}^n$ of a solution of this system. We construct sequentially a subset $L \subset J = \{1, 2, \ldots, m\}$ and, consequently, the matrix $G_L$ which has the properties: $G_L$ has a full column rank and $y := \left(G_L^\top G_L\right)^{-1} \left(G_L^\top x_k - b_L\right) \geq 0$. To simplify the notation, we denote $\underbrace{A}_{n\times l} := G_L$, $\underbrace{d}_{l\times 1} := b_L$, where $l = |L|$.

Let $A = [\underbrace{A_1}_{n\times(l-1)}, \underbrace{a}_{n\times 1}]$. Denote by $r$ the residual vector, i.e.,

$$r = \begin{bmatrix} r_1 \\ \rho \end{bmatrix} = A^\top \overline{x} - d = \begin{bmatrix} A_1^\top \\ a^\top \end{bmatrix} \overline{x} - \begin{bmatrix} d_1 \\ \delta_l \end{bmatrix}, \tag{5}$$

where $r_1 \in \mathbb{R}^{l-1}$ and $\rho \in \mathbb{R}$.

The following theorem enables a sequential construction of a full column rank submatrix $A$ of $G$ for which $y := \left(A^\top A\right)^{-1} r \geq 0$, where the residual vector $r$ is not necessarily nonnegative. Therefore, we call a model obtained by such a construction a *residual selection model.*

**Theorem 1.** *Suppose that there exists $z \in \mathbb{R}^n$ such that $A^\top z \leq d$. If*

(i) *$A_1$ has a full column rank,*

(ii) *$\left(A_1^\top A_1\right)^{-1} r_1 \geq 0$,*

(iii) *$A_1^+ a \leq 0$,*

(iv) *$(A_1^+ a)^\top r_1 < \rho$,*

*then*

(I) *$A$ has a full column rank,*

(II) *$\left(A^\top A\right)^{-1} r \geq 0$.*

***Proof.*** See [6, Theorem 1]. ■

**Corollary 2.** *Let $\overline{x} \in \mathbb{R}^n$ be arbitrary. Suppose that the assumptions of Theorem 1 are satisfied and let*

$$t = -A\left(A^\top A\right)^{-1}\left(A^\top \overline{x} - d\right). \tag{6}$$

*Then*

$$x^+ = \overline{x} + t = P_{\{x: A^\top x \leq d\}} \overline{x}\,. \tag{7}$$

***Proof.*** By (3) and by (6) we have $\overline{x}+t = P_{\{x:A^\top x=d\}}\overline{x}$. By Theorem 1 and by (4) we obtain equality (7). ■

## 3. PROJECTION METHOD WITH RESIDUAL SELECTION

In this section, we present a projection method with a residual selection for the linear feasibility problem presented in Section 1. We do not suppose that the system (1) is consistent.

**Iterative scheme 3.**

Choose:

$x_0 \in \mathbb{R}^n$ (*starting point*), $\varepsilon \geq 0$ (*optimality tolerance*).

For $x_k \in \mathbb{R}^n$:

1. (*stopping criterion*)

set $i_k = \arg\max_{1\leq i\leq m}\{G_i^\top x_k - b_i\}$,
where $G_i$ is the $i$th column of $G$;
if $G_{i_k}^\top x_k - b_{i_k} \leq \varepsilon$, then terminate;
otherwise

2. (*residual selection*)

select $L_k \subset \{1, 2, \dots, m\}$ such that:

$i_k \in L_k$, $G_{L_k} = [G_i : i \in L_k]$ has a full column rank and $\left(G_{L_k}^\top G_{L_k}\right)^{-1} r_{L_k} \geq 0$,
where $r_{L_k} = G_{L_k}^\top x_k - b_{L_k}$,

3. make a Cholesky factorization $C_{L_k} C_{L_k}^\top$ of the matrix $G_{L_k}^\top G_{L_k}$;
if the Cholesky procedure breaks down, then terminate ($\{x : G^\top x \leq b\} = \emptyset$),

4. evaluate $t_k = -G_{L_k}\left(C_{L_k} C_{L_k}^\top\right)^{-1} r_{L_k}$,

5. set $x_{k+1} = x_k + \lambda_k t_k$,
where the *relaxation parameter* $\lambda_k \in [\alpha, 2-\alpha]$, $0 < \alpha < 1$.

**Remark 4.** We apply sequentially Theorem 1 in order to construct the subset $L_k$ and, consequently, the matrix $G_{L_k}$ in Step 2.

If the Cholesky procedure detects a linear dependency of the columns of $G_{L_k}$ in Step 3, then we obtain a contradiction, which proves by Theorem 1 the inconsistency of the system $G^\top x \leq b$. If the inconsistency is not detected, then the matrix $G_{L_k}$ has a full column rank and, by Corollary 2, the vector $t_k$ determined in Step 4 is the projection vector of $x_k$ onto the subset $\{x : G_{L_k}^\top x \leq b_{L_k}\}$.

Now, we show that any sequence generated by Iterative scheme 3 converges to a solution $x^* \in M_0$.

**Theorem 5.** *Suppose that there exists $z \in \mathbb{R}^n$ such that $G^\top z \leq b$. If the sequence $(x_k)$ is generated by Iterative scheme 3, then*

$$\max\{0, G_i^\top x_k - b_i : i = 1, 2, \dots, m\} \longrightarrow 0. \tag{8}$$

***Proof.*** For all $z \in M_0$ and $k \geq 0$ we have

$$(z - x_k)^\top t_k \geq \|t_k\|^2$$

and, consequently,

$$\|x_{k+1} - z\|^2 = \|x_k + \lambda_k t_k - z\|^2 = \|x_k - z\|^2 - 2\lambda_k (z - x_k)^\top t_k + (\lambda_k)^2 \|t_k\|^2$$

$$\leq \|x_k - z\|^2 - 2\lambda_k \|t_k\|^2 + (\lambda_k)^2 \|t_k\|^2 = \|x_k - z\|^2 - \lambda_k (2 - \lambda_k) \|t_k\|^2.$$

Hence,

$$\sum_{k=0}^{\infty} \|t_k\|^2 < \infty$$

since $\lambda_k \in [\alpha, 2 - \alpha]$, $0 < \alpha < 1$ and, consequently,

$$\|t_k\| \longrightarrow 0. \tag{9}$$

We denote $\beta = \max_{i=1,2,\dots,m} \|G_i\|$. If $L_k = \{i_k\}$, then

$$t_k = -\left(G_{i_k}^\top x_k - b_{i_k}\right)\left(G_{i_k} / \|G_{i_k}\|^2\right).$$

Hence, if $i_k \in L_k$, then $\left\{x : G_{L_k}^{\top} x \le b_{L_k}\right\} \subset \left\{x : G_{i_k}^{\top} x \le b_{i_k}\right\}$ and

$$\|t_k\| \ge \left(G_{i_k}^{\top} x_k - b_{i_k}\right) / \|G_{i_k}\| \ge \left(G_{i_k}^{\top} x_k - b_{i_k}\right) / \beta.$$

Consequently,

$$\|t_k\| \ge \left(G_i^{\top} x_k - b_i\right) / \beta, \tag{10}$$

for all $i = 1, 2, \ldots, m$. Now, we obtain (8) from (9) and (10). ∎

## 4. Numerical results

In this section, we present the computation results of the projection method with a residual selection for linear feasibility problems.

In the numerical experiments we have tested the method for the randomly generated linear feasibility problems

$$G^{\top} x \le b,$$

where $G$ is a matrix of size $n \times m$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

In these problems, the coordinates of columns of matrix $G$ are randomly generated in the interval $(-0.5, 0.5)$. For $i = 1, 2, \ldots, l$ we admit $b_i = 0$ and for $i = l+1, \ldots, m$ the coordinates of vector $b$ are randomly generated in the interval $(0, 1)$. We have guaranteed that the system $G^{\top} x \le b$ is consistent. The coordinates of the starting point are generated in the interval $(0, 1)$.

Ten examples were solved for each system of parameters $n, m, l$. In Tables 1 and 2 we present the average number of iterations $k_i, i = 1, 2, 3$ which are necessary to get an $\varepsilon$-optimal solution. We set the optimality tolerance $\varepsilon = 10^{-6}$. The method was programmed in Fortran 90 (Lahey Fortran 90 v.3.5). All floating point calculations were performed with double precision, allowing the relative accuracy of $2.2 * 10^{-16}$.

In Table 1 we present the results of numerical tests for the method presented in Section 3 (Iterative scheme 3) with relaxation parameter $\lambda_k = 1$ and $\lambda_k = 1.5$. In the last column we present results where we use the so called largest residuum strategy (*l.r.s.*) in Step 2 of Iterative scheme 3, (see [5, Section 3]).

Table 1

| | | $\lambda_k = 1$ | $\lambda_k = 1.5$ | $\lambda_k = 1.5$ $(l.r.s.)$ |
|---|---|---|---|---|
| $n \times m$ | $l$ | $k_1$ | $k_2$ | $k_3$ |
| $20 \times 20$ | 12 | 6 | 6 | 5 |
| | 20 | 7 | 6 | 5 |
| $20 \times 40$ | 12 | 8 | 7 | 7 |
| | 24 | 14 | 13 | 11 |
| $20 \times 80$ | 12 | 11 | 10 | 10 |
| | 24 | 18 | 14 | 12 |
| $50 \times 50$ | 30 | 19 | 12 | 10 |
| | 50 | 33 | 16 | 14 |
| $50 \times 100$ | 30 | 35 | 19 | 14 |
| | 60 | 145 | 37 | 28 |
| $50 \times 200$ | 30 | 69 | 28 | 21 |
| | 60 | 122 | 37 | 27 |
| $100 \times 100$ | 60 | 49 | 20 | 17 |
| | 100 | 77 | 25 | 19 |
| $200 \times 200$ | 120 | 97 | 34 | 25 |
| | 200 | 164 | 38 | 30 |

In Table 2 we present the results of numerical tests for the projection method with a residual selection (see [6]) for convex minimization problems of the form

$$\text{minimize} \qquad f(x) = \max\{0, G_i^\top x - b_i : i = 1, 2, \ldots, m\}$$

which is equivalent to the problem (1).

Table 2

| | | $\lambda_k = 1$ | $\lambda_k = 1.5$ | $\lambda_k = 1.5$ $(l.r.s.)$ |
|---|---|---|---|---|
| $n \times m$ | $l$ | $k_1$ | $k_2$ | $k_3$ |
| $20 \times 20$ | 12 | 10 | 9 | 9 |
| | 20 | 11 | 10 | 10 |
| $50 \times 50$ | 30 | 29 | 24 | 24 |
| | 50 | 51 | 33 | 32 |
| $100 \times 100$ | 60 | 71 | 49 | 47 |
| | 100 | 99 | 57 | 54 |

We can see that for each system of parameters $n, m, l$ the results for the projection method with a residual selection for the linear feasibility problem (Table 1) are better than for the projection method with a residual selection for the convex minimization problem (Table 2). The influence of the relaxation parameter $\lambda_k$ on the convergence is essential for both methods. If the parameter $l$ is greater then the solution set $M_0 = \{x : G^\top x \leq b\}$ is flatter and the number of iterations is greater.

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