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SET-VALUED FRACTIONAL ORDER DIFFERENTIAL EQUATIONS IN THE SPACE OF SUMMABLE FUNCTIONS

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Abstract

In this paper, we study the existence of integrable solutions for the set-valued differential equation of fractional type

$$\left(D^{\alpha_n} - \sum_{i=1}^{n-1} a_i D^{\alpha_i}\right) x(t) \in F(t, x(\varphi(t))),$$
$$\text{a.e. on } (0,1), \qquad I^{1-\alpha_n} x(0) = c, \quad \alpha_n \in (0,1),$$

where $F(t,\cdot)$ is lower semicontinuous from $\mathbb{R}$ into $\mathbb{R}$ and $F(\cdot,\cdot)$ is measurable. The corresponding single-valued problem will be considered first.

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1. Introduction

Recently, much attention has been paid to the existence of solutions for the fractional order differential equations (see [1, 7, 12, 17, 18] and the references therein). Our aim in this paper is to prove the existence of solutions (in the class of summable functions) for the set-valued differential equation of the fractional type

$$(1)\qquad L(D)x(t) \in F(t, x(\varphi(t))), \quad \text{a.e. on } (0,1), \quad I^{1-\alpha_n}x(0) = c, \ \alpha_n \in (0,1),$$

where $L(D) := D^{\alpha_n} - a_{n-1}D^{\alpha_{n-1}} - \cdots - a_1 D^{\alpha_1}$, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$ and D^{α_i} denotes the standard Riemann-Liouville fractional derivatives. Here F is a set-valued function defined on $[0,1] \times \mathbb{R}$, with nonempty closed values. In order to achieve our aim we first consider the corresponding (single-valued) differential equation

$$(2)\qquad \begin{aligned} &L(D)x(t) = f(t, x(\varphi(t))),\\ &\text{a.e. on } (0,1), \quad I^{1-\alpha_n}x(0) := I^{1-\alpha_n}x(t)|_{t=0} = c, \quad \alpha_n \in (0,1). \end{aligned}$$

Our investigation is based on reducing the problem (2) to the Volterra integral equation

$$(3)\quad x(t) = x_0 t^{\alpha_n - 1} + \sum_{i=1}^{n-1} a_i I^{\alpha_n - \alpha_i} x(t) + I^{\alpha_n} f(t, x(\varphi(t))), \quad \text{a.e. on } (0,1),$$

where

$$x_0 = \frac{1}{\Gamma(\alpha_n)} \left\{ c - \sum_{i=1}^{n-1} a_i I^{1-\alpha_i} x(0) \right\}.$$

The set-valued problem (1) has been considered in [10, 11] and [13] but in [10] the set-valued function $F(\cdot, x(\cdot))$ was assumed to satisfy the Lipschitz condition and in [13] it was assumed to have Carathéodory selections and considered that $\alpha_n > 1$. Here, we study the problem (1), where $F(t, \cdot)$ is lower semicontinuous from $\mathbb{R}$ into $\mathbb{R}$ and $F(\cdot, \cdot)$ is measurable. Furthermore, we point out that particular cases of the problems (2) and (3) have provoked some interest in the literature (cf. [1, 3, 7, 12, 15, 17, 18] and [22] for instance). Our paper is a continuation of the results mentioned above. In comparison with the earlier results of type (2) and (3) we get more general assumptions. In [1, 3, 7, 15, 18] and [22] the equations (2) and (3) have been studied in view of obtaining the existence of continuous solutions, so the function f is assumed to be continuous and $\phi(t) = t$ and in [17], f is assumed to satisfy the Lipschitz condition. The case of monotonic functions which satisfy Carathéodory conditions, equation (3), was studied in [10, 12, 13]. Here, in Theorem 3.1, we assume that the function f satisfies only Carathéodory conditions. Let us remark that most of the above investigations have not been complete, however, most researchers have obtained results not for the set-valued problems but for the corresponding set-valued integral equations. Some of the earlier results of this type contain errors in the proof of equivalence of the initial value problems and the corresponding Volterra integral equations (see survey paper by Kilbas and Trujillo [16], Sections 4 and 5). In the present paper we focus on avoiding such a problem.

2. Notations and auxiliary results

Let $L_1(a,b)$ be the space of Lebesgue integrable functions on the interval $I = (a,b)$. Define $B_r := \{x \in L^1(I) : \|x\| < r, \, r > 0\}$. We recall that the fractional integral operator of order $\alpha > 0$ with left-hand point a is defined by

$$I_a^\alpha x(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s) ds.$$

Using the known relations between the Beta- and Gamma-function, a well-known calculation with the Fubini-Tonelli theorem shows that $I_a^{\alpha+\beta} x = I_a^\alpha I_a^\beta x$ for each $x \in L_1(a,b)$ and each $\alpha, \beta > 0$. In particular, I_a^n is the n-th iterate of the usual integral operator, and so I_a^α may indeed be considered as a corresponding fractional integral. We define the corresponding (Riemann-Liouville) differential operator

$$D_a^\alpha x(t) := D I_a^{1-\alpha} x(t), \; 0 < \alpha < 1.$$

Here, D denotes the usual differential operator. The following Proposition is obvious:

Proposition 2.1. *Let $\alpha, \beta \in \mathbb{R}^+$, $f \in L_1(0,1)$ and $n = 1, 2, 3, \cdots$. Then we have:*

1. $I^\alpha : L_1(0,1) \to L_1(0,1)$ *is a continuous operator,*
2. $\lim_{\alpha \to n} I^\alpha \; f(t) = I^n f(t)$,

3. $D^{\alpha}I^{\alpha}f(t) = f(t)$. *If the fractional derivative* $D^{\beta}f$ *is integrable,*

$$I^{\alpha}D^{\beta}f(t) = I^{\alpha-\beta}f(t) - \left[I^{1-\beta}f(t)\right]_{t=0}\frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad 0<\beta\le\alpha<1.$$

For more remarks concerning the fractional calculus, we refer the readers to ([17, 18] and [21]). Now, let us conclude the introduction by stating main results that will be used in the sequel (cf. [8] and [9]).

Theorem 2.1 (Rothe's Fixed Point Theorem)**.** *Let* U *be an open, bounded, convex subset of a Banach space* X*,* $0 \in U$ *and let* $T : \bar{U} \to X$ *be completely continuous. If*

$$T(\partial U) \subseteq \bar{U} \tag{4}$$

then T has a fixed point.

Theorem 2.2 (Kolmogorov's Compactness Criterion)**.** *Let* $\Omega \subseteq L_p(0,1)$, $1 \le p < \infty$. *If*

1. Ω *is bounded in* $L_P(0,1)$ *and*
2. $x_h \to x$ *as* $h \to 0$ *uniformly with respect to* $x \in \Omega$, *then* Ω *is relatively compact in* $L_p(0,1)$, *where*

$$x_h(t) = \frac{1}{h}\int_t^{t+h} x(s)\ ds.$$

Lemma 2.1 [23]**.** *Let* f *be Lebesgue integrable on* $[0,1]$. *then*

$$\frac{1}{h}\int_t^{t+h} |f(\tau)-f(t)|\ d\tau \to 0 \quad \textit{for a.e. } t \in [0,1].$$

3. Single-valued problem

In this section, we prove that the integral equation (3) has a summable solution. We begin by showing that the Cauchy problem (2) and the Volterra integral equation (3) are equivalent in the sense that if $x \in L_1(0,1)$ satisfies one of these relations, then it also satisfies the other. We prove such a result by assuming that for every $x \in L_1(0,1)$ the function $f(\cdot, x(\cdot)) \in L_1(0,1)$. To facilitate our discussion, let us first state the following assumptions:

1. $f : (0,1) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function with the following properties:
   (a) for each $t \in (0,1)$, $f(t,\cdot)$ is continuous,
   (b) for each $x \in \mathbb{R}$, $f(\cdot,x)$ is measurable,
   (c) there exist two real functions $t \to a(t), t \to b(t)$ such that:
   $$|f(t,x)| \le a(t) + b(t)|x| \quad \text{for each } t \in (0,1) \text{ and } x \in \mathbb{R},$$
   where $a(\cdot) \in L^1(0,1)$ and $b(\cdot)$ is measurable and bounded,
2. $\varphi : (0,1) \longrightarrow (0,1)$ is nondecreasing, absolutely continuous and there is a constant $M > 0$ such that $\varphi^{'} \ge M$ a.e. on $(0,1)$.

Thus, we are in a position to formulate and prove the following

Lemma 3.1. *Let $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$. Assume that the assumptions* (1) *and* (2) *are satisfied. If $x \in L_1(0,1)$, then x satisfies a.e. the problem* (2) *if, and only if, x satisfies the integral equation* (3).

Proof. First, we prove the necessity. Let $x \in L_1(0,1)$ satisfy a.e. the problem (2). Since f satisfies Carathéodory conditions (a), (b) and since φ satisfies assumption (2), $f(\cdot, x(\varphi(\cdot)))$ is measurable and from (c) we have

$$\begin{aligned} \int_0^1 |f(s,x(\phi(s))|\, ds &\le \int_0^1 \{|a(s)| + |b(s)|\ |x(\phi(s))|\}\, ds \\ &\le \|a\| + \frac{\sup|b(t)|}{M}\, \|x\|. \end{aligned} \tag{5}$$

Thus, $f(\cdot, x(\varphi(\cdot))) \in L_1(0,1)$ and consequently equation (2) means that there exist a.e. on $(0,1)$ the fractional derivatives $D^{\alpha_i}x \in L_1(0,1)$, $i = 1,2,\cdots$. According to ([21], Lemma 2.4), $I^{1-\alpha_i}x$ is absolutely continuous on $[0,1]$ for every i. Thanks to Proposition 2.1 we deduce

$$\begin{aligned} &I^{\alpha_n} L(D)x(t) = \\ &= I^{\alpha_n} D^{\alpha_n} x(t) - \sum_{i=1}^{n-1} a_i I^{\alpha_n} D^{\alpha_i} x(t) \\ &= x(t) - \frac{t^{\alpha_n - 1}}{\Gamma(\alpha_n)} I^{1-\alpha_n} x(0) - \sum_{i=1}^{n-1} a_i \left\{ I^{\alpha_n - \alpha_i} x(t) - \frac{t^{\alpha_n - 1}}{\Gamma(\alpha_n)} I^{1-\alpha_i} x(0) \right\} \\ &= x(t) - \sum_{i=1}^{n-1} a_i I^{\alpha_n - \alpha_i} x(t) - \frac{t^{\alpha_n - 1}}{\Gamma(\alpha_n)} \left\{ c - \sum_{i=1}^{n-1} a_i I^{1-\alpha_i} x(0) \right\}. \end{aligned}$$

since $f(\cdot, x(\varphi(\cdot))) \in L_1(0,1)$, Proposition 2.1 results in $I^{\alpha_n} f \in L_1(0,1)$ a.e. on $(0,1)$. Applying the operator I^{α_n} on both sides of (2) we have

$$x(t) - \sum_{i=1}^{n-1} a_i I^{\alpha_n - \alpha_i} x(t) - \frac{t^{\alpha_n - 1}}{\Gamma(\alpha_n)} \left\{ c - \sum_{i=1}^{n-1} a_i I^{1-\alpha_i} x(0) \right\} = I^{\alpha_n} f(t, x(\varphi(t))),$$

therefore, we obtain the integral equation 3.

Conversely, let $x \in L_1(0,1)$ satisfy the integral equation (3) a.e. on $(0,1)$. Applying the operator D^{α_n} on both sides of (3) and using Proposition 2.1 we obtain

$$\begin{aligned} D^{\alpha_n} x(t) &= \sum_{i=1}^{n-1} a_i D I^{1-\alpha_n} I^{\alpha_n - \alpha_i} x(t) + f(t, x(\varphi(t))) \\ &= \sum_{i=1}^{n-1} a_i D^{\alpha_i} x(t) + f(t, x(\varphi(t))). \end{aligned}$$

From here, we arrive at the equation 2. Now, we show that the initial conditions of the problem (2) also hold. To see this we transform both sides of (3) by the operator $I^{1-\alpha_n}$ and obtain:

$$\begin{aligned} I^{1-\alpha_n} x(t) &= x_0 \Gamma(\alpha_n) + \sum_{i=1}^{n-1} a_i I^{1-\alpha_j} x(t) + \int_0^t f(s, x(\varphi(s)))\, ds \\ &= c - \sum_{i=1}^{n-1} a_i I^{1-\alpha_j} x(0) + \sum_{i=1}^{n-1} a_i I^{1-\alpha_j} x(t) + \int_0^t f(s, x(\varphi(s)))\, ds. \end{aligned}$$

Taking the limit as $t \to 0^+$, we obtain the initial condition of the problem 2. Thus the sufficiency is proved, which completes the proof. ■

Now, we formulate and prove the following result

Theorem 3.1. *Suppose that the assumptions of Lemma* 3.1 *hold along with*

$$\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1 + \alpha_n - \alpha_i)} + \frac{\sup |b(t)|}{M\Gamma(\alpha_n + 1)} < 1. \tag{6}$$

Then equation (2) *has at least one solution s.t.* $x(\cdot) \in B_r$, *where*

$$r \leq \frac{\frac{|x_0|}{\alpha_n} + \frac{\|a\|}{\Gamma(1+\alpha_n)}}{1 - \left[\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{\sup|b(s)|}{M\Gamma(1+\alpha_n)}\right]}.$$

Proof. Let us define the operator T as

$$(Tx)(t) := x_0 t^{\alpha_n - 1} + \sum_{i=1}^{n-1} a_i I^{\alpha_n-\alpha_i} x(t) + I^{\alpha_n} f(t, x(\varphi(t))), \quad \text{a.e. on } (0,1). \tag{7}$$

We claim

$$T : L_1(0,1) \to L_1(0,1), \text{ continuously.}$$

To prove our claim, first note that, as in the proof of Lemma 3.1, for each $x \in L_1(0,1)$, $f(\cdot, x(\varphi(\cdot))) \in L_1(0,1)$. That is, the operator T makes sense. Further, f is continuous in x (assumption 1) and I^α maps $L_1(0,1)$ continuously into itself (Proposition 2.1), $x \to I^\alpha f(t, x(\varphi(t)))$ is continuous in x. Since x is an arbitrary element in $L_1(0,1)$, then T is well-defined and maps $L_1(0,1)$ continuously into $L_1(0,1)$. Now, we will show that $T : \bar{B}_r \to L_1(0,1)$ is a completely continuous operator. To achieve this goal we will let x be an arbitrary element in the open set B_r. Then from assumptions (1) and (2) we have

$$\|Tx\| \leq |x_0| \int_0^1 t^{\alpha_n-1} dt$$
$$+ \int_0^1 \int_0^t \left[\sum_{i=1}^{n-1} |a_i| \frac{(t-s)^{\alpha_n-\alpha_i-1}}{\Gamma(\alpha_n-\alpha_i)} |x(s)| + \frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)} |f(s, x(\varphi(s)))|\right] ds\ dt.$$

By interchanging the order of integration, we get

$$\|Tx\| \leq \frac{|x_0|}{\alpha_n} + \int_0^1 \int_s^1 \left[\sum_{i=1}^{n-1} |a_i| \frac{(t-s)^{\alpha_n-\alpha_i-1}}{\Gamma(\alpha_n-\alpha_i)} |x(s)| \right.$$
$$\left. + \frac{(t-s)^{\alpha_n-1}}{\Gamma(\alpha_n)} |f(s, x(\varphi(s)))|\right] dt\, ds$$

$$\leq \frac{|x_0|}{\alpha_n} + \int_0^1 \left[\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} |x(s)| \right.$$
$$\left. + \frac{1}{\Gamma(1+\alpha_n)} \{|a(s)| + |b(s)||x(\varphi(s))|\} \right] ds$$

$$\leq \frac{|x_0|}{\alpha_n} + \sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} \|x\| + \frac{\|a\|}{\Gamma(1+\alpha_n)}$$
$$+ \frac{\sup |b(s)|}{M\Gamma(1+\alpha_n)} \int_0^1 |x(\varphi(s))||\varphi'(s)| \, ds$$

$$\leq \frac{|x_0|}{\alpha_n} + \sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} \|x\| + \frac{\|a\|}{\Gamma(1+\alpha_n)}$$
$$+ \frac{\sup |b(s)|}{M\Gamma(1+\alpha_n)} \int_{\varphi(0)}^{\varphi(1)} |x(u)| \, du \, .$$

Therefore

$$(8) \quad \|Tx\| \leq \frac{|x_0|}{\alpha_n} + \frac{\|a\|}{\Gamma(1+\alpha_n)} + \left[\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n-\alpha_i)} + \frac{\sup |b(s)|}{M\Gamma(1+\alpha_n)} \right] \|x\| \, .$$

The above inequality means that the operator T maps B_r into $L_1(0,1)$. Moreover, if $x \in \partial B_r$, then from inequality (8) we have $\|Tx\| \leq r$ i.e., the condition (4) of Theorem 2.1 is satisfied. It remains to show that T is compact, so, let Ω be a bounded subset of B_r. We will show that $(Tx)_h \to Tx$ in $L_1(0,1)$ as $h \to 0$ uniformly with respect to $Tx \in \Omega$. We have the following estimation:

$$\begin{aligned} \|(Tx)_h - (Tx)\| &= \int_0^1 |(Tx)_h(t) - (Tx)(t)| \, dt \\ &= \int_0^1 \left| \frac{1}{h} \int_t^{t+h} (Tx)_h(\tau) \, d\tau - (Tx)(t) \right| \, dt \\ &\leq \int_0^1 \left(\frac{1}{h} \int_t^{t+h} |(Tx)_h(\tau) - (Tx)(t)| \, d\tau \right) \, dt \end{aligned}$$

$$\leq \int_0^1 \frac{|x_0|}{h} \int_t^{t+h} |\tau^{\alpha_n - 1} - t^{\alpha_n - 1}| \, d\tau \, dt$$

$$+ \int_0^1 \frac{1}{h} \int_t^{t+h} \sum_{i=1}^{n-1} |a_i||I^{\alpha_n - \alpha_i} x(t) - I^{\alpha_n - \alpha_i} x(\tau)| d\tau \, dt$$

$$+ \int_0^1 \frac{1}{h} \int_t^{t+h} |I^{\alpha_n} \; f(\tau, x(\phi(\tau))) - I^{\alpha_n} \; f(t, x(\phi(t)))| \; d\tau \; dt.$$

Since f, x and $t^{\alpha_n - 1}$ are in $L_1(0,1)$, Proposition 2.1 and Lemma 2.1 imply that $T(\Omega)$ is relatively compact, that is, T is a compact operator. Set $U = B_r$ and $X = L_1(0,1)$. Then Theorem 2.1 implies that T has a fixed point. Therefore, equation (3) has a solution $x \in L_1(0,1)$. In view of Lemma 3.1, equation (2) has a solution $x \in L_1(0,1)$. This completes the proof. ■

4. Differential inclusions

In this section, we present our main result by proving the existence of solutions of equation (1). Consider the multivalued equation (1), where $F : [0,1] \times R \to 2^{\mathbb{R}}$ has nonempty closed values. As an important consequence of the main result we can present the following:

Theorem 4.1. *Assume that the multifunction F satisfies the following assumptions:*

1° *$F(t,x)$ are nonempty, closed and convex for all $(t,x) \in [0,1] \times \mathbb{R}$,*

2° *$F(t,\cdot)$ is lower semicontinuous from $\mathbb{R}$ into $\mathbb{R}$,*

3° *$F(\cdot,\cdot)$ is measurable,*

4° *$|F(t,x)| := \sup\{|y| : y \in F(t,x)\} \leq a(t) + b(t)|x|$ for each $t \in (0,1)$ and $x \in \mathbb{R}$, where $a(\cdot) \in L^1(0,1)$ and $b(\cdot)$ is measurable and bounded,*

5° *$\varphi : (0,1) \longrightarrow (0,1)$ is nondecreasing and there is a constant $M > 0$ such that $\varphi' \geq M$ for a.e. $t \in (0,1)$,*

6° $\sum_{i=1}^{n-1} \frac{|a_i|}{\Gamma(1+\alpha_n - \alpha_i)} + \frac{\sup |b(t)|}{M\Gamma(\alpha_n + 1)} < 1.$

Then the equation (1) has at least one continuous solution.

Proof. By the Kuratowski Selection Theorem (cf. [5], for instance), for each continuous function $x(\cdot)$ we can find a measurable selection for $F(\cdot, x(\varphi(\cdot))$. By assumption 4° this selection is integrable. Consider a new multifunction $G(x) := \{f \in L_1(I) : f(t) \in F(t, x(\varphi(t))$ for a.e. $t \in [0,1]\}$. Its values are nonempty and since F is lower semicontinuous, G is also lower semicontinuous. Thus, we can repeat our argumentation from Theorem 3.1 to obtain *a priori* boundedness of solutions by r. Then $G : B_r \to 2^{L_1(I)}$ has decomposable values. By the result of Bressan and Colombo ([4]), we get a continuous selection of G, namely $g(x)(t) \in F(t, x(\varphi(t)))$ for a.e. $t \in [0,1]$. Define $f(t,x) := g(x)(t)$. Now, we are able to repeat the rest of the proof of Theorem 3.1.

Finally, let us remark that from the proof of Theorem 3.1 it follows that we can replace assumptions 1°–4° by an arbitrary set of assumptions which guarantee the existence of Carathéodory selections (see [2]). In particular, the continuity hypothesis can be weakened replacing lower semi-continuity with quasi lower-semicontinuity (cf. [14, 19] or [20]). ■

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