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ON THE EXISTENCE OF A FUZZY INTEGRAL EQUATION OF URYSOHN-VOLTERRA TYPE

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Abstract

We present an existence theorem for integral equations of Urysohn-Volterra type involving fuzzy set valued mappings. A fixed point theorem due to Schauder is the main tool in our analysis.

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1. Introduction

Dubois and Prade [4, 5] introduced the concept of integration of fuzzy functions. Alternative approaches were later suggested by Goetschel and Voxman [8], Kaleva [9], Nanda [11] and others. While Goetschel and Voxman preferred a Riemann integral type approach, Kaleva chose to define the integral of a fuzzy function, using the Lebesgue-type concept of integration. For more information about integration of fuzzy functions and fuzzy integral equations, for instance, see [1–5, 7–14] and references therein.

By means of the fuzzy integral due to Kaleva [9], we study the fuzzy integral equation of Urysohn-Volterra, for the fuzzy set-valued mappings of a real variable whose values are normal, convex, upper semicontinuous and compactly supported fuzzy sets in $\mathbb{R}^n$. This equation takes the form

$$x(t) = f(t) + \int_0^t u(t, s, g(s, x(s)))\, ds, \quad t \in [0, \mathrm{T}]. \tag{1.1}$$

In the special case when $g(t,x) = x$, we obtain the nonlinear integral equation involving fuzzy set valued mappings, namely

$$x(t) = f(t) + \int_0^t u(t,s,x(s))\, ds, \quad t \in [0, \mathrm{T}]. \tag{1.2}$$

Existence theorems for equation (1.2) have been studied by several authors, see for examples [12, 13] and references therein. In [14], the authors established the unique solvability of equation (1.2) by using the Contraction Mapping Theorem.

In this paper, we prove the existence theorem of a solution to the fuzzy integral equation (1.1). The fixed point theorem due to Schauder is the main tool in carrying out our proof.

2. Auxiliary facts and results

This section is devoted to collect some definitions and results which will be needed further on.

Definition 1. Let X be a nonempty set. A *fuzzy set* A in X is characterized by its membership function $\mathrm{A} : \mathrm{X} \to [0,1]$ and $\mathrm{A}(x)$, called the membership function of fuzzy set A, is interpreted as the degree of membership of element x in fuzzy set A for each $x \in \mathrm{X}$.

The value zero is used to represent complete non-membership, the value one is used to represent complete membership and values between them are used to represent intermediate degrees of membership.

Example 1. The membership function of a fuzzy set of real numbers, close to zero, can be defined as follows

$$\mathrm{A}(x) = \frac{1}{1+x^3}\,.$$

Example 2. Let the membership function of a fuzzy set of real numbers be close to one defined as follows

$$\mathrm{B}(x) = \exp(-\gamma(x-1)^2),$$

where γ is a positive real number.

Let $P_k(\mathbb{R}^n)$ denote the collection of all nonempty compact convex subsets of $\mathbb{R}^n$ and define the addition and scalar multiplication in $P_k(\mathbb{R}^n)$ as usual. Define the Hausdorff metric

$$d(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\right\}$$

where $d(b,A) = \inf\{d(b,a) \ : \ a \in A\}$, A, B are nonempty bounded subsets of $\mathbb{R}^n$. It is clear that $(P_k(\mathbb{R}^n), d)$ is a metric space.

A fuzzy set $u \in \mathbb{R}^n$ is a function $u : \mathbb{R}^n \to [0,1]$ for which

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex,

(iii) u is upper semicontinuous, and

(iv) the closure of $\{x \in \mathbb{R}^n : u(x) > 0\}$, denoted by $[u]^0$, is compact.

For $0 < \alpha \leq 1$, the $\alpha-$level set $[u]^\alpha$ is define by $[u]^\alpha = \{x \in \mathbb{R}^n \ : \ u(x) \geq \alpha\}$. Then from $(i) - (iv)$, it follows that $[u]^\gamma \in P_k(\mathbb{R}^n)$ for all $0 \leq \alpha \leq 1$.

By Zadeh's extension principle, we can define addition and scalar multiplication in E^n as follows:

$$\begin{aligned} [u+v]^\gamma &= [u]^\gamma + [v]^\gamma, \\ [\lambda\, u]^\gamma &= \lambda\, [u]^\gamma, \end{aligned}$$

where $u,\ v \in E^n$, $\lambda \in \mathbb{R}$ and $0 \leq \gamma \leq 1$. Define $\hat{0} : \mathbb{R}^n \to [0,1]$ by

$$\hat{0}(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise}. \end{cases}$$

We call $\hat{0}$ the null element of E^n.

Let $D : E^n \times E^n \to [0,\infty)$ be define by

$$D(u,v) = \sup_{0\leq\gamma\leq 1} d\left([u]^\gamma,\ [v]^\gamma\right)$$

where d is the Hausdorff metric defined in $P_k(\mathbb{R}^n)$. Then (E^n, D) is a complete metric space [13]. Also, we know that [13]

(1) $D(u+w, v+w) = D(u,v)$ for $u,\ v,\ w \in E^n$

(2) $D(\lambda u, \lambda v) = |\lambda|\ D(u,v)$ for all $u,\ v \in E^n$ and $\lambda \in \mathbb{R}$.

Now, we recall some definitions and theorems concerning integrability properties for the set-valued mapping of a real variable whose values are in (E^n, D) [9, 13].

Definition 2. A mapping $F : J \to E^n$ is strongly measurable if for $\gamma \in [0, 1]$ the set-valued mapping $F_\gamma : J \to P_k(\mathbb{R}^n)$ defined by $F_\gamma(t) = [f(t)]^\gamma$ is Lebesgue measurable, when $P_k(\mathbb{R}^n)$ is endowed with the topology generated by the Hausdorff metric d.

Definition 3. A mapping $F : J \to E^n$ is called strongly bounded if there exists an integrable function h such that $\|x\| \le h(t)$ for all $x \in F_0(t)$.

Definition 4. Let $F : J \to E^n$. The integral of F over J, defined by $\int_J F(t)\,dt$, is defined levelwise by

$$\left(\int_J F(t)\,dt\right)^\gamma = \int_J F_\gamma(t)\,dt$$
$$= \{\, f(t)\,dt \mid f : J \to \mathbb{R}^n \text{ is a measurable selection for } F_\gamma\}\,.$$

A strongly measurable and integrably bounded mapping $F : J \to E^n$ is said to be integrable over J if $\int_J F(t)\,dt \in E^n$.

Theorem 1. *If $F : J \to E^n$ is strongly measurable and integrably bounded, then F is integrable.*

Theorem 2. *If $F : J \to E^n$ is continuous, then it is integrable.*

Theorem 3. *If $F : J \to E^n$ is integrable and $b \in J$. Then*

$$\int_{t_0}^{t_0+a} F(t)\,dt = \int_{t_0}^{b} F(t)\,dt + \int_{b}^{t_0+a} F(t)\,dt.$$

Theorem 4. *If $F, G : J \to E^n$ is integrable and $\lambda \in \mathbb{R}$. Then*

(1) $\int_J (F(t) + G(t))\,dt = \int_J F(t)\,dt + \int_J G(t)\,dt$,

(2) $\int_J \lambda\, F(t)\,dt = \lambda \int_J F(t)\,dt$,

(3) $D(F, G)$ *is integrable,*

(4) $D\left(\int_J F(t)\,dt, \int_J G(t)\,dt\right) \le \int_J D(F(t), G(t))\,dt$.

For our purposes, we will need the following fixed point theorem [6]

Theorem 5 (Schauder's Fixed Point Theorem)**.** *Let C be a convex subset of a Banach space* X *and $\mathcal{F}$ be a completely continuous mapping of C into C. Then $\mathcal{F}$ has at least one fixed point in C.*

3. Main theorem

Let b, M and T be positive numbers. Take U to the set of all $x \in E^n$ for which there exists an $t \in [0,T]$ such that $D(x(t), f(t)) \leq b$. In this section, we will study equation (1.1) assuming that the following assumptions are satisfied.

(a_1) $f : [0,T] \to E^n$ is continuous and bounded.

(a_2) $u : [0,T] \times [0,T] \times \mathrm{U} \to E^n$ is continuous and

$$D\left(u(t,s,x),\ \hat{0}\right) \leq M$$

for all $(t,s,x) \in [0,T] \times [0,T] \times \mathrm{U}$.

(a_3) $g : [0,T] \times E^n \to E^n$ is continuous and bounded.

Now, we are in a position to state and prove our main result.

Theorem 6. *Let the assumptions* (a_1)–(a_3) *be satisfied. Then equation* (1.1) *has at least one solution x on* $[0,\tau]$*, where* $\tau = \min\left\{\mathrm{T},\ Mb^{-1}\right\}$.

Proof. Define $\Psi_u :\ [0,\infty) \to \mathbb{R}$ by

$$\Psi_u(\delta) = \sup\{D(u(t_2,y_2,w_2), u(t_1,y_1,w_1)) \mid (t_i,s_i,y_i) \in \Omega;\ i = 1,2,$$
$$\max\{d(t_2,t_1), d(s_2,s_1), D(y_2,y_1)\} \leq\ \delta\}\,.$$

By the uniform continuity of u on the compact set $[0,T] \times [0,T] \times \mathrm{U}$, Ψ_u is continuous at $\delta = 0$ and $\Psi_u(0) = 0$.

Now, let

$$\Omega := \{y \mid y \in C([0,\tau]; E^n),\ y(0) = f(0),\ \text{ and }\ \mathcal{D}(y,f) \leq b\}$$

be a subset of $C([0,\tau];E^n)$ and

$$(\mathcal{F}y)(t) = f(t) + \int_0^t u(t,s,g(s,y(s)))\,ds,\ t\in[0,\tau], \tag{3.1}$$

where $\mathcal{D}(x,y) = \sup_{0\le t\le\tau} D(x(t),y(t))$.

Solving equation (1.1) is equivalent to finding a fixed point of the operator $\mathcal{F}$.

It is easy to see, by the aid of our assumptions, that $\mathcal{F}$ is continuous. We claim the operator $\mathcal{F}:\Omega\to\Omega$ is completely continuous. Once the claim is established, then Theorem 5 with $\mathrm{X}=C([0,\tau];\mathrm{E}^n)$ and $C=\Omega$ guarantees the existence of a fixed point of $\mathcal{F}$ in Ω, and hence equation (1.1) has a solution in $C([0,\tau];\mathrm{E}^n)$.

We begin by showing that condition $\mathcal{F}$ maps Ω into itself. To see this, take $y\in\Omega$ and $0\le t\le\tau$. Thus

$$\begin{aligned} D(\mathcal{F}y(t),f(t)) &= D\left(f(t)+\int_0^t u(t,s,g(s,y(s)))\,ds, f(t)\right)\\ &\le D\left(\int_0^t u(t,s,g(s,y(s)))\,ds, \hat{0}\right)\\ &\le \int_0^t D\left(u(t,s,g(s,y(s))),\hat{0}\right)\,ds \qquad (3.2)\\ &\le M\,t, \end{aligned}$$

thanks to assumption (a_2). In particular, $(\mathcal{F}y)(0)=f(0)$ and the estimate

$$D(x(t),f(t)) \le M\,t \tag{3.3}$$

holds for any solution x of equation (1.1) in $[0,\tau]$. Moreover,

$$\mathcal{D}(\mathcal{F}y,f)\le M\,t\le b. \tag{3.4}$$

Hence $\mathcal{F}:\Omega\to\Omega$ is continuous. Also $\mathcal{F}:\Omega\to\Omega$ is completely continuous. To see this, due to the theorem of Arzèla-Ascoli, the uniform boundedness and the equicontinuity of $\{\mathcal{F}y_m\}$ is to be checked, where $\{y_m\}$ is a bounded sequence in Ω. Let $0\le t_1\le t_2\le\tau$. Then

$$
\begin{aligned}
D(\mathcal{F}y_m)(t_2) - (\mathcal{F}y_m)(t_1)) \leq\ & D(f(t_2), f(t_1)) \\
& + D\left(\int_0^{t_2} u(t_2, s, g(s, y_m(s)))\, ds, \int_0^{t_1} u(t_1, s, g(s, y_m(s)))\, ds\right) \\
\leq\ & D(f(t_2), f(t_1)) \\
& + D\left(\int_0^{t_2} u(t_2, s, g(s, y_m(s)))\, ds, \int_0^{t_2} u(t_1, s, g(s, y_m(s)))\, ds\right) \\
& + D\left(\int_0^{t_2} u(t_1, s, g(s, y_m(s)))\, ds, \int_0^{t_1} u(t_1, s, g(s, y_m(s)))\, ds\right) \qquad (3.5) \\
\leq\ & D(f(t_2), f(t_1)) \\
& + \int_0^{t_2} D\left(u(t_2, s, g(s, y_m(s))), u(t_1, s, g(s, y_m(s)))\right)\, ds \\
& + \int_{t_1}^{t_2} D\left(u(t_1, s, g(s, y_m(s))), \hat{0}\right)\, ds \\
\leq\ & D(f(t_2), f(t_1)) + \Psi_u(d(t_2, t_1))\, t_2 + M\,(t_2 - t_1).
\end{aligned}
$$

Inequality (3.5), by symmetry, is valid for all t_1, $t_2 \in [0, \tau]$ regardless whether or not $t_2 \geq t_1$. Therefore, the equicontinuity follows. Now, we have

$$
\begin{aligned}
D(\mathcal{F}y_m(t), \hat{0}) \ &\leq\ D(\mathcal{F}y_m(t), f(t)) + D(f(t), \hat{0}) \\
&\leq\ b + D(f(t), \hat{0}).
\end{aligned}
$$

This means that $\{\mathcal{F}y_m\}$ is uniformly bounded. Lemma 5 guarantees that (1.1) has a solution $y \in \Omega$. This completes the proof.

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