

ABSTRACT INCLUSIONS IN BANACH SPACES WITH BOUNDARY CONDITIONS OF PERIODIC TYPE

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Abstract

We study in the space of continuous functions defined on $[0, T]$ with values in a real Banach space E the periodic boundary value problem for abstract inclusions of the form

$$\begin{cases} x \in S(x(0), \text{sel}_F(x)) \\ x(T) = x(0), \end{cases}$$

where, $F : [0, T] \times \mathcal{K} \rightarrow 2^E \setminus \emptyset$ is a multivalued map with convex compact values, $\mathcal{K} \subset E$, sel_F is the superposition operator generated by F , and $S : \mathcal{K} \times L^1([0, T]; E) \rightarrow C([0, T]; \mathcal{K})$ an abstract operator. As an application, some results are given to the periodic boundary value problem for nonlinear differential inclusions governed by m -accretive operators generating not necessarily a compact semigroups.

Keywords: measure of noncompactness, condensing operator, nonlinear abstract inclusion, accretive operator, integral solution, nonlinear semigroup.

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1. INTRODUCTION

Let $C([0, T]; E)$ be the space of continuous functions defined on $[0, T]$ with values in a real Banach space $(E, \|\cdot\|)$ endowed with the uniform convergence norm, and let \mathcal{K} be a nonempty closed convex subset of E . The propose of this paper is to study in $C([0, T]; \mathcal{K})$ nonlinear abstract inclusions satisfying boundary conditions of periodic type described in the form

$$(1) \quad \begin{cases} x \in S(x(0), \text{sel}_F(x)) \\ x(T) = x(0), \end{cases}$$

where, $F : [0, T] \times \mathcal{K} \rightarrow 2^E \setminus \emptyset$ is a multivalued map with compact convex values satisfying upper Carathéodory conditions, $S : \mathcal{K} \times L^1([0, T]; E) \rightarrow C([0, T]; \mathcal{K})$ an abstract operator, sel_F is the superposition operator generated by F , and

$$S(x(0), \text{sel}_F(x)) = \{S(x(0), f) ; f \in \text{sel}_F(x)\}.$$

A fundamental example of such abstract problems is given by the following problem

$$(2) \quad \begin{cases} x'(t) \in -Ax(t) + F(t, x(t)), & t \in [0, T], \\ x(0) = x(T), \end{cases}$$

where A is a single valued or a multivalued operator not necessarily linear. In the semilinear case, when $A : D(A) \subset E \rightarrow E$ is a (single valued) linear operator such that $-A$ generates a C_0 -semigroup $\{\mathcal{T}(t)\}_{t \geq 0}$, the operator S is the mild solutions operator. More precisely, for $x_0 \in \mathcal{K} = E$ and $f \in L^1([0, T]; E)$, the value $S(x_0, f)$ stands for the (unique) mild solution of the Cauchy problem

$$(3) \quad \begin{cases} x'(t) \in -Ax(t) + f(t), & t \in [0, T], \\ x(0) = x_0. \end{cases}$$

Moreover, by means of the variation of constants formula, S can be expressed explicitly by

$$S : E \times L^1([0, T]; E) \rightarrow C([0, T]; E)$$

$$S(x_0, f)(t) = \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t-s)f(s) ds.$$

In the nonlinear case with A m -accretive, S is the integral solutions operator, i.e., for $x_0 \in \mathcal{K} = \overline{D(A)}$ and $f \in L^1([0, T]; E)$, the value $S(x_0, f)$ stands for the (unique) integral solution of the Cauchy problem (3). In both cited cases the operator S has been studied by many authors, see for example, [22, 26, 28] for the semilinear case, and [11, 9, 19, 20, 26] for the nonlinear case.

In [4] and [13], the authors developed a fixed point approach which can be used in the study of the Cauchy problem for various classes of differential inclusions by considering abstract inclusions in $C([0, T]; E)$ of the form

$$(4) \quad x \in \Upsilon \circ \text{sel}_F(x)$$

where $\Upsilon : L^1([0, T]; E) \rightarrow C([0, T]; E)$ is an operator not necessarily linear. Our paper can be considered as a nontrivial extension of their approach to the periodic problem for linear and nonlinear differential inclusions. More precisely, fix $x_0 \in E$ and consider the abstract inclusion

$$x \in S(x_0, \text{sel}_F(x)).$$

By taking $\Upsilon(\cdot) = S(x_0, \cdot)$, we are in the situation of [4] and [13]. But it is clear that to study the periodic problem, one has to allow x_0 varying in $\mathcal{K} \subset E$.

In the present work, we construct a multioperator for whose fixed points are solutions of the inclusion (1) and we give sufficient conditions under which this multioperator is upper semi continuous with closed contractible values and condensing with respect to a monotone, nonsingular, regular measure of noncompactness.

As an application of our result, we study the nonlinear periodic problem (2), where A is supposed to be m -accretive such that $-A$ generates an equicontinuous semigroup.

One motivation to consider such problems is that in the case when E is a Hilbert space and $A = \partial\phi$ is the subdifferential of a proper convex lower semi continuous function $\phi : \mathcal{D}_\phi \subset E \rightarrow \mathbb{R}$, the semigroup generated by $-A$ is always equicontinuous and it is compact iff ϕ has compact sublevel sets, i.e., $\{x \in E : \|x\|^2 + \phi(x) \leq r\}$ is compact for all $r > 0$; (see for example [28] p. 42).

Finally we mention that there are many works in the study of the periodic problem for differential equations and inclusions governed by m -accretive operators generating compact semigroups, see introductions in [29] and [1] and the references therein.

The paper is organized as follows. In Section 2 we give some basic notations as well as some preliminary lemmas which will play essential roles in this paper. In Section 3 we formulate our problem, we give the construction of a multioperator associated to the problem (1) and we define a measure of noncompactness for which this operator is condensing. In Section 4 we give the proof of the main result. Finally, an abstract application of the main result is presented in Section 5.

2. PRELIMINARIES

Let X, Y be two topological vector spaces. We denote by $\mathcal{P}(Y)$ the family of all nonempty subsets of Y and by $K(X)$ (resp. $Kv(X)$) we denote the collection of

all nonempty compact (resp. nonempty compact convex) subsets of X .

- A multivalued map $F : X \rightarrow \mathcal{P}(Y)$ is said to be:
 - (i) upper semicontinuous (u.s.c) if $F^{-1}(O) = \{x \in X : F(x) \subset O\}$ is an open subset of X for every open $O \subset Y$;
 - (ii) closed if its graph $\Gamma_F = \{(x, y) \in X \times Y : y \in F(x)\}$ is a closed subset of $X \times Y$;
 - (iii) compact if $\overline{F(X)}$ is compact in Y ;
 - (vi) quasicompact if its restriction to every compact subset $A \subset X$ is compact.

Lemma 2.1 [22]. *Let X and Y be metric spaces and $F : X \rightarrow K(Y)$ a closed quasicompact multimap. Then F is upper semicontinuous.*

Let E be a real Banach space and (Y, \leq) a partially ordered set.

- A function $\Psi : \mathcal{P}(E) \rightarrow Y$ is called a measure of noncompactness in E if

$$\Psi(\Omega) = \Psi(\bar{co} \Omega)$$

for every $\Omega \subset \mathcal{P}(E)$, where $\bar{co} \Omega$ denotes the closed convex hull of Ω .

- The measure Ψ is called:
 - (i) nonsingular if for every $a \in E$, $\Omega \in \mathcal{P}(E)$, $\Psi(\{a\} \cup \Omega) = \Psi(\Omega)$;
 - (ii) monotone, if $\Omega_0, \Omega_1 \in \mathcal{P}(E)$ and $\Omega_0 \subseteq \Omega_1$ imply $\Psi(\Omega_0) \leq \Psi(\Omega_1)$;
 - (iii) If Y is a cone in a Banach space we will say that Ψ is regular if $\Psi(\Omega) = 0$ is equivalent to the relative compactness of the set Ω .

One of the most important examples of a measure of noncompactness possessing all these properties is the Hausdorff measure of noncompactness defined by:

$$\chi(\Omega) = \inf\{\varepsilon > 0; \Omega \text{ has a finite } \varepsilon\text{-net in } E\}.$$

- Let $Z \subset E$ be a closed subset. A multimap $G : Z \rightarrow K(E)$ is called Ψ -condensing, where $\Psi : \mathcal{P}(E) \rightarrow (Y, \leq)$ is a measure of noncompactness in E , if for every bounded set $\Omega \subset Z$, the relation $\Psi(G(\Omega)) \geq \Psi(\Omega)$ implies the relative compactness of Ω .

- A multifunction $F : [0, T] \rightarrow K(E)$ is said to be strongly measurable if there exists a sequence $\{F_n\}_{n=1}^{\infty}$ of step multifunctions such that $Haus(F(t), F_n(t)) \rightarrow 0$ as $n \rightarrow \infty$ for μ -a.e. $t \in [0, T]$ where μ denotes a Lebesgue measure on $[0, T]$ and $Haus$ is the Hausdorff metric on $K(E)$.

Every strongly measurable multivalued map F admits a strongly measurable selection f i.e., $f : [0, T] \rightarrow E$ is measurable and such that $f(t) \in F(t)$ for a.e. $t \in [0, T]$.

- A subset $\Lambda \subset E$ is said to be contractible if for some $x_0 \in E$, there is a continuous map H from $[0, 1] \times \Lambda$ to Λ satisfying $H(0, x) = x_0$ and $H(1, x) = x$, for each $x \in \Lambda$. For more details, see for example [22, 18].

By the symbol $L^1([0, T]; E)$ we denote the space of all Bochner summable functions equipped with the usual norm.

Definition 2.2. A sequence $\{f_n\}_{n=1}^\infty \subset L^1([0, T]; E)$ is semicompact if:

- (i) it is integrably bounded: $\|f_n(t)\| \leq q(t)$ for a.e. $t \in [0, T]$ and for every $n \geq 1$ where $q(\cdot) \in L^1([0, T], \mathbb{R}^+)$;
- (ii) the set $\{f_n(t)\}_{n=1}^\infty$ is relatively compact for almost every $t \in [0, T]$.

Lemma 2.3 [15]. *Any semicompact sequence in $L^1([0, T]; E)$ is weakly compact in $L^1([0, T]; E)$.*

Lemma 2.4 [17]. *Let X be a Banach space and $D \subset X$ be a nonempty compact convex set. Suppose that $\mathcal{G} : D \rightarrow \mathcal{P}(D)$ is u.s.c. with closed contractible values. Then \mathcal{G} has a fixed point.*

We give now some basic concepts and results concerning m -accretive operators.

- A multi-valued map A with domain $D(A)$ and range $R(A)$ in E is said to be:

- (i) accretive if $\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\|$, for all $\lambda > 0$ and $y_i \in Ax_i$, $i = 1, 2$;
- (ii) m -accretive if it is accretive and $R(I + A) = E$, (Here I stands for the identity on E).

- If A is m -accretive, the resolvents $J_\lambda = (I + \lambda A)^{-1} : E \rightarrow D(A)$ are nonexpansive mappings, i.e., $\|J_\lambda(x) - J_\lambda(y)\| \leq \|x - y\|$ on $E \times E$, for all $\lambda > 0$.

- If A is m -accretive, it generates a semigroup $\{T(t)\}_{t \geq 0}$ of nonexpansive mappings $T(t) : \overline{D(A)} \rightarrow \overline{D(A)}$, given by the exponential formula, i.e.,

$$T(t)x = \lim_{n \rightarrow \infty} J_{t/n}^n x \quad \text{for all } t \geq 0 \text{ and } x \in \overline{D(A)}$$

and $T(t)x$ is the integral solution of the initial value problem:

$$\begin{cases} y'(t) \in -Ay(t), & t \in [0, T], \\ y(0) = x. \end{cases}$$

The semigroup $\{T(t)\}_{t \geq 0}$ is said to be compact if $T(t)B$ is compact for all $t > 0$ and bounded $B \subset \overline{D(A)}$, while $\{T(t)\}_{t \geq 0}$ is called equicontinuous if the family of functions $\{T(\cdot)x : x \in B\}$ is equicontinuous at every $t > 0$, for all bounded $B \subset \overline{D(A)}$.

The semigroup $\{T(t)\}_{t \geq 0}$ is compact iff $\{T(t)\}_{t \geq 0}$ is equicontinuous and J_λ is a compact map for some (or, equivalently, for all) $\lambda > 0$.

For more details on the previous definitions and facts, see for example [6].

3. FORMULATION OF THE PROBLEM, STATEMENT OF THE RESULT

Notations

Throughout, $0 < T < +\infty$ is a fixed time, E an arbitrary real Banach space with the norm $\|\cdot\|$, \mathcal{K} a nonempty closed convex subset of E , $C([0, T]; E)$ denotes the space of continuous functions defined on $[0, T]$ with values in E and endowed with the uniform convergence norm, $L^1([0, T]; E)$ the space of all Bochner summable functions, χ the Hausdorff measure of noncompactness in E and $C([0, T]; \mathcal{K})$ the set of all continuous functions defined on $[0, T]$ with values in \mathcal{K} .

It is clear that $C([0, T]; \mathcal{K})$ is a closed convex subset of $C([0, T]; E)$.

Hypotheses

We shall consider the following hypotheses:

The multimap $F : [0, T] \times \mathcal{K} \rightarrow Kv(E)$ satisfies the following hypotheses:

(F_1) the multimap $F : (\cdot, u) \rightarrow Kv(E)$ has a strongly measurable selector for every $u \in \mathcal{K}$;

(F_2) the multimap $F : (t, \cdot) \rightarrow Kv(E)$ is u.s.c. for a.e. $t \in [0, T]$;

(F_3) for any nonempty bounded set $\Omega \subset \mathcal{K}$ there exists a function $U_\Omega(\cdot) \in L^1([0, T]; \mathbb{R}^+)$ such that, for all $x \in \Omega$ and a.e. $t \in [0, T]$

$$\|F(t, x)\| \leq U_\Omega(t);$$

(F_4) there exists a function $\kappa(\cdot) \in L^1([0, T]; \mathbb{R}^+)$ such that for every bounded $\Omega \subset \mathcal{K}$

$$\chi(F(t, \Omega)) \leq \kappa(t) \chi(\Omega), \text{ a.e. } t \in [0, T].$$

The abstract operator $S : \mathcal{K} \times L^1([0, T]; E) \rightarrow C([0, T]; \mathcal{K})$ satisfies the following conditions:

(S₀) for all $x_0 \in \mathcal{K}$ and $f \in L^1([0, T]; E)$:

$$S(x_0, f)(0) = x_0;$$

(S₁) there exists $M > 0$ and $p > 0$ such that

$$\|S(x_0, f)(t) - S(y_0, g)(t)\| \leq M \int_0^t \|f(s) - g(s)\| ds + e^{-pt} \|x_0 - y_0\|$$

for all $f, g \in L^1([0, T]; E)$, $0 \leq t \leq T$ and $x_0, y_0 \in \mathcal{K}$;

(S₂) for any compact $K \subset E$ and sequence $\{f_n\}_{n=1}^\infty \subset L^1([0, T]; E)$ such that $\{f_n(t)\}_{n=1}^\infty \subset K$ for a.e. $t \in [0, T]$ the weak convergence $f_0 \xrightarrow{w} f_n$ implies $S(x_0, f_n) \rightarrow S(x_0, f_0)$ in $C([0, T]; \mathcal{K})$ for every $x_0 \in \mathcal{K}$;

(S₃) for all $g_0, g_1, g_2 \in L^1([0, T], E)$ and $x_0 \in \mathcal{K}$ if $S(x_0, g_1) = S(x_0, g_2)$

$$S(x_0, \mathbf{1}_{[0, \theta]}g_1 + \mathbf{1}_{[\theta, T]}g_0) = S(x_0, \mathbf{1}_{[0, \theta]}g_2 + \mathbf{1}_{[\theta, T]}g_0),$$

for all $\theta \in [0, T]$, where $\mathbf{1}_{[a, b]}$ denotes the characteristic function of the interval $[a, b]$.

Remark 3.1. Recall that, under conditions (F₁)–(F₃), for every continuous function $x : [0, T] \rightarrow \mathcal{K}$ there exists a summable selection $f : [0, T] \rightarrow E$ of $F(\cdot, x(\cdot))$ (see Theorem 1.3.5 in [22]). Consequently, the superposition operator

$$\text{sel}_F : C([0, T]; \mathcal{K}) \rightarrow L^1([0, T]; E)$$

$$\text{sel}_F(x) = \{f \in L^1([0, T]; E) : f(t) \in F(t, x(t)), \text{ a.e. } t \in [0, T]\}$$

is correctly defined. Moreover, as $C([0, T]; \mathcal{K})$ is closed, according to Lemma 5.1.1 in [22] the superposition operator sel_F is weakly closed. More precisely:

Lemma 3.2. *If the sequences $\{x^n\}_{n=1}^\infty \subset C([0, T]; \mathcal{K})$, $\{f_n\}_{n=1}^\infty \subset L^1([0, T]; E)$, $f_n(t) \in F(t, x^n(t))$, a.e. $t \in [0, T]$, $n \geq 1$ are such that $x^n \rightarrow x^0$, $f_n \xrightarrow{w} f_0$, then $f_0(t) \in F(t, x^0(t))$ a.e. $t \in [0, T]$.*

Construction of an operator associated with the problem (1)

In $C([0, T]; \mathcal{K})$ define the multivalued operator \mathcal{F} in the following way

$$(5) \quad \begin{cases} \mathcal{F} : C([0, T]; \mathcal{K}) \rightarrow \mathcal{P}(C([0, T]; \mathcal{K})), \\ \mathcal{F}(x) = \{S(S(x(0), f)(T), f) : f \in \text{sel}_F(x)\}. \end{cases}$$

Remark that the fixed points of the operator \mathcal{F} coincide with the solutions set of the problem (1). Indeed, let $x \in \mathcal{F}(x)$. Then, there exists $f \in \text{sel}_F(x)$, such that

$$(6) \quad x = S(S(x(0), f)(T), f).$$

By condition (S_0) , we have

$$x(0) = S(S(x(0), f)(T), f)(0) = S(x(0), f)(T).$$

Hence,

$$x = S(x(0), f) \text{ with } x(T) = S(x(0), f)(T) = x(0).$$

Now, let x be a solution of the problem (1). Then, there exists $f \in \text{sel}_F(x)$ such that,

$$x = S(x(0), f) \text{ and } x(0) = x(T).$$

It results that $x = S(x(0), f) = S(x(T), f)$. But $x(T) = S(x(0), f)(T)$. Then, $x = S(S(x(0), f)(T), f)$, which means that x is a fixed point of \mathcal{F} .

Such operator was considered in [29] in the study of the periodic problem for fully nonlinear differential equation.

Measures of noncompactness

Let $\chi_{\mathcal{K}}$ be a function defined on bounded subsets of \mathcal{K} in the following way

$$\chi_{\mathcal{K}}(\Omega) = \inf\{\varepsilon > 0; \Omega \text{ has a finite } \varepsilon\text{-net in } \mathcal{K}\};$$

Since \mathcal{K} is a nonempty closed convex subset of E , the function $\chi_{\mathcal{K}}$ defines a measure of noncompactness in \mathcal{K} . Indeed, the invariance of $\chi_{\mathcal{K}}$ under passage to the closure is obvious and the invariance under passage to the convex hull is a consequence of the fact that if $S \subset \mathcal{K}$ is a finite ε -net of the set Ω , then $\text{co}S \subset \mathcal{K}$ is a totally bounded ε -net of the set $\text{co}\Omega$. It is readily seen that $\chi_{\mathcal{K}}$ is a monotone, nonsingular, regular measure of noncompactness in \mathcal{K} and $\chi(\Omega) \leq \chi_{\mathcal{K}}(\Omega) \leq 2\chi(\Omega)$ for all $\Omega \subset \mathcal{K}$.

Now let Ψ be a function defined on bounded subsets of $C([0, T]; \mathcal{K})$ in the following way

$$(7) \quad \Psi(\Omega) = \max_{\mathcal{D} \in \Delta(\Omega)} \left(\chi_{\mathcal{K}}(\mathcal{D}(0)), \vartheta(\mathcal{D}), \text{mod}_c(\mathcal{D}) \right),$$

where

$$(8) \quad \vartheta(\mathcal{D}) = \sup_{t \in [0, T]} \chi_{\mathcal{K}}(\mathcal{D}(t)),$$

$$\text{mod}_c(\mathcal{D}) = \limsup_{\delta \rightarrow 0} \max_{x \in \mathcal{D}} \max_{|t_1 - t_2| \leq \delta} \|x(t_1) - x(t_2)\|,$$

and $\Delta(\Omega)$ is the collection of all denumerable subsets of Ω . The range of the function Ψ is a cone \mathbb{R}_+^3 , max is taken in the sense of the ordering induced by this cone.

Since $C([0, T]; \mathcal{K})$ is a closed convex subset of $C([0, T]; E)$, from Example 2.1.4 in [22] and the definition of $\chi_{\mathcal{K}}$, one can easily see that Ψ is well defined and is a monotone, nonsingular, regular measure of noncompactness in $C([0, T]; \mathcal{K})$.

Main result

We can now state the main result of this paper.

Theorem 3.3. *Suppose that conditions (F_1) – (F_4) are satisfied. Then the following are valid:*

- (i) *if the operator S satisfies conditions (S_1) and (S_2) , then the multioperator \mathcal{F} is u.s.c with compact values;*
- (ii) *if the operator S satisfies conditions (S_0) – (S_2) and the estimation*

$$(9) \quad 4M \|\kappa(\cdot)\|_{L^1} + e^{-pT} < 1,$$

holds, then \mathcal{F} is Ψ -condensing;

- (iii) *if the operator S satisfies conditions (S_1) – (S_3) , then the multioperator \mathcal{F} has contractible values.*

4. PROOF OF THE MAIN RESULT

Auxiliary results

We need some auxiliary results.

Lemma 4.1. *Let Υ be an abstract operator*

$$\Upsilon : L^1([0, T]; E) \rightarrow C([0, T]; \mathcal{K})$$

satisfying the following conditions:

(Υ_1) *there exists $D > 0$ such that*

$$\|\Upsilon f(t) - \Upsilon g(t)\| \leq D \int_0^t \|f(s) - g(s)\| ds, \quad 0 \leq t \leq T$$

for every $f, g \in L^1([0, T]; E)$;

(Υ_2) *for any compact $K \subset E$ and sequence $\{f_n\}_{n=1}^\infty \subset L^1([0, T]; E)$ such that $\{f_n(t)\}_{n=1}^\infty \subset K$ for a.e. $t \in [0, T]$, the weak convergence $f_0 \xrightarrow{w} f_n$ implies $\Upsilon f_n \rightarrow \Upsilon f_0$ in $C([0, T]; \mathcal{K})$.*

Then:

- (i) If the sequence of functions $\{f_n\}_{n=1}^\infty \subset L^1([0, T]; E)$ is such that $\|f_n(t)\| \leq \delta(t)$ for all $n = 1, 2, \dots$ a.e. $t \in [0, T]$ and $\chi(\{f_n\}_{n=1}^\infty) \leq \zeta(t)$ a.e. $t \in [0, T]$, where $\delta, \zeta \in L_+^1([0, T])$, then

$$(10) \quad \chi_{\mathcal{K}}(\Upsilon\{f_n(t)\}_{n=1}^\infty) \leq 2D \int_0^t \zeta(s) ds;$$

- (ii) for every semicompact sequence $\{f_n\}_{n=1}^\infty \subset L^1([0, T]; E)$ the sequence $\{\Upsilon f_n\}_{n=1}^\infty$ is relatively compact in $C([0, T]; \mathcal{K})$, and moreover, if $f_n \xrightarrow{w} f_0$ then $\Upsilon f_n \rightarrow \Upsilon f_0$.

Proof. This Lemma is a direct consequence of Theorem 4.2.2 and Theorem 5.1.1 in [22]. Since Υ is with values in $C([0, T]; \mathcal{K})$, one has only to observe that the net constructed for the set $\{\Upsilon f_n\}_{n=1}^\infty$ in the cited theorems is in $C([0, T]; \mathcal{K})$ and keep in mind that $C([0, T]; \mathcal{K})$ is a closed subset of $C([0, T]; E)$. ■

Remark 4.2. The factor 2 in the estimation (10) can be dropped if the space E is separable (see Corollary 4.2.4 in [22]).

Remark 4.3. For $x_0 \in \mathcal{K}$ fixed, by conditions (S_1) and (S_2) , we deduce immediately that the operator

$$S(x_0, \cdot) : L^1([0, T]; E) \rightarrow C([0, T]; \mathcal{K})$$

satisfies the conditions (Υ_1) and (Υ_2) of Lemma 4.1.

Lemma 4.4. Let the sequence $\{f_n\} \subset L^1([0, T], E)$ be integrably bounded, i.e.,

$$(11) \quad \|f_n(t)\| \leq v(t) \text{ for all } n = 1, 2, \dots \text{ and a.e. } t \in [0, T],$$

for some $v \in L_+^1([0, T])$.

Suppose that

$$(12) \quad \chi(\{f_n(t)\}) \leq q(t) \text{ for a.e. } t \in [0, T], \text{ where } q(\cdot) \in L_+^1[0, T].$$

Then for every bounded subset $\Omega \subset \mathcal{K}$ and for all $t \in [0, T]$:

$$(13) \quad \chi_{\mathcal{K}}\{S(\Omega, \{f_n\}_{n=1}^\infty)(t)\} \leq 2M \int_0^t q(s) ds + e^{-pt} \chi_{\mathcal{K}}(\Omega),$$

where

$$\{S(\Omega, \{f_n\}_{n=1}^\infty)(t)\} = \bigcup_{\substack{x \in \Omega \\ n \geq 1}} S(x, f_n)(t).$$

Proof. Let $t \in [0, T]$ be fixed. For arbitrary $\varepsilon > 0$, let $\{x_i\}_{i=1}^m \subset \mathcal{K}$ be a finite $(\chi_{\mathcal{K}}(\Omega) + \varepsilon)$ -net of the set Ω . Invoking Remark 4.3 and Lemma 4.1(ii), we obtain

$$\chi_{\mathcal{K}} \{S(x_i, \{f_n\}_{n=1}^\infty)(t)\} \leq 2M \int_0^t q(s) ds, \quad i = 1, \dots, m.$$

Now, for $1 \leq i \leq m$, let $\{y_i^j, 1 \leq j \leq k(i)\} \subset \mathcal{K}$ be a finite $(2M \int_0^t q(s) ds + \varepsilon)$ net of $\{S(x_i, \{f_n\}_{n=1}^\infty)(t)\}$ such that

$$\|S(x_i, f_n)(t) - y_i^j\| \leq 2M \int_0^t q(s) ds + \varepsilon, \quad \forall n \in \alpha_{i,j}, \text{ where, } \alpha_{i,j} \subset \mathbb{N}, \bigcup_{j=1}^{k(i)} \alpha_{i,j} = \mathbb{N}^*.$$

Then, the set $\{y_i^j, 1 \leq i \leq m, 1 \leq j \leq k(i)\}$ forms a finite $e^{-pt}(\chi_{\mathcal{K}}(\Omega) + \varepsilon) + 2M \int_0^t q(s) ds + \varepsilon$ -net of the set $\{S(\Omega, \{f_n\}_{n=1}^\infty)(t)\}$.

Indeed, let $x \in \Omega$ and $x_{i_0}, 1 \leq i_0 \leq m$, be the corresponding point such that

$$\|x - x_{i_0}\| \leq \chi_{\mathcal{K}}(\Omega) + \varepsilon.$$

Using the last inequality and the condition (S_1) , we get for all $n \geq 1$

$$\|S(x, f_n)(t) - S(x_{i_0}, f_n)(t)\| \leq e^{-pt}(\chi_{\mathcal{K}}(\Omega) + \varepsilon).$$

Now, choose $y_{i_0}^{j_0}, 1 \leq j_0 \leq k(i_0)$ such that

$$(14) \quad \|S(x_{i_0}, f_n)(t) - y_{i_0}^{j_0}\| \leq 2M \int_0^t q(s) ds + \varepsilon, \quad \forall n \in \alpha_{i_0, j_0}.$$

Then, we get, for all $n \in \alpha_{i_0, j_0}$

$$\begin{aligned} \|S(x, f_n)(t) - y_{i_0}^{j_0}\| &\leq \|S(x, f_n)(t) - S(x_{i_0}, f_n)(t)\| + \|S(x_{i_0}, f_n)(t) - y_{i_0}^{j_0}\| \\ &\leq e^{-pt}(\chi_{\mathcal{K}}(\Omega) + \varepsilon) + 2M \int_0^t q(s) ds + \varepsilon. \end{aligned}$$

Since the choice of ε is arbitrary and $\bigcup_{j=1}^{k(i_0)} \alpha_{i_0, j} = \mathbb{N}^*$ for all $1 \leq i_0 \leq m$, the proof is complete. ■

Lemma 4.5. *For every bounded subset $Z \subset \mathcal{K}$ such that $\chi_{\mathcal{K}}(Z) = 0$ and semicom-
pact sequence $\{f_n\}_{n=1}^\infty \subset L^1([0, T]; E)$, the set $\{S(Z, f_n)\}_{n=1}^\infty$ is relatively compact
in $C([0, T]; \mathcal{K})$, and moreover, if $f_n \xrightarrow{w} f_0$ in $L^1([0, T]; E)$ and $x_n \rightarrow x_0$ in \mathcal{K} , then
 $S(x_n, f_n) \rightarrow S(x_0, f_0)$ in $C([0, T]; \mathcal{K})$.*

Proof. For arbitrary $\varepsilon > 0$, let $\{z_i\}_{i=1}^m \subset \mathcal{K}$ be an ε -net of Z . From Re-
mark 4.3 and Lemma 4.1 it follows that for every $1 \leq i \leq m$, the sequence
 $\{S(z_i, f_n)\}_{n=1}^\infty$ is relatively compact in $C([0, T]; \mathcal{K})$. Now, by condition (S_1) , it is
easy to see that the relatively compact set $\bigcup_{i=1}^m \{S(z_i, f_n)\}_{n=1}^\infty$ in $C([0, T]; \mathcal{K})$ is an
 ε -net of $\{S(Z, f_n)\}_{n=1}^\infty$. Again from Remark 4.3 and Lemma 4.1, we know that
 $S(x_0, f_n) \rightarrow S(x_0, f_0)$ in $C([0, T]; \mathcal{K})$. Thus, applying condition (S_1) , we obtain

$$\begin{aligned} \|S(x_n, f_n) - S(x_0, f_0)\| &\leq \|S(x_n, f_n) - S(x_0, f_n)\| + \|S(x_0, f_n) - S(x_0, f_0)\| \\ &\leq \|x_n - x_0\| + \|S(x_0, f_n) - S(x_0, f_0)\| \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Proof of Theorem 3.3. (i) We will prove that the multivalued operator \mathcal{F} is
u.s.c. with compact values. First we will show that \mathcal{F} is closed with compact
values. Let $\{x_n\}_n, \{z_n\}_n \subset C([0, T]; \mathcal{K})$, $x_n \rightarrow x_0$, $z_n \in \mathcal{F}(x_n)$, $n \geq 1$, and
 $z_n \rightarrow z_0$. Let $\{f_n\}_n$ be a sequence from $L^1([0, T], E)$ such that $f_n \in \text{sel}_F(x_n)$
and $z_n = S(S(x_n(0), f_n)(T), f_n)$, $n \geq 1$. By hypothesis (F_3) we have $\|f_n(t)\| \leq$
 $\|F(t, x_n(t))\| \leq U_{\Omega_0}(t)$ a.e. $t \in [0, T]$, where Ω_0 is a bounded subset of \mathcal{K} ,
containing the set $\{x_n(t), n \geq 1, t \in [0, T]\}$, and by hypothesis (F_4) we have

$$\chi(\{f_n(t)\}_{n=1}^{+\infty}) \leq \kappa(t)\chi(\{x_n(t)\}_{n=1}^{+\infty}) = 0 \quad \text{a.e. } t \in [0, T].$$

Then, the sequence $\{f_n\}_n$ is semicom-compact. Consequently it is weakly com-
pact in $L^1([0, T]; E)$. Without loss of generality one can suppose that $f_n \xrightarrow{w}$
 f_0 . Since $\chi_{\mathcal{K}}(\{x_n(0)\}_{n=1}^\infty) = 0$, from Lemma 4.5 it follows that the sequence
 $\{S(x_n(0), f_n)\}_n$ is relatively compact in $C([0, T]; \mathcal{K})$ and

$$S(x_n(0), f_n) \rightarrow S(x_0(0), f_0).$$

Now, applying Lemma 3.2, we get $f_0 \in \text{sel}_F(x_0)$. Thus

$$S(x_n(0), f_n)(T) \rightarrow S(x_0(0), f_0)(T) \text{ in } \mathcal{K}, \text{ with } f_0 \in \text{sel}_F(x_0).$$

Using again Lemma 4.5, we deduce that the sequence $\{S(S(x_n(0), f_n)(T), f_n)\}_{n=1}^\infty$
is relatively compact in $C([0, T]; \mathcal{K})$ and

$$z_n = S(S(x_n(0), f_n)(T), f_n) \rightarrow S(S(x_0(0), f_0)(T), f_0) \text{ with } f_0 \in \text{sel}_F(x_0).$$

Hence, $z_0 = S(S(x_0(0), f_0)(T), f_0) \in \mathcal{F}(x_0)$, which yields the closedness of \mathcal{F} .

Let $x(\cdot) \in C([0, T]; \mathcal{K})$. By the same reasoning as above, hypotheses (F_3) and (F_4) imply that every sequence $\{f_n\}_n$, $f_n \in \text{sel}_F(x)$ is semicompact, which implies by Lemma 4.5 that $\{S(x(0), f_n)(T)\}_{n=1}^\infty$ is relatively compact in \mathcal{K} , which implies again by the same lemma that $S\{(S(x(0), f_n)(T), f_n)\}_{n=1}^\infty$ is relatively compact in $C([0, T]; \mathcal{K})$. The compactness of $\mathcal{F}(x)$ follows from its closedness.

Finally, let us prove that \mathcal{F} is u.s.c. By Lemma 2.1 it is enough to prove its quasi-compactness. Let us consider a convergent sequence $\{x_n\}_n \subset C([0, T]; \mathcal{K})$ and an arbitrary sequence $\{f_n\}_{n=1}^\infty \subset L^1([0, T]; E)$ such that $f_n \in \text{sel}_F(x_n)$, $n \geq 1$. By hypotheses (F_3) and (F_4) it follows that the sequence $\{f_n\}_n$ is semicompact. Since $\chi_{\mathcal{K}}(\{x_n(0)\}_{n=1}^\infty) = 0$, from Lemma 4.5 it follows that the sequence $\{S(x_n(0), f_n)\}_{n=1}^\infty$ is relatively compact in $C([0, T]; \mathcal{K})$. Hence $\{S(x_n(0), f_n)(T)\}_{n=1}^\infty$ is relatively compact in \mathcal{K} . Using again Lemma 4.5, we get that the sequence $\{S(S(x_n(0), f_n)(T), f_n)\}_{n=1}^\infty$ is relatively compact in $C([0, T]; \mathcal{K})$. Therefore, the multioperator \mathcal{F} is quasicompact.

(ii) Let $\Omega \subset C([0, T]; \mathcal{K})$ be a bounded subset such that

$$(15) \quad \Psi(\mathcal{F}(\Omega)) \geq \Psi(\Omega),$$

where the inequality is taken in the sense of the order \mathbb{R}^3 , induced by the positive cone \mathbb{R}_+^3 . We will show that (15) implies that Ω is relatively compact. Let the maximum on the left-hand side of the inequality (15) be achieved for the countable set $D' = \{z_n\}_{n=1}^\infty$ with

$$z_n(t) = S(S(x_n(0), f_n)(T), f_n), \quad f_n \in \text{sel}_F(x_n), \quad n \geq 1, \quad \{x_n\}_n \subset \Omega.$$

Using condition (S_0) we have

$$z_n(0) = S(S(x_n(0), f_n)(T), f_n)(0) = S(x_n(0), f_n)(T).$$

From (15) we get

$$(16) \quad \Psi(\{x_n\}_{n=1}^\infty) \leq \Psi(\{z_n\}_{n=1}^\infty).$$

Then

$$(17) \quad \chi_{\mathcal{K}}(\{x_n(0)\}_{n=1}^\infty) \leq \chi_{\mathcal{K}}(\{z_n(0)\}_{n=1}^\infty) = \chi_{\mathcal{K}}(\{S(x_n(0), f_n)(T)\}_{n=1}^\infty);$$

$$(18) \quad \vartheta(\{x_n\}_{n=1}^\infty) \leq \vartheta(\{z_n\}_{n=1}^\infty) = \vartheta(\{S(S(x_n(0), f_n)(T), f_n)\}_{n=1}^\infty).$$

By hypothesis (F_4) we have

$$(19) \quad \begin{aligned} \chi(\{f_n(t)\}_{n=1}^\infty) &\leq \kappa(t)\chi(\{x_n(t)\}_{n=1}^\infty) \\ &\leq \kappa(t)\vartheta(\{x_n\}_{n=1}^\infty) \quad \text{a.e. } t \in [0, T], \end{aligned}$$

and by (F_3) the sequence $\{f_n\}_n$ is integrably bounded. Hence, by Lemma 4.4, we get

$$(20) \quad \begin{aligned} \chi_{\mathcal{K}}(\{z_n(0)\}_{n=1}^\infty) &= \chi_{\mathcal{K}}(\{S(x_n(0), f_n)(T)\}_n) \\ &\leq 2M \|\kappa\|_{L^1} \vartheta(\{x_n\}_{n=1}^\infty) + e^{-pT} \chi_{\mathcal{K}}(\{x_n(0)\}_{n=1}^\infty); \end{aligned}$$

$$(21) \quad \begin{aligned} \vartheta(\{z_n\}_{n=1}^\infty) &= \vartheta(\{S(S(x_n(0), f_n))(T), f_n\}_{n=1}^\infty) \\ &\leq 2M \|\kappa\|_{L^1} \vartheta(\{x_n\}_{n=1}^\infty) + e^{-pt} \chi_{\mathcal{K}}(\{S(x_n(0), f_n)(T)\}_{n=1}^\infty) \\ &\leq 4M \|\kappa\|_{L^1} \vartheta(\{x_n\}_{n=1}^\infty) + e^{-pT} \chi_{\mathcal{K}}(\{x_n(0)\}_{n=1}^\infty) \end{aligned}$$

Set

$$\gamma_0 = \chi_{\mathcal{K}}(\{x_n(0)\}_{n=1}^\infty) \quad \text{and} \quad \nu = \vartheta(\{x_n\}_{n=1}^\infty).$$

From inequalities (17), (18), (20) and (21), we get

$$(22) \quad \begin{cases} \gamma_0 \leq 2M \|\kappa(\cdot)\|_{L^1} \nu + e^{-pT} \gamma_0, \\ \nu \leq 4M \|\kappa(\cdot)\|_{L^1} \nu + e^{-pT} \gamma_0. \end{cases}$$

Since $\gamma_0 \leq \nu$ then (22) together with (9) give $\nu \leq (4M \|\kappa(\cdot)\|_{L^1} \nu + e^{-pT}) \nu$ implying $\nu = 0$. Hence, $\gamma_0 \leq e^{-pT} \gamma_0$, and we obtain $\gamma_0 = 0$. Therefore,

$$(23) \quad \chi(\{x_n(t)\}_{n=1}^\infty) \leq \chi_{\mathcal{K}}(\{x_n(t)\}_{n=1}^\infty) = 0 \quad \text{for all } t \in [0, T].$$

By (F_3) and (F_4) , the sequence $\{f_n\}_{n=1}^\infty$ is semicompact in $L^1([0, T]; E)$. By Lemma 4.5, the set $\{S(x_n(0), f_n)(T)\}_{n=1}^\infty = \{z_n(0)\}_{n=1}^\infty$ is relatively compact in \mathcal{K} . We can apply again Lemma 4.5, to deduce that the sequence $\{z_n\}_{n=1}^\infty = \{S(S(x_n(0), f_n)(T), f_n)\}_{n=1}^\infty$ is relatively compact in $C([0, T]; \mathcal{K})$. Consequently

$$\text{mod}_c(\{z_n\}_{n=1}^\infty) = 0.$$

Then, by (15), we get

$$\text{mod}_c(\{x_n\}_{n=1}^\infty) = 0.$$

The last equality and (23) imply that

$$\Psi(\Omega) = (0, 0, 0).$$

Then Ω is relatively compact.

If the space E is separable then using Remark 4.2, the estimation (9) in the point (ii) of Theorem 3.3 can be weakened as

$$2M \|\kappa(\cdot)\|_{L^1} + e^{-pT} < 1.$$

(iii) Let us prove that \mathcal{F} has contractible values. Let $x \in C([0, T]; \mathcal{K})$ and $v_0 \in \mathcal{F}(x)$, with $v_0 = S(S(x(0), f_0)(T), f_0)$ for some $f_0 \in \text{sel}_F(x)$.

Consider the function $H : [0, 1] \times \mathcal{F}(x) \longrightarrow C([0, T]; \mathcal{K})$ given by

$$H(\lambda, v) = S(S(x(0), \mathbf{1}_{[0, \lambda T]}(\cdot)f + \mathbf{1}_{[\lambda T, T]}(\cdot)f_0)(T), \mathbf{1}_{[0, \lambda T]}(\cdot)f + \mathbf{1}_{[\lambda T, T]}(\cdot)f_0),$$

where $v \in \mathcal{F}(x)$, $v = S(S(x(0), f)(T), f)$, for some $f \in \text{sel}_F(x)$. By condition (S_3) , the value $H(\lambda, v)$ does not depend on the choice of f and therefore, the function H it is correctly defined. Moreover,

$$H(0, v) = v_0, \quad H(1, v) = v$$

and

$$H(\lambda, v) \in \mathcal{F}(x), \quad \forall \lambda \in [0, 1] \text{ and } \forall v \in \mathcal{F}(x).$$

The last point is due to the fact that

$$\mathbf{1}_{[0, \lambda T]}(\cdot)f + \mathbf{1}_{[\lambda T, T]}(\cdot)f_0 \in \text{sel}_F(x), \quad \forall \lambda \in [0, 1].$$

It remains to show that H is continuous. Let sequences $\{\lambda_n\}_n \subset [0, 1]$ and $\{v_n\}_n \subset \mathcal{F}(x)$ be such that $\lambda_n \longrightarrow \lambda_0$, $v_n \longrightarrow v_\infty$, with $v_n = S(S(x(0), f_n)(T), f_n)$ and $f_n \in \text{sel}_F(x)$. By conditions (F_3) and (F_4) the sequence $\{f_n\}_n \subset L^1([0, T], E)$ is semicompact and hence weakly compact. Without loss of generality, we can assume that $f_n \xrightarrow{w} f_\infty$. By Lemma 3.2, $f_\infty \in \text{sel}_F(x)$.

Now, applying Lemma 4.5, we have

$$S(x_n(0), \mathbf{1}_{[0, \lambda_n T]}(\cdot)f_n + \mathbf{1}_{[\lambda_n T, T]}(\cdot)f_0) \xrightarrow{C([0, T]; \mathcal{K})} S(x(0), \mathbf{1}_{[0, \lambda_0 T]}(\cdot)f_\infty + \mathbf{1}_{[\lambda_0 T, T]}(\cdot)f_0),$$

which implies that,

$$S(x_n(0), \mathbf{1}_{[0, \lambda_n T]}(\cdot)f_n + \mathbf{1}_{[\lambda_n T, T]}(\cdot)f_0)(T) \xrightarrow{\mathcal{K}} S(x(0), \mathbf{1}_{[0, \lambda_0 T]}(\cdot)f_\infty + \mathbf{1}_{[\lambda_0 T, T]}(\cdot)f_0)(T).$$

Using again Lemma 4.5, we get

$$\begin{aligned} & S \left(S \left(x_n(0), \mathbf{1}_{[0, \lambda_n T]}(\cdot) f_n + \mathbf{1}_{[\lambda_n T, T]}(\cdot) f_0 \right) (T), \mathbf{1}_{[0, \lambda_n T]}(\cdot) f_n + \mathbf{1}_{[\lambda_n T, T]}(\cdot) f_0 \right) \\ & \quad \xrightarrow{C([0, T]; \mathcal{K})} \\ & S \left(S \left(x(0), \mathbf{1}_{[0, \lambda_0 T]}(\cdot) f_\infty + \mathbf{1}_{[\lambda_0 T, T]}(\cdot) f_0 \right) (T), \mathbf{1}_{[0, \lambda_0 T]}(\cdot) f_\infty + \mathbf{1}_{[\lambda_0 T, T]}(\cdot) f_0 \right). \end{aligned}$$

Therefore, $H(\lambda_n, v_n) \rightarrow H(\lambda_0, v_\infty)$. ■

5. APPLICATION

As an application of our result (Theorem 3.3), we study in a real Banach space E the existence of integral solutions to abstract periodic problems of the form

$$(24) \quad \begin{cases} x'(t) \in -Ax(t) + F(t, x(t)), 0 < t \leq T, \\ x(0) = x(T), \end{cases}$$

where

- (e) the topological dual E^* of E is uniformly convex;
- (A₁) $A : D(A) \subset E \rightarrow \mathcal{P}(E)$ is an operator, with $0 \in A(0)$ and such that $-A$ generates an equicontinuous semigroup;
- (A₂) there exists $\varepsilon > 0$ such that $A - \varepsilon I$ is m -accretive;
- (A₃) $\overline{D(A)}$ is a convex subset of E .

The multimap $F : [0, T] \times \overline{D(A)} \rightarrow Kv(E)$ satisfies (F_1) , (F_2) , (F_4) and

(F'₃) there exists a function $\alpha(\cdot) \in L^1([0, T]; \mathbb{R}^+)$ such that

$$\|F(t, x)\| \leq \alpha(t)$$

for all $x \in \mathcal{K}$ and a.e. $t \in [0, T]$.

Theorem 5.1. *Let the assumptions (e), (A₁), (A₂), (A₃), (F₁), (F₂), (F'₃) and (F₄) be satisfied. If*

$$(25) \quad 4M \|\kappa(\cdot)\|_{L^1} + e^{-\varepsilon T} < 1,$$

then the problem (24) has at least one integral solution.

Proof. Consider the Cauchy problem

$$(26) \quad \begin{cases} x'_\varepsilon(t) \in -Ax_\varepsilon(t) + f^\varepsilon(t), & 0 < t \leq T, \\ x_\varepsilon(0) = x_0, \end{cases}$$

where, $x_0 \in \overline{D(A)}$ and $f^\varepsilon \in L^1([0, T]; E)$. It is well known that, the problem (26) has a unique integral solution x_ε with $x_\varepsilon(t) \in \overline{D(A)}$ for $t \in [0, T]$. Moreover, if y_ε is a mild solution to the following differential inclusion

$$(27) \quad \begin{cases} -y'_\varepsilon(t) \in Ay_\varepsilon(t) + \bar{f}^\varepsilon(t), & 0 < t \leq T \\ y_\varepsilon(0) = y_0, & y_0 \in \overline{D(A)} \end{cases}$$

then

$$(28) \quad \|x_\varepsilon(t) - y_\varepsilon(t)\| \leq e^{-\varepsilon(t-s)} \|x_\varepsilon(s) - y_\varepsilon(s)\| + \int_s^t e^{-\varepsilon(t-\tau)} \|f^\varepsilon(\tau) - \bar{f}^\varepsilon(\tau)\| d\tau,$$

for all $0 \leq s \leq t \leq T$ (see for example [7]).

In this situation, the integral solutions operator S_ε

$$S_\varepsilon : \overline{D(A)} \times L^1([0, T]; E) \rightarrow C([0, T], \overline{D(A)}),$$

where $S_\varepsilon(x_0, f^\varepsilon)$, is the unique integral solution of the problem (26).

The operator S_ε checks the conditions (S_0) – (S_3) . Indeed,

- (i) the condition (S_0) is trivial;
- (ii) the condition (S_1) follows from (28), with $p = \varepsilon$ and $M = 1$;
- (iii) as the operator A is m-accretive and generates a strongly equicontinuous semigroup (see [11]), and as the dual space E^* is uniformly convex (by hypothesis e)), one can invoke Proposition 1 and Lemma 4 from [11] to infer that the operator S_ε satisfies the condition (S_2) ;
- (iii) The condition (S_3) follows easily from the inequality (28).

Now let the operator \mathcal{F}_ε be given by

$$(29) \quad \begin{cases} \mathcal{F}_\varepsilon : C([0, T]; \mathcal{K}) \rightarrow \mathcal{P}(C([0, T]; \mathcal{K})), \\ \mathcal{F}_\varepsilon(x_\varepsilon) = \{S_\varepsilon(S_\varepsilon(x_\varepsilon(0), f^\varepsilon)(T), f^\varepsilon) : f^\varepsilon \in \text{sel}_F(x_\varepsilon)\}. \end{cases}$$

It is clear that a fixed point of \mathcal{F}_ε is an integral solution of (24).

From Theorem 3.3 it follows that the operator \mathcal{F}_ε is u.s.c with compact (hence closed) contractile values. According to Lemma 2.4 to ensure the existence of at

least one integral solution of the inclusion (24), we have to look for some compact convex set $K^\varepsilon \subset C([0, T]; \mathcal{K})$ such that $\mathcal{F}_\varepsilon(K^\varepsilon) \subset K^\varepsilon$. To do that, take the set

$$W_0^\varepsilon = \{x_\varepsilon \in C([0, T]; \mathcal{K}) : \|x_\varepsilon(t)\| \leq \psi(t) \text{ on } [0, T]\}$$

where ψ is the solution of

$$(30) \quad \begin{cases} \psi'(t) = -\varepsilon\psi(t) + \alpha(t), \text{ a.e., } t \in [0, T], \\ \psi(0) = \psi(T). \end{cases}$$

The set W_0^ε is well defined. Indeed if ψ_i , $i = 1, 2$ are solutions of the Cauchy problem

$$(31) \quad \begin{cases} \psi'_i(t) = -\varepsilon\psi_i(t) + \alpha(t), \text{ a.e., } t \in [0, T], \\ \psi_i(0) = \psi_i^0, \end{cases}$$

then

$$\|\psi_1(t) - \psi_2(t)\| \leq e^{-\varepsilon t} \|\psi_1^0 - \psi_2^0\|.$$

It follows that the Poincaré map $\psi^0 = \psi(T)$ is a strict contraction on \mathcal{K} , hence the problem (30) has a unique solution. Moreover,

$$(32) \quad \begin{aligned} \psi(t) &= e^{-\varepsilon t} e^{-\varepsilon T} \psi(0) + e^{-\varepsilon t} \int_0^T e^{-\varepsilon(T-s)} \alpha(s) ds + \int_0^t e^{-\varepsilon(t-s)} \alpha(s) ds \\ &= e^{-\varepsilon t} (1 - e^{-\varepsilon T})^{-1} \int_0^T e^{-\varepsilon(T-s)} \alpha(s) ds + \int_0^t e^{-\varepsilon(t-s)} \alpha(s) ds. \end{aligned}$$

The first expression of ψ follows from the fact that ψ is a fixed point of the operator \mathcal{F}_ε with $F(t, x) = \{\alpha(t)\}$, the second one follows from the fact that $\psi(0) = \psi(T)$.

Using (28) and the fact that $0 \in A(0)$, we get for all $f \in \text{sel}_F(x_\varepsilon)$

$$S_\varepsilon(x_\varepsilon(0), f)(T) \leq e^{-\varepsilon T} \|x_\varepsilon(0)\| + \int_0^T e^{-\varepsilon(T-s)} \alpha(\tau) d\tau.$$

For the same reason we have

$$\begin{aligned} &S_\varepsilon(S_\varepsilon(x_\varepsilon(0), f)(T), f)(t) \\ &\leq e^{-\varepsilon t} e^{-\varepsilon T} \|x_\varepsilon(0)\| + e^{-\varepsilon t} \int_0^T e^{-\varepsilon(T-s)} \alpha(\tau) d\tau + \int_0^t e^{-\varepsilon(t-s)} \alpha(\tau) d\tau \\ &\leq e^{-\varepsilon t} e^{-\varepsilon T} \psi(0) + e^{-\varepsilon t} \int_0^T e^{-\varepsilon(T-s)} \alpha(\tau) d\tau + \int_0^t e^{-\varepsilon(t-s)} \alpha(\tau) d\tau = \psi(t). \end{aligned}$$

Hence $\mathcal{F}_\varepsilon(W_0^\varepsilon) \subset W_0^\varepsilon$.

From (20) and (21), we have

$$\begin{pmatrix} \chi(\mathcal{F}_\varepsilon(W_0^\varepsilon(0))) \\ \vartheta(W_0^\varepsilon) \end{pmatrix} \leq \Xi \begin{pmatrix} \chi(W_0^\varepsilon(0)) \\ \vartheta(W_0^\varepsilon) \end{pmatrix}$$

with

$$\Xi = \begin{pmatrix} e^{-pT} & 4M \|\kappa(\cdot)\|_{L^1} \\ e^{-pT} & 4M \|\kappa(\cdot)\|_{L^1} \end{pmatrix}.$$

Define $W_1^\varepsilon = \overline{\text{co}} \mathcal{F}_\varepsilon(W_0^\varepsilon)$. It is easy to see that W_1^ε is a nonempty, closed, convex subset and

$$W_1^\varepsilon = \overline{\text{co}} \mathcal{F}_\varepsilon(W_0^\varepsilon) \subset \overline{\text{co}} W_0^\varepsilon = W_0^\varepsilon.$$

For the same reason we get

$$\begin{pmatrix} \chi(\mathcal{F}_\varepsilon(W_1^\varepsilon(0))) \\ \vartheta(\mathcal{F}_\varepsilon(W_1^\varepsilon)) \end{pmatrix} \leq \Xi \begin{pmatrix} \chi(W_1^\varepsilon(0)) \\ \vartheta(W_1^\varepsilon) \end{pmatrix} = \Xi \begin{pmatrix} \chi(\mathcal{F}_\varepsilon(W_0^\varepsilon(0))) \\ \vartheta(\mathcal{F}_\varepsilon(W_0^\varepsilon)) \end{pmatrix}.$$

Hence

$$\begin{pmatrix} \chi(\mathcal{F}_\varepsilon(W_1^\varepsilon(0))) \\ \vartheta(\mathcal{F}_\varepsilon(W_1^\varepsilon)) \end{pmatrix} \leq \Xi^2 \begin{pmatrix} \chi(W_0^\varepsilon(0)) \\ \vartheta(W_0^\varepsilon) \end{pmatrix}.$$

Define $W_2^\varepsilon = \overline{\text{co}} \mathcal{F}_\varepsilon(W_1^\varepsilon)$. W_2^ε is a nonempty, closed, convex subset and

$$W_2^\varepsilon \subset W_1^\varepsilon \subset W_0^\varepsilon;$$

$$\begin{pmatrix} \chi(\mathcal{F}_\varepsilon(W_2^\varepsilon(0))) \\ \vartheta(\mathcal{F}_\varepsilon(W_2^\varepsilon)) \end{pmatrix} \leq \Xi^3 \begin{pmatrix} \chi(W_0^\varepsilon(0)) \\ \vartheta(W_0^\varepsilon) \end{pmatrix}.$$

Continuing this procedure, we get a decreasing sequence $(W_n^\varepsilon)_n$ of nonempty, closed, convex, bounded subsets such that

$$(33) \quad \begin{pmatrix} \chi(\mathcal{F}_\varepsilon(W_n^\varepsilon(0))) \\ \vartheta(\mathcal{F}_\varepsilon(W_n^\varepsilon)) \end{pmatrix} \leq \Xi^{n+1} \begin{pmatrix} \chi(W_0^\varepsilon(0)) \\ \vartheta(W_0^\varepsilon) \end{pmatrix}.$$

As $\det(\Xi - \lambda I) = 0$ for $\lambda = 0$ or $\lambda = 4M \|\kappa(\cdot)\|_{L^1} + e^{-\varepsilon T} < 1$, from (33), it follows that

$$(34) \quad \begin{pmatrix} \chi \mathcal{K}(\mathcal{F}_\varepsilon(W_n^\varepsilon(0))) \\ \vartheta(\mathcal{F}_\varepsilon(W_n^\varepsilon)) \end{pmatrix} \xrightarrow{n \rightarrow +\infty} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We claim that

$$\text{mod}_c(\mathcal{F}_\varepsilon(W_n^\varepsilon)) \xrightarrow{n \rightarrow +\infty} 0.$$

Indeed, let first show that for each $\zeta \in]0, T]$

$$\text{mod}_c\left(\mathcal{F}_\varepsilon(W_0^\varepsilon) \mid_{[\zeta, T]}\right) = 0.$$

Remark that if $B \subset \overline{D(A)}$ is a nonempty bounded subset and $G \subset L^1([0, T]; E)$ is an integrably bounded subset, then using relation (28) and the fact that $0 \in A(0)$, we deduce that the set

$$S(B, g)[0, T] = \{S(b, g)(t), b \in B, g \in G, t \in [0, T]\}$$

is bounded in $\overline{D(A)}$. Therefore, by a minor adaptation of the proof of Theorem 2.5.1 in [7], p. 57 we conclude that the set $\{S(b, g)(\cdot), b \in B, g \in G\}$ is equicontinuous in $C([\zeta, T], \overline{D(A)})$. Now, by (F'_3) the set

$$\{f_0^\varepsilon, f_0^\varepsilon \in \text{sel}_F(x_\varepsilon^0), x_\varepsilon \in W_0^\varepsilon\}$$

is integrably bounded subset in $L^1([0, T]; E)$. Since W_1^ε is bounded in $C([0, T], \overline{D(A)})$, then the set

$$\{S(x_0^\varepsilon(0), f_0^\varepsilon)(T), x_\varepsilon \in W_0^\varepsilon, f_0^\varepsilon \in \text{sel}_F(x_\varepsilon^0)\} \subset W_1^\varepsilon(0)$$

is bounded in $\overline{D(A)}$.

Hence, the set

$$\begin{aligned} \mathcal{F}_\varepsilon(W_0^\varepsilon)(\cdot) &= \{S(S(x_0^\varepsilon(0), f_0^\varepsilon)(T), f_\varepsilon)(\cdot), x_\varepsilon \in W_0^\varepsilon, f_0^\varepsilon \in \text{sel}_F(x_\varepsilon^0)\} \\ &\subset \{S(W_0^\varepsilon, f_0^\varepsilon)(\cdot), f_0^\varepsilon \in \text{sel}_F(x_\varepsilon^0), x_\varepsilon \in W_0^\varepsilon\} \end{aligned}$$

is equicontinuous in $C([\zeta, T], \overline{D(A)})$. By the monotonicity of the function $\text{mod}_c(\cdot)$, we get

$$(35) \quad \text{mod}_c\left(\mathcal{F}_\varepsilon(W_n^\varepsilon) \mid_{[\zeta, T]}\right) = 0 \text{ for all } n \geq 0 \text{ and for each } \zeta \in]0, T].$$

Let us prove now that

$$\text{mod}_c(\mathcal{F}_\varepsilon(W_n^\varepsilon)) \xrightarrow{n \rightarrow \infty} 0 \text{ at the origin.}$$

By formula (34) we obtain

$$\chi\mathcal{K}(\mathcal{F}_\varepsilon(W_n^\varepsilon(0))) = \chi\mathcal{K}\{S(x_n^\varepsilon(0), f_\varepsilon)(T) : x_n^\varepsilon \in W_n^\varepsilon, f_n^\varepsilon \in \text{sel}_F(x_n^\varepsilon)\} \xrightarrow{n \rightarrow \infty} 0.$$

Then, for $\delta > 0$, there exist n_0 such that

$$n \geq n_0 \Rightarrow \chi_K \{S(x_n^\varepsilon(0), f_\varepsilon)(T) : x_n^\varepsilon \in W_n^\varepsilon, f_n^\varepsilon \in \text{sel}_F(x_n^\varepsilon)\} \leq \delta.$$

For $n \geq n_0$, let $\{y_i\}_{i=1}^m \subset \overline{D(A)}$ be a finite 2δ -net of the set

$$\{S(x_n^\varepsilon(0), f_\varepsilon)(T) : x_n^\varepsilon \in W_n^\varepsilon, f_n^\varepsilon \in \text{sel}_F(x_n^\varepsilon)\}.$$

Let $x_n^\varepsilon \in W_n^\varepsilon$, $f_n^\varepsilon \in \text{sel}_F(x_n^\varepsilon)$ and let y_{i_0} be the corresponding point such that

$$(36) \quad \|S(x_n^\varepsilon, f_n^\varepsilon)(T) - y_{i_0}\| \leq 2\delta.$$

Using the relation (28), condition (F'_3) and the fact that the operator $T(t)$ is nonexpansive for all $t \in [0, T]$, we have

$$\begin{aligned} & \|S(S(x_n^\varepsilon(0), f_n^\varepsilon)(T), f_\varepsilon)(h) - S(S(x_0^\varepsilon(0), f_n^\varepsilon)(T), f_n^\varepsilon)(0))\| \\ &= \|S(S(x_0^\varepsilon(0), f_n^\varepsilon)(T), f_\varepsilon)(h) - S(x_0^\varepsilon, f_n^\varepsilon)(T))\| \\ &\leq \|S(S(x_0^\varepsilon(0), f_n^\varepsilon)(T), f_\varepsilon)(h) - T(h)S(x_0^\varepsilon, f_n^\varepsilon)(T))\| \\ &\quad + \|T(h)S(x_0^\varepsilon, f_n^\varepsilon)(T) - S(x_0^\varepsilon, f_n^\varepsilon)(T)\| \\ &\leq \int_0^h e^{-\varepsilon(h-s)} \|f_n^\varepsilon(s)\| ds + \|T(h)S(x_0^\varepsilon, f_n^\varepsilon)(T) - S(x_0^\varepsilon, f_n^\varepsilon)(T)\| \\ &\leq \int_0^h \alpha(s) ds + \|T(h)S(x_0^\varepsilon, f_n^\varepsilon)(T) - S(x_0^\varepsilon, f_n^\varepsilon)(T)\| \\ &\leq \|T(h)S(x_0^\varepsilon, f_n^\varepsilon)(T) - T(h)y_{i_0}\| + \|T(h)y_{i_0} - y_{i_0}\| \\ &\quad + \|S(x_0^\varepsilon, f_n^\varepsilon)(T) - y_{i_0}\| + \int_0^h \alpha(s) ds \\ &\leq 2\|S(x_0^\varepsilon, f_n^\varepsilon)(T) - y_{i_0}\| + \|T(h)y_{i_0} - y_{i_0}\| + \int_0^h \alpha(s) ds. \end{aligned}$$

Taking into account that the operator $t \rightarrow T(t)y_{i_0}$ is continuous at the origin, and that the set $\{y\}_{i=1}^m$ is finite, (36) and the last inequality imply that

$$\|S(S(x_n^\varepsilon, f_n^\varepsilon)(T), f_\varepsilon)(h) - S(S(x_n^\varepsilon, f_\varepsilon)(T), f_\varepsilon)(0))\| \leq 5\delta \text{ as } h \rightarrow 0 \text{ and } n \geq n_0,$$

for all $x_n^\varepsilon \in W_n^\varepsilon$ and $f_n^\varepsilon \in \text{sel}_F(x_n^\varepsilon)$. Therefore,

$$\text{mod}_c(\mathcal{F}_\varepsilon(W_n^\varepsilon)) \xrightarrow{n \rightarrow \infty} 0 \text{ at the origin.}$$

Last relation together with (35) imply

$$\text{mod}_c(\mathcal{F}_\varepsilon(W_n^\varepsilon)) \xrightarrow[n \rightarrow \infty]{} 0.$$

Hence,

$$\Psi(\mathcal{F}_\varepsilon(W_n^\varepsilon)) \xrightarrow[n \rightarrow \infty]{} (0, 0, 0).$$

This proves our claim.

Since Ψ is a nonsingular, monotone, regular measure of noncompactness defined on subsets of $C([0, T]; \overline{D(A)})$, subsets $(W_n^\varepsilon)_n$ are nonempty, closed and such that $W_{n+1}^\varepsilon \subset W_n^\varepsilon$, $n \geq 0$,

$$\Psi(W_n^\varepsilon) = \Psi(\overline{\text{co}}(\mathcal{F}_\varepsilon(W_{n-1}^\varepsilon))) = \Psi(\mathcal{F}_\varepsilon(W_{n-1}^\varepsilon)) \xrightarrow[n \rightarrow \infty]{} (0, 0, 0),$$

then the set

$$K^\varepsilon = \bigcap_{n \geq 0} W_n^\varepsilon$$

is nonempty and compact. Moreover, since $(W_n^\varepsilon)_{n \geq 0}$ are convex, the set K^ε is convex too. It is clear that

$$\mathcal{F}_\varepsilon(K^\varepsilon) \subset K^\varepsilon.$$

Thus, K^ε is the required one. The proof of Theorem 5.1 is complete.

Remark 5.2. It is well known that condition (A_2) implies that a mild solution to (24) is also an integral solution to this problem. In other words, for the problem (24), under condition (A_2) the notion of a mild solution and the notion of an integral solution coincide (see Theorem 1.3 in [7], p. 204).

Corollary 5.3. *Suppose conditions (e) , (A_1) , (A_2) , (A_3) , (F_1) , (F_2) , (F'_3) are satisfied, and that*

(F'_4) for every bounded subset $\Omega \subset \mathcal{K}$

$$\chi(F(t, \Omega)) = 0, \text{ a.e., } t \in [0, T].$$

Then for all $\varepsilon > 0$ the problem (24) has at least one integral (mild) solution.

Proof. It is a consequence of Theorem 5.1 and the fact that $e^{-\varepsilon T} < 1$ for all $\varepsilon > 0$.

Corollary 5.4. *Suppose that conditions (e) , (A_1) , (A_3) , (F_1) , (F_2) , (F'_3) , (F'_4) are satisfied and that*

(\tilde{A}_2) $A : D(A) \subset E \rightarrow \mathcal{P}(E)$ is m -accretive.

Then for all $\varepsilon > 0$, the perturbed problem

$$(37) \quad \begin{cases} x'_\varepsilon(t) \in -(A + \varepsilon I)x_\varepsilon(t) + F(t, x_\varepsilon(t)), & 0 < t \leq T, \\ x_\varepsilon(0) = x_\varepsilon(T) \end{cases}$$

has at least one integral (mild) solution.

Proof. From hypothesis (\tilde{A}_2) , it follows that for all $\varepsilon > 0$, the operator $(A + \varepsilon I)$ is m -accretive and generates an equicontinuous semigroup (see for example [12], Remark 3.5, p. 47). Moreover, $(A + \varepsilon I) - \varepsilon I = A$ is m -accretive. It remains to apply Corollary 5.3 in order to deduce that the problem (37) has at least one integral (mild) solution. \square

Remark 5.5. To obtain the estimation better than (25) we tried to define measures of noncompactness for equivalent norms in E . From the inequalities (20) and (21), it follows that Ψ is the best measure of noncompactness because it requires less conditions.

Remark 5.6. Theorem 5.1 generalizes Theorem 1 in [7] in the sense that in our case F is a multivalued map and A generates only an equicontinuous semigroup.

Remark 5.7. When the operator A in (24) is m accretive and generates an equicontinuous semigroup, then under conditions of Corollary 5.4, the perturbed problem (37) has a least one solution x_ε for every $\varepsilon > 0$.

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