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EXISTENCE OF POSITIVE SOLUTIONS FOR A FRACTIONAL BOUNDARY VALUE PROBLEM WITH LOWER-ORDER FRACTIONAL DERIVATIVE DEPENDENCE ON THE HALF-LINE

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Abstract

The aim of this paper is to study the existence of solutions to a boundary value problem associated to a nonlinear fractional differential equation where the nonlinear term depends on a fractional derivative of lower order posed on the half-line. An appropriate compactness criterion and suitable Banach spaces are used and so a fixed point theorem is applied to obtain fixed points which are solutions of our problem.

Keywords: fractional differential equation, half-line, Krasnoselsk'ii's fixed point theorem, existence, positive solution.

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1. Introduction

Boundary value problems associated to ordinary differential equations on unbounded domains have been studied widely. Most of them have studied the case where the nonlinear term does not depend on a lower order fractional derivative and when it depends on it, the domain of the problem is bounded. Among the few articles that have studied boundary value problem associated to fractional differential equations where the nonlinear term depends on a lower order fractional derivative we find the paper of Su and Zhang [5] who have studied the problem

$$(1) \qquad \begin{cases} D^{\alpha}_{0^+}u(t) = f(t, u(t), D^{\alpha-1}_{0^+}u(t)), \quad t \in [0, +\infty), \\ u(0) = 0, \quad D^{\alpha-1}_{0^+}u(\infty) = u_\infty, \quad u_\infty \in \mathbb{R}, \end{cases}$$

where $1 < \alpha \leq 2, f \in C([0,+\infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $D^{\alpha}_{0^+}, D^{\alpha-1}_{0^+}$ are the standard Riemann-Liouville fractional derivatives. The authors have obtain existence of sign-changing solutions under a sublinear growth condition by using Schauder's fixed point theorem. Motivated by the work of [5], the aim of this paper is to obtain existence of positive solutions for the problem (2) below under new growth conditions by using Krasnoselskii's fixed point theorem.

We consider the following boundary value problem

$$(2) \qquad \begin{cases} -D^{\alpha}_{0^+}u(t) = a(t)g(u(t), D^{\beta}_{0^+}u(t)), & t \in [0,+\infty), \\ u(0) = D^{\alpha-1}_{0^+}u(\infty) = 0, \end{cases}$$

where $1 < \alpha \leq 2, \beta > 0, \alpha - \beta \geq 1, g \in C([0,+\infty) \times \mathbb{R}^+, \mathbb{R}^+)$, $a(.)(1+t^{\alpha-1})^{\eta} \in L^1([0,+\infty[)$, for some $\eta > 0$ and when x, y are bounded, then $\frac{1}{(1+t^{\alpha-1})^{\eta}}g((1+t^{\alpha-1})x, (1+t^{\alpha-\beta-1})y)$ is uniformly bounded with respect to $t \in \mathbb{R}^+$. We put

$$S_R = \sup\left\{\frac{1}{(1+t^{\alpha-1})^{\eta}}g((1+t^{\alpha-1})y, (1+t^{\alpha-\beta-1})z), \ \ (y,z) \in [0,R]^2\right\} \text{ for } R > 0.$$

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Fore more details, see for example [3] and [4].

Definition 1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function u is defined by

$$I^{\alpha}_{a^+}u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)ds, \quad t > a,$$

provided that the right-hand side is pointwise defined and $\Gamma(\alpha)$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}e^{-t}dt$.

Definition 2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function u is defined by

$$\begin{aligned} D^{\alpha}_{a^+}u(t) &= \frac{d^n}{dt^n}I^{n-\alpha}_{a^+}u(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1}u(s)ds, \ t > a, \end{aligned}$$

where n is the smallest integer greater than or equal to α, provided that the right-hand side is pointwise defined. In particular, for $\alpha = n$, $D^{\alpha}_{a^+}u = D^n u$.

Lemma 1 ([3, 4])**.** *If* $D^{\alpha}_{a^+}u \in L^1(a,+\infty)$, *then*

$$I^{\alpha}_{a^+}D^{\alpha}_{a^+}u(t) = u(t) + \sum_{j=1}^{n} c_j(t-a)^{\alpha-j}$$

where $c_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$.

We define the spaces

$$X = \Big\{u \in C([0,+\infty),\mathbb{R}), \lim_{t\to+\infty} \frac{|u(t)|}{1+t^{\alpha-1}} = 0\Big\}$$

endowed with the norm $\|u\|_X = \sup\limits_{t\geq 0} \dfrac{|u(t)|}{1+t^{\alpha-1}}$,

$$X' = \{u \in C([0,+\infty),\mathbb{R}), \lim_{t\to+\infty} \frac{|u(t)|}{1+t^{\alpha-\beta-1}} = 0\}$$

endowed with the norm $\|u\|_{X'} = \sup\limits_{t\geq 0} \dfrac{|u(t)|}{1+t^{\alpha-\beta-1}}$ and

$$Y = \Big\{u \in X : D^{\beta}_{0^+}u \in C([0,+\infty),\mathbb{R}), \lim_{t\to+\infty} \frac{|D^{\beta}_{0^+}u(t)|}{1+t^{\alpha-\beta-1}} = 0\Big\}$$

endowed with the norm

$$\|u\|_Y = \sup_{t\geq 0} \frac{|u(t)|}{1+t^{\alpha-1}} + \sup_{t\geq 0} \frac{|D^{\beta}_{0^+}u(t)|}{1+t^{\alpha-\beta-1}}\,.$$

One can prove easily the following lemma.

Lemma 2. $(X, \|.\|_X)$ *and* $(Y, \|.\|_Y)$ *are Banach spaces.*

We need also the following lemma which is a compactness criterion called Corduneanu-like compactness criterion. Its proof is easy and similar to the classical one (see [1]).

Lemma 3. *Let* $Z \subseteq Y$ *be a bounded set. Then* Z *is relatively compact in* Y *if the following conditions hold:*

(i) $Z \subseteq Y$ *is equicontinuous on any compact interval of* $\mathbb{R}^+$, *i.e.,* $\forall J \subset [0,+\infty)$ *compact,* $\forall \varepsilon > 0$, $\exists\, \delta > 0$, $\forall t_1, t_2 \in J$,

$$|t_1 - t_2| < \delta \Rightarrow \Big|\frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}}\Big| < \varepsilon$$

and

$$\Big|\frac{D^{\beta}_{0^+}u(t_1)}{1+t_1^{\alpha-\beta-1}} - \frac{D^{\beta}_{0^+}u(t_2)}{1+t_2^{\alpha-\beta-1}}\Big| < \varepsilon,\ \forall u \in Z,$$

(ii) $Z \subseteq Y$ *is equiconvergent at* $(+\infty)$, *i.e.*, $\forall \varepsilon > 0$, $\exists T(\varepsilon) > 0$, *such that* $\forall t_1, t_2$, $t_1, t_2 \geq T(\varepsilon) \Rightarrow \left|\frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}}\right| < \varepsilon$ *and*

$$\left|\frac{D^{\beta}_{0^+}u(t_1)}{1+t_1^{\alpha-\beta-1}} - \frac{D^{\beta}_{0^+}u(t_2)}{1+t_2^{\alpha-\beta-1}}\right| < \varepsilon,\ \forall u \in Z.$$

Lemma 4 ([5]). *Let* $1 < \alpha \leq 2$ *and* $h \in L^1([0, +\infty), \mathbb{R})$. *Then the unique solution of*

$$\begin{cases} -D^{\alpha}_{0^+}u(t) = h(t), & t \in [0, +\infty), \\ u(0) = 0, \quad D^{\alpha-1}_{0^+}u(\infty) = 0, \end{cases} \tag{3}$$

is $u(t) = \int_0^{+\infty} G(t, s)h(s)ds$, *where*

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & s \leq t, \\ t^{\alpha-1}, & s \geq t. \end{cases} \tag{4}$$

By an easy computation, we obtain

$$D^{\beta}_{0^+}G_t(t, s) = \frac{1}{\Gamma(\alpha-\beta)} \begin{cases} t^{\alpha-\beta-1} - (t-s)^{\alpha-\beta-1}, & s \leq t, \\ t^{\alpha-\beta-1}, & s \geq t, \end{cases}$$

$G(t, s)$ and $D^{\beta}_{0^+}G_t(t, s)$ satisfy

$$\left|\frac{G(t, s)}{1+t^{\alpha-1}}\right| \leq \frac{1}{\Gamma(\alpha)}, \quad \text{for } t, s \in [0, +\infty)$$

and

$$\left|\frac{D^{\beta}_{0^+}G_t(t, s)}{1+t^{\alpha-\beta-1}}\right| \leq \frac{1}{\Gamma(\alpha-\beta)}, \quad \text{for } t, s \in [0, +\infty).$$

We define the operator T by

$$\begin{aligned} Tu(t) &= \int_0^{+\infty} G(t, s)a(s)g(u(s), D^{\beta}_{0^+}u(s))ds \\ &= \frac{\int_0^{+\infty} a(s)g(u(s), D^{\beta}_{0^+}u(s))ds}{\Gamma(\alpha)} t^{\alpha-1} \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}a(s)g(u(s), D^{\beta}_{0^+}u(s))ds. \end{aligned} \tag{5}$$

Then, we have

$$
\begin{aligned}
D_{0^+}^{\beta} Tu(t) &= \int_0^{+\infty} D_{0^+}^{\beta} G_t(t,s) a(s) g(u(s), D_{0^+}^{\beta} u(s)) ds \\
&= \frac{\int_0^{+\infty} a(s) g(u(s), D_{0^+}^{\beta} u(s)) ds}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} \\
&+ \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} a(s) g(u(s), D_{0^+}^{\beta} u(s)) ds.
\end{aligned} \tag{6}
$$

The integral equation (5) indicates that fixed points of the operator T coincide with the solutions of the problem (2).

Our arguments will be based on fixed point theory. So let us recall for the sake of completeness the following fixed point theorem.

Theorem 1 ([2, 6])**.** (Krasnoselskii's fixed point theorem) *Let E be a Banach space and let $P \subseteq E$ be a cone. Assume that Ω_1, Ω_2 are two open subsets of E with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ and let $T : P \cap (\overline{\Omega_2} - \Omega_1) \longrightarrow P$ be a completely continuous operator such that either*

(i) $\|Tu\|_E \leq \|u\|_E$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\|_E \geq \|u\|_E$ *for* $u \in P \cap \partial\Omega_2$,

or

(ii) $\|Tu\|_E \geq \|u\|_E$ *for* $u \in P \cap \partial\Omega_1$ *and* $\|Tu\|_E \leq \|u\|_E$ *for* $u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3. MAIN RESULTS

Let the following quantities be given

$$
\frac{G(t,s)}{1+t^{\alpha-1}} = \frac{1}{\Gamma(\alpha)} \begin{cases} h_1(t,s) = \frac{t^{\alpha-1}}{1+t^{\alpha-1}} - \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}, & s \leq t, \\ h_2(t,s) = \frac{t^{\alpha-1}}{1+t^{\alpha-1}}, & s \geq t, \end{cases} \tag{7}
$$

and

$$
\frac{D_{0^+}^{\beta} G_t(t,s)}{1+t^{\alpha-\beta-1}} = \frac{1}{\Gamma(\alpha-\beta)} \begin{cases} H_1(t,s) = \frac{t^{\alpha-\beta-1}}{1+t^{\alpha-\beta-1}} - \frac{(t-s)^{\alpha-\beta-1}}{1+t^{\alpha-\beta-1}}, & s \leq t, \\ H_2(t,s) = \frac{t^{\alpha-\beta-1}}{1+t^{\alpha-\beta-1}}, & s \geq t. \end{cases} \tag{8}
$$

The proof of the following two lemmas is easy.

Lemma 5. *The functions $h_1(t,s)$ and $h_2(t,s)$ defined in* (7) *satisfy*

1. $h_1(t,s)$ *is decreasing with respect to its first variable,*
2. $h_1(t,s)$ *is increasing with respect to the second variable,*
3. $h_2(t,s)$ *is increasing with respect to the first variable.*

Lemma 6. *The functions $H_1(t,s)$ and $H_2(t,s)$ defined in* (8) *satisfy*

1. $H_1(t,s)$ *is decreasing with respect to the first variable,*
2. $H_1(t,s)$ *is increasing with respect to the second variable,*
3. $H_2(t,s)$ *is increasing with respect to the first variable.*

Lemma 7. *The Green's function $G(t,s)$ satisfies the following property*

$$\min_{t\in[\frac{1}{\sigma},\sigma]} \frac{G(t,s)}{1+t^{\alpha-1}} \geq \theta_1 \sup_{t\geq 0} \frac{G(t,s)}{1+t^{\alpha-1}}$$

with

$$\theta_1 = \min\left\{\frac{\sigma^{2\alpha-2}-(\sigma^2-1)^{\alpha-1}}{\sigma^{\alpha-1}[1+\sigma^{\alpha-1}]}, \frac{1}{1+\sigma^{\alpha-1}}\right\} \text{ for some } \sigma > 1.$$

Proof. Indeed, we have

$$\begin{aligned}
\min_{t\in[\frac{1}{\sigma},\sigma]} h_1(t,s) &= h_1(\sigma,s) \\
&= \frac{\sigma^{\alpha-1}}{1+\sigma^{\alpha-1}} - \frac{(\sigma-s)^{\alpha-1}}{1+\sigma^{\alpha-1}} \\
&\geq \frac{\sigma^{\alpha-1}}{1+\sigma^{\alpha-1}} - \frac{(\sigma-\frac{1}{\sigma})^{\alpha-1}}{1+\sigma^{\alpha-1}} \\
&\geq \frac{1}{\Gamma(\alpha)} \cdot \frac{\sigma^{2\alpha-2}-(\sigma^2-1)^{\alpha-1}}{\sigma^{\alpha-1}[1+\sigma^{\alpha-1}]}.
\end{aligned}$$

Then $\min_{t\in[\frac{1}{\sigma},\sigma]} h_1(t,s) \geq \frac{1}{\Gamma(\alpha)} \cdot \frac{\sigma^{2\alpha-2}-(\sigma^2-1)^{\alpha-1}}{\sigma^{\alpha-1}[1+\sigma^{\alpha-1}]}$ and

$$\begin{aligned}
\min_{t\in[\frac{1}{\sigma},\sigma]} h_2(t,s) &= h_2(\frac{1}{\sigma},s) \\
&\geq \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{1+\sigma^{\alpha-1}}.
\end{aligned}$$

We obtain

$$\min_{t\in[\frac{1}{\sigma},\sigma]} \frac{G(t,s)}{1+t^{\alpha-1}} \geq \min\left\{\frac{\sigma^{2\alpha-2}-(\sigma^2-1)^{\alpha-1}}{\sigma^{\alpha-1}[1+\sigma^{\alpha-1}]}, \frac{1}{1+\sigma^{\alpha-1}}\right\}$$

and since we have

$$\sup_{t\geq 0} \frac{G(t,s)}{1+t^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha)},$$

then the desired result is obtained. ■

Lemma 8. *The Green's function $G(t,s)$ satisfies the following property*

$$\min_{t\in[\frac{1}{\sigma},\sigma]} \frac{D^{\beta}_{0^+}G_t(t,s)}{1+t^{\alpha-\beta-1}} \geq \theta_2 \sup_{t\geq 0} \frac{D^{\beta}_{0^+}G_t(t,s)}{1+t^{\alpha-\beta-1}}$$

with

$$\theta_2 = \min\left\{\frac{\sigma^{2\alpha-2\beta-2}-(\sigma^2-1)^{\alpha-\beta-1}}{\sigma^{\alpha-\beta-1}[1+\sigma^{\alpha-\beta-1}]}, \frac{1}{1+\sigma^{\alpha-\beta-1}}\right\}.$$

Proof. We have

$$\begin{aligned}
\min_{t\in[\frac{1}{\sigma},\sigma]} H_1(t,s) &= h_1(\sigma,s) \\
&= \frac{\sigma^{\alpha-\beta-1}}{1+\sigma^{\alpha-\beta-1}} - \frac{(\sigma-s)^{\alpha-\beta-1}}{1+\sigma^{\alpha-\beta-1}} \\
&\geq \frac{\sigma^{\alpha-\beta-1}}{1+\sigma^{\alpha-\beta-1}} - \frac{(\sigma-\frac{1}{\sigma})^{\alpha-\beta-1}}{1+\sigma^{\alpha-\beta-1}} \\
&\geq \frac{1}{\Gamma(\alpha-\beta)} \cdot \frac{\sigma^{2\alpha-2\beta-2}-(\sigma^2-1)^{\alpha-\beta-1}}{\sigma^{\alpha-\beta-1}[1+\sigma^{\alpha-\beta-1}]}.
\end{aligned}$$

Then $\min_{t\in[\frac{1}{\sigma},\sigma]} H_1(t,s) \geq \frac{1}{\Gamma(\alpha-\beta)} \cdot \frac{\sigma^{2\alpha-2\beta-2}-(\sigma^2-1)^{\alpha-\beta-1}}{\sigma^{\alpha-\beta-1}[1+\sigma^{\alpha-\beta-1}]}$, and

$$\begin{aligned}
\min_{t\in[\frac{1}{\sigma},\sigma]} H_2(t,s) &= H_2(\frac{1}{\sigma},s) \\
&\geq \frac{1}{\Gamma(\alpha-\beta)} \cdot \frac{1}{1+\sigma^{\alpha-\beta-1}}.
\end{aligned}$$

We obtain

$$\min_{t\in[\frac{1}{\sigma},\sigma]} \frac{D^{\beta}_{0^+}G_t(t,s)}{1+t^{\alpha-\beta-1}} \geq \min\left\{\frac{\sigma^{2\alpha-2\beta-2}-(\sigma^2-1)^{\alpha-\beta-1}}{\sigma^{\alpha-\beta-1}[1+\sigma^{\alpha-\beta-1}]}, \frac{1}{1+\sigma^{\alpha-\beta-1}}\right\},$$

and since we have

$$\sup_{t\geq 0}\frac{D^{\beta}_{0^+}G_t(t,s)}{1+t^{\alpha-\beta-1}}\leq\frac{1}{\Gamma(\alpha-\beta)},$$

then the desired result is obtained. ■

Define the cone P by

$$P=\Big\{u\in Y,\ u(t)\geq 0,\ D^{\beta}_{0^+}u(t)\geq 0,\ t\in\mathbb{R}^+,\ \frac{u(t)}{1+t^{\alpha-1}}\geq\theta_1\|u\|_X,\frac{D^{\beta}_{0^+}u(t)}{1+t^{\alpha-\beta-1}}$$
$$\geq\theta_2\|D^{\beta}_{0^+}u\|_{X'},\ \forall\, t\in[\frac{1}{\sigma},\sigma]\Big\}.$$

We remark that $T(P)\subset P$. Indeed, we have

$$\begin{aligned}
\min_{t\in[\frac{1}{\sigma},\sigma]}\frac{Tu(t)}{1+t^{\alpha-1}} &= \min_{t\in[\frac{1}{\sigma},\sigma]}\int_0^{+\infty}\frac{G(t,s)}{1+t^{\alpha-1}}a(s)g(u(s),D^{\beta}_{0^+}u(s))ds\\
&= \int_0^{+\infty}\min_{t\in[\frac{1}{\sigma},\sigma]}\frac{G(t,s)}{1+t^{\alpha-1}}a(s)g(u(s),D^{\beta}_{0^+}u(s))ds\\
&\geq \int_0^{+\infty}\theta_1\sup_{t\geq 0}\frac{G(t,s)}{1+t^{\alpha-1}}a(s)g(u(s),D^{\beta}_{0^+}u(s))ds\\
&\geq \theta_1\sup_{t\geq 0}\frac{1}{1+t^{\alpha-1}}\int_0^{+\infty}G(t,s)a(s)g(u(s),D^{\beta}_{0^+}u(s))ds\\
&\geq \theta_1\sup_{t\geq 0}\frac{\int_0^{+\infty}G(t,s)a(s)g(u(s),D^{\beta}_{0^+}u(s))ds}{1+t^{\alpha-1}}.
\end{aligned}$$

Then

$$\min_{t\in[\frac{1}{\sigma},\sigma]}\frac{Tu(t)}{1+t^{\alpha-1}}\geq\theta_1\|Tu\|_X$$

and

$$\begin{aligned}
\min_{t\in[\frac{1}{\sigma},\sigma]}\frac{D^{\beta}_{0^+}Tu(t)}{1+t^{\alpha-\beta-1}} &= \min_{t\in[\frac{1}{\sigma},\sigma]}\int_0^{+\infty}\frac{D^{\beta}_{0^+}G_t(t,s)}{1+t^{\alpha-\beta-1}}a(s)g(u(s),D^{\beta}_{0^+}u(s))ds\\
&= \int_0^{+\infty}\min_{t\in[\frac{1}{\sigma},\sigma]}\frac{D^{\beta}_{0^+}G_t(t,s)}{1+t^{\alpha-\beta-1}}a(s)g(u(s),D^{\beta}_{0^+}u(s))ds
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_0^{+\infty} \theta_2 \sup_{t\geq 0} \frac{1}{1+t^{\alpha-\beta-1}} \int_0^{+\infty} D_{0^+}^{\beta} G_t(t,s) a(s) g(u(s), D_{0^+}^{\beta} u(s)) ds \\
&\geq \theta_2 \sup_{t\geq 0} \frac{\int_0^{+\infty} D_{0^+}^{\beta} G_t(t,s) a(s) g(u(s), D_{0^+}^{\beta} u(s)) ds}{1+t^{\alpha-\beta-1}}.
\end{aligned}
$$

Therefore,

$$
\min_{t\in[\frac{1}{\sigma},\sigma]} \frac{D_{0^+}^{\beta} Tu(t)}{1+t^{\alpha-\beta-1}} \geq \theta_2 \|D_{0^+}^{\beta} Tu\|_{X'}.
$$

In the sequel, we put

$$
M = \Big(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-\beta)}\Big) \int_0^{+\infty} a(s)(1+s^{\alpha-1})^{\eta} ds,
$$

$$
N = \theta(\theta_1+\theta_2) \int_{\frac{1}{\sigma}}^{\sigma} a(s) ds,
$$

where $\theta = \min(\theta_1, \theta_2)$.

Proposition 1. *The operator T defined in* (5) *is completely continuous.*

Proof. Let V a bounded set in Y, i.e., $V \subset \{u \in Y; \|u\|_Y < \mu\}$ for some $\mu > 0$.

1. $T(V)$ is uniformly bounded. Indeed, we have

$$
\begin{aligned}
\Big|\frac{Tu(t)}{1+t^{\alpha-1}}\Big| &= \Big|\int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) g(u(s), D_{0^+}^{\beta} u(s)) ds\Big| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} |a(s)||g(u(s), D_{0^+}^{\beta} u(s)) ds| \\
&\leq \frac{S_\mu}{\Gamma(\alpha)} \int_0^{+\infty} a(s)(1+s^{\alpha-1})^{\eta} ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\Big|\frac{D_{0^+}^{\beta} Tu(t)}{1+t^{\alpha-\beta-1}}\Big| &= \Big|\frac{D_{0^+}^{\beta} G_t(t,s)}{1+t^{\alpha-\beta-1}} a(s) g(u(s), D_{0^+}^{\beta} u(s)) ds\Big| \\
&\leq \frac{1}{\Gamma(\alpha-\beta)} \int_0^{+\infty} |a(s)||g(u(s), D_{0^+}^{\beta} u(s)) ds| \\
&\leq \frac{S_\mu}{\Gamma(\alpha-\beta)} \int_0^{+\infty} a(s)(1+s^{\alpha-1})^{\eta} ds.
\end{aligned}
$$

Then

$$\|Tu\|_Y \leq S_\mu\Big(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-\beta)}\Big)\int_0^{+\infty} a(s)(1+s^{\alpha-1})^\eta ds = S_\mu M.$$

Thus we have

$$\|Tu\|_Y \leq S_\mu M, \ \ \forall u \in V.$$

2. $T(V)$ is equicontinuous.
Indeed, let $J\subset[0,+\infty)$ be a compact interval, $t_1,t_2\in J$ with $t_1<t_2$. Then for any $u\in V$, we have

$$\begin{aligned}\Big|\frac{Tu(t_2)}{1+t_2^{\alpha-1}}-\frac{Tu(t_1)}{1+t_1^{\alpha-1}}\Big| &\leq \int_0^{+\infty}\Big|\frac{G(t_2,s)}{1+t_2^{\alpha-1}}-\frac{G(t_1,s)}{1+t_1^{\alpha-1}}\Big||a(s)g(u(s),D_{0+}^\beta u(s))|ds\\ &\leq S_\mu\int_0^{+\infty}\Big|\frac{G(t_2,s)}{1+t_2^{\alpha-1}}-\frac{G(t_1,s)}{1+t_1^{\alpha-1}}\Big||a(s)|(1+s^{\alpha-1})^\eta ds\end{aligned}$$

and

$$\begin{aligned}&\Big|\frac{D_{0+}^\beta Tu(t_2)}{1+t_2^{\alpha-\beta-1}}-\frac{D_{0+}^\beta Tu(t_1)}{1+t_1^{\alpha-\beta-1}}\Big|\\ &\leq \int_0^{+\infty}\Big|\frac{D_{0+}^\beta G_t(t_2,s)}{1+t_2^{\alpha-\beta-1}}-\frac{D_{0+}^\beta G_t(t_1,s)}{1+t_1^{\alpha-\beta-1}}\Big||a(s)g(u(s),D_{0+}^\beta u(s))|ds\\ &\leq S_\mu\int_0^{+\infty}\Big|\frac{D_{0+}^\beta G_t(t_2,s)}{1+t_2^{\alpha-\beta-1}}-\frac{D_{0+}^\beta G_t(t_1,s)}{1+t_1^{\alpha-\beta-1}}\Big||a(s)|(1+s^{\alpha-1})^\eta ds\,.\end{aligned}$$

The Lebesgue's dominated convergence theorem asserts the equicontinuous of T.

3. $T(V)$ is equiconvergent.
Using Lebesgue's dominated theorem, we obtain $\lim_{t\to+\infty}\frac{Tu(t)}{1+t^{\alpha-1}}=0$ and $\lim_{t\to+\infty}\frac{D_{0+}^\beta Tu(t)}{1+t^{\alpha-\beta-1}}=0$. We have also

$$\int_0^{+\infty}|a(s)||g(u(s),D_{0+}^\beta u(s))|ds \leq S_\mu\int_0^{+\infty}a(s)(1+s^{\alpha-1})^\eta ds<+\infty.$$

We know that for a given $\varepsilon>0$, there exists a constant $L>0$ such that

$$\int_L^{+\infty}|a(s)g(u(s),D_{0+}^\beta u(s))|ds<\varepsilon. \tag{9}$$

On the other hand, since $\lim\limits_{t\to+\infty}\frac{t^{\alpha-1}}{1+t^{\alpha-1}}=1$, there exists a constant $T_1>0$ such that for any $t_1,t_2\geq T_1$, we have

$$\Big|\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}\Big|\leq\Big|1-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\Big|+\Big|1-\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}\Big|<\varepsilon. \tag{10}$$

Similarly, $\lim\limits_{t\to+\infty}\frac{(t-L)^{\alpha-1}}{1+t^{\alpha-1}}=1$ and thus, there exists a constant $T_2>L>0$ such that for all $t_1,t_2\geq T_2$ and $0\leq s\leq L$, we have

$$\begin{aligned}\Big|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\Big|&\leq\Big(1-\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\Big)+\Big(1-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\Big)\\&\leq\Big(1-\frac{(t_1-L)^{\alpha-1}}{1+t_1^{\alpha-1}}\Big)+\Big(1-\frac{(t_2-L)^{\alpha-1}}{1+t_2^{\alpha-1}}\Big)<\varepsilon.\end{aligned} \tag{11}$$

Now choose $T_3>\max\{T_1,T_2\}$; then for $t_1,t_2\geq T_3$ and by (9)–(11) we obtain

$$\begin{aligned}&\Big|\frac{Tu(t_2)}{1+t_2^{\alpha-1}}-\frac{Tu(t_1)}{1+t_1^{\alpha-1}}\Big|\\&=\frac{1}{\Gamma(\alpha)}\Big|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}a(s)g(u(s),D_{0+}^{\beta}u(s))ds\\&-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}a(s)g(u(s),D_{0+}^{\beta}u(s))ds\Big|\\&\leq\frac{1}{\Gamma(\alpha)}\int_0^{L}\Big|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\Big||a(s)g(u(s),D_{0+}^{\beta}u(s))|ds\\&+\frac{1}{\Gamma(\alpha)}\int_L^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}|a(s)g(u(s),D_{0+}^{\beta}u(s))|ds\\&+\frac{1}{\Gamma(\alpha)}\int_L^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}|a(s)g(u(s),D_{0+}^{\beta}u(s))|ds\\&\leq\frac{\max\limits_{t\in[0,L],u\in V}|a(t)g(u(t),D_{0+}^{\beta}u(t))|}{\Gamma(\alpha)}\int_0^L\Big|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\Big|ds\\&+\frac{2}{\Gamma(\alpha)}\int_L^{+\infty}|a(t)g(u(t),D_{0+}^{\beta}u(t))|dt\\&<\frac{2}{\Gamma(\alpha)}\varepsilon+\frac{\max\limits_{t\in[0,L],u\in V}|a(t)g(u(t),D_{0+}^{\beta}u(t))|}{\Gamma(\alpha)}L\varepsilon,\ \forall\,u\in V.\end{aligned}$$

On the other hand, since $\lim\limits_{t\to+\infty}\frac{t^{\alpha-\beta-1}}{1+t^{\alpha-\beta-1}}=1$, there exists a constant $T_1'>0$ such that for any $t_1,t_2\geq T_1'$, we have

$$\Big|\frac{t_1^{\alpha-\beta-1}}{1+t_1^{\alpha-\beta-1}}-\frac{t_2^{\alpha-\beta-1}}{1+t_2^{\alpha-\beta-1}}\Big|\leq\Big|1-\frac{t_1^{\alpha-\beta-1}}{1+t_1^{\alpha-\beta-1}}\Big|+\Big|1-\frac{t_2^{\alpha-\beta-1}}{1+t_2^{\alpha-\beta-1}}\Big|<\varepsilon. \tag{12}$$

Similarly, $\lim\limits_{t\to+\infty}\frac{(t-L)^{\alpha-\beta-1}}{1+t^{\alpha-\beta-1}}=1$ and thus there exists a constant $T_2'>L'>0$ such that for all $t_1,t_2\geq T_2'$ and $0\leq s\leq L'$, we have

$$\begin{aligned}\Big|\frac{(t_1-s)^{\alpha-\beta-1}}{1+t_1^{\alpha-\beta-1}}-\frac{(t_2-s)^{\alpha-\beta-1}}{1+t_2^{\alpha-\beta-1}}\Big| &\leq\Big(1-\frac{(t_1-s)^{\alpha-\beta-1}}{1+t_1^{\alpha-\beta-1}}\Big)+\Big(1-\frac{(t_2-s)^{\alpha-\beta-1}}{1+t_2^{\alpha-\beta-1}}\Big)\\ &\leq\Big(1-\frac{(t_1-L)^{\alpha-\beta-1}}{1+t_1^{\alpha-\beta-1}}\Big)+\Big(1-\frac{(t_2-L)^{\alpha-\beta-1}}{1+t_2^{\alpha-\beta-1}}\Big)\\ &<\varepsilon.\end{aligned} \tag{13}$$

Choose $T_3'>\max\{T_1',T_2'\}$. Then for $t_1,t_2\geq T_3'$ and by (9)–(13) we obtain

$$\begin{aligned}&\Big|\frac{D_{0^+}^{\beta}Tu(t_2)}{1+t_2^{\alpha-\beta-1}}-\frac{D_{0^+}^{\beta}Tu(t_1)}{1+t_1^{\alpha-\beta-1}}\Big|\\ =&\ \frac{1}{\Gamma(\alpha)}\Big|\int_0^{t_2}\frac{(t_2-s)^{\alpha-\beta-1}}{1+t_2^{\alpha-\beta-1}}a(s)g(u(s),D_{0^+}^{\beta}u(s))ds\\ -&\int_0^{t_1}\frac{(t_1-s)^{\alpha-\beta-1}}{1+t_1^{\alpha-\beta-1}}a(s)g(u(s),D_{0^+}^{\beta}u(s))ds\Big|\\ +&\ \frac{1}{\Gamma(\alpha)}\int_0^{L'}\Big|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\Big||a(s)g(u(s),D_{0^+}^{\beta}u(s))|ds\\ +&\ \frac{1}{\Gamma(\alpha)}\int_{L'}^{t_1}\frac{(t_1-s)^{\alpha-\beta-1}}{1+t_1^{\alpha-\beta-1}}|a(s)g(u(s),D_{0^+}^{\beta}u(s))|ds\\ +&\ \frac{1}{\Gamma(\alpha)}\int_{L'}^{t_2}\frac{(t_2-s)^{\alpha-\beta-1}}{1+t_2^{\alpha-\beta-1}}|a(s)g(u(s),D_{0^+}^{\beta}u(s))|ds\\ \leq&\ \frac{\max\limits_{t\in[0,L'],u\in V}|a(t)g(u(t),D_{0^+}^{\beta}u(t))|}{\Gamma(\alpha)}\int_0^{L'}\Big|\frac{t_2^{\alpha-\beta-1}}{1+t_2^{\alpha-\beta-1}}\\ -&\ \frac{t_1^{\alpha-\beta-1}}{1+t_1^{\alpha-\beta-1}}\Big|ds+\frac{2}{\Gamma(\alpha)}\int_{L'}^{+\infty}|a(t)g(u(t),D_{0^+}^{\beta}u(t))|dt\\ \leq&\ \frac{2}{\Gamma(\alpha)}\varepsilon+\frac{\max\limits_{t\in[0,L'],u\in V}|a(t)g(u(t),D_{0^+}^{\beta}u(t))|}{\Gamma(\alpha)}L'\varepsilon,\ \forall\,u\in V.\end{aligned}$$

The Lebesgue's dominated convergence theorem asserts the equiconvergency of $T(V)$. Therefore, Lemma 3 ensures that the set $A(V)$ is relatively compact.

Claim. *Operator T is continuous.*

Let $u_n, u \in P \cap (\overline{\Omega_2} - \Omega_1)$ with $u_n \to u$, $\|u_n\| \leq C$ and $\|u\| \leq C \ \ \forall\, n \in \mathbb{N}^*$. For each $t \in [0, +\infty)$, we have

$$\begin{aligned}
\Big|\frac{Tu_n(t)}{1+t^{\alpha-1}} - \frac{Tu(t)}{1+t^{\alpha-1}}\Big| &\leq \int_0^{+\infty} \Big|\frac{G(t,s)}{1+t^{\alpha-1}}\Big| |a(s) g(u_n(s), D^{\beta}_{0^+} u_n(s))| ds \\
&+ \int_0^{+\infty} \Big|\frac{G(t,s)}{1+t^{\alpha-1}}\Big| |a(s) g(u(s), D^{\beta}_{0^+} u(s))| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} |a(s)| \Big| g\Big((1+s^{\alpha-1})\frac{u_n(s)}{1+s^{\alpha-1}}, \\
&(1+s^{\alpha-\beta-1})\frac{D^{\beta}_{0^+} u_n(s)}{1+s^{\alpha-\beta-1}}\Big)\Big| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} |a(s)| \Big| g\Big((1+s^{\alpha-1})\frac{u(s)}{(1+s^{\alpha-1})}, \\
&(1+s^{\alpha-\beta-1})\frac{D^{\beta}_{0^+} u(s)}{1+s^{\alpha-\beta-1}}\Big)\Big| \\
&\leq \frac{2S_\mu}{\Gamma(\alpha)} \int_0^{+\infty} a(s)(1+s^{\alpha-1})^{\eta} ds < +\infty,
\end{aligned}$$

and

$$\begin{aligned}
\Big|\frac{D^{\beta}_{0^+} Tu_n(t)}{1+t^{\alpha-\beta-1}} - \frac{D^{\beta}_{0^+} Tu(t)}{1+t^{\alpha-\beta-1}}\Big| &\leq \int_0^{+\infty} \Big|\frac{D^{\beta}_{0^+} G_t(t,s)}{1+t^{\alpha-\beta-1}}\Big| |a(s) g(u_n(s), D^{\beta}_{0^+} u_n(s))| ds \\
&+ \int_0^{+\infty} \Big|\frac{D^{\beta}_{0^+} G_t(t,s)}{1+t^{\alpha-\beta-1}}\Big| |a(s) g(u(s), D^{\beta}_{0^+} u(s))| ds \\
&\leq \frac{1}{\Gamma(\alpha-\beta)} \int_0^{+\infty} |a(s)| \Big| g\Big(1+s^{\alpha-1})\frac{u_n(s)}{1+s^{\alpha-1}}, \\
&(1+s^{\alpha-\beta-1})\frac{D^{\beta}_{0^+} u_n(s)}{1+s^{\alpha-\beta-1}}\Big)\Big| ds \\
&+ \frac{1}{\Gamma(\alpha-\beta)} \int_0^{+\infty} |a(s)| |g(1+s^{\alpha-1})\frac{u(s)}{(1+s^{\alpha-1}}, \\
&(1+s^{\alpha-\beta-1})\frac{D^{\beta}_{0^+} u(s)}{1+s^{\alpha-\beta-1}})| ds \\
&\leq \frac{2S_\mu}{\Gamma(\alpha-\beta)} \int_0^{+\infty} a(s)(1+s^{\alpha-1})^{\eta} ds < +\infty.
\end{aligned}$$

By the Lebesgue dominated convergence theorem, we conclude that the operator T is continuous. ■

Theorem 2. *Assume that the conditions*

(a_1) $\limsup\limits_{x+y\to 0^+} \frac{1}{(1+t^{\alpha-1})^\eta} \cdot \frac{g((1+t^{\alpha-1})x,(1+t^{\alpha-\beta-1})y)}{x+y} = 0$, *uniformly with respect to* $t \in \mathbb{R}^+$,

(a_2) $\liminf\limits_{x+y\to+\infty} \frac{g(x,y)}{x+y} = +\infty$

hold. Then the boundary value problem (2) *has at least one positive solution.*

Proof. The proof is based on the Krasnoselskii's fixed point Theorem 1. From the condition (a_1), it follows that there exists $R_1 > 0$ such that $\frac{1}{(1+t^{\alpha-1})^\eta} g((1+t^{\alpha-1})x,(1+t^{\alpha-\beta-1})y) \le \varepsilon(x+y)$ with $x+y \le R_1$ for all $t \in [0,+\infty[$ and for some $\varepsilon > 0$. Then for $u \in P \cap \partial\Omega_1$ with $\Omega_1 = \{u \in Y : \|u\|_Y < R_1\}$, we have

$$
\begin{aligned}
&\frac{Tu(t)}{1+t^{\alpha-1}}\\
&= \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) g(u(s), D_{0^+}^\beta u(s)) ds\\
&= \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) \frac{(1+s^{\alpha-1})^\eta}{(1+s^{\alpha-1})^\eta} g\Big(\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}, \frac{(1+s^{\alpha-\beta-1})D_{0^+}^\beta u(s)}{1+s^{\alpha-\beta-1}}\Big) ds\\
&\le \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} a(s)\varepsilon(1+s^{\alpha-1})^\eta \Big(\frac{u(s)}{1+s^{\alpha-1}} + \frac{D_{0^+}^\beta u(s)}{1+s^{\alpha-\beta-1}}\Big) ds\\
&\le \varepsilon\Big[\frac{1}{\Gamma(\alpha)} \int_0^{+\infty} a(s)(1+s^{\alpha-1})^\eta ds\Big] \|u\|_Y.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\frac{D_{0^+}^\beta Tu(t)}{1+t^{\alpha-\beta-1}}\\
&= \int_0^{+\infty} \frac{D_{0^+}^\beta G_t(t,s)}{1+t^{\alpha-\beta-1}} a(s) g(u(s), D_{0^+}^\beta u(s)) ds\\
&= \int_0^{+\infty} \frac{D_{0^+}^\beta G_t(t,s)}{1+t^{\alpha-\beta-1}} a(s) \frac{(1+s^{\alpha-1})^\eta}{(1+s^{\alpha-1})^\eta} g\Big(\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}, \frac{(1+s^{\alpha-\beta-1})D_{0^+}^\beta u(s)}{1+s^{\alpha-\beta-1}}\Big) ds
\end{aligned}
$$

$$\leq \frac{1}{\Gamma(\alpha-\beta)} \int_0^{+\infty} a(s)\varepsilon(1+s^{\alpha-1})^{\eta}\Big(\frac{u(s)}{1+s^{\alpha-1}} + \frac{D_{0^+}^{\beta}u(s)}{1+s^{\alpha-\beta-1}}\Big)ds$$
$$\leq \varepsilon\Big[\frac{1}{\Gamma(\alpha-\beta)} \int_0^{+\infty} a(s)(1+s^{\alpha-1})^{\eta}ds\Big]\|u\|_Y.$$

If we choose ε such that $\varepsilon M \leq 1$, then $\|Tu\|_Y \leq \|u\|_Y$.

On the other hand, by condition (a_2), there exists $\overline{R_2} > 0$, such that $g(x,y) \geq \delta(x+y)$ with $x+y \geq \overline{R_2}$ and $\delta > 0$. Then for $u \in P \cap \partial\Omega_2$, where $\Omega_2 = \{u \in Y : \|u\|_Y < R_2\}$ with $R_2 = \max\{2R_1, \frac{\overline{R_2}}{\theta}\}$, we have

$$\begin{aligned} u(t) + D_{0^+}^{\beta}u(t) &\geq \frac{u(t)}{1+t^{\alpha-1}} + \frac{D_{0^+}^{\beta}u(t)}{1+t^{\alpha-\beta-1}} \\ &\geq \theta_1\|u\|_X + \theta_2\|D_{0^+}^{\beta}u\|_{X'} \\ &\geq \theta(\|u\|_X + \|D_{0^+}^{\beta}u\|_{X'}) \\ &\geq \theta R_2 \geq \overline{R_2}, \ \ \forall \ t \in [\frac{1}{\sigma}, \sigma] \end{aligned} \tag{14}$$

with $\theta = \min(\theta_1, \theta_2)$. We obtain

$$\begin{aligned} \frac{Tu(t)}{1+t^{\alpha-1}} &= \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s)g(u(s), D_{0^+}^{\beta}u(s))ds \\ &\geq \int_{\frac{1}{\sigma}}^{\sigma} \frac{G(t,s)}{1+t^{\alpha-1}} a(s)g(u(s), D_{0^+}^{\beta}u(s))ds \\ &\geq \int_{\frac{1}{\sigma}}^{\sigma} \frac{G(t,s)}{1+t^{\alpha-1}} a(s)\delta(u(s) + D_{0^+}^{\beta}u(s))ds \\ &\geq \int_{\frac{1}{\sigma}}^{\sigma} \frac{G(t,s)}{1+t^{\alpha-1}} a(s)\delta\Big(\frac{u(s)}{1+s^{\alpha-1}} + \frac{D_{0^+}^{\beta}u(s)}{1+s^{\alpha-\beta-1}}\Big)ds \\ &\geq \delta\Big[\theta \int_{\frac{1}{\sigma}}^{\sigma} \theta_1 a(s)ds\Big]\|u\|_Y. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \frac{D_{0^+}^{\beta}Tu(t)}{1+t^{\alpha-\beta-1}} &= \int_0^{+\infty} \frac{D_{0^+}^{\beta}G_t(t,s)}{1+t^{\alpha-\beta-1}} a(s)g(u(s), D_{0^+}^{\beta}u(s))ds \\ &\geq \int_{\frac{1}{\sigma}}^{\sigma} \frac{D_{0^+}^{\beta}G_t(t,s)}{1+t^{\alpha-\beta-1}} a(s)g(u(s), D_{0^+}^{\beta}u(s))ds \end{aligned}$$

$$\begin{aligned}&\geq \int_{\frac{1}{\sigma}}^{\sigma} \frac{D_{0^+}^{\beta} G_t(t,s)}{1+t^{\alpha-\beta-1}} a(s)\delta(u(s)+D_{0^+}^{\beta}u(s))ds\\ &\geq \int_{\frac{1}{\sigma}}^{\sigma} \frac{D_{0^+}^{\beta} G_t(t,s)}{1+t^{\alpha-\beta-1}} a(s)\delta\Big(\frac{u(s)}{1+s^{\alpha-1}}+\frac{D_{0^+}^{\beta}u(s)}{1+s^{\alpha-\beta-1}}\Big)ds\\ &\geq \delta\Big[\theta\int_{\frac{1}{\sigma}}^{\sigma}\theta_2 a(s)ds\Big]\|u\|_Y.\end{aligned}$$

If we choose δ such that $\delta N \geq 1$, then $\|Tu\|_Y \geq \|u\|_Y$. Therefore, by Theorem 1, the operator T has at least one fixed point which is a positive solution of the boundary value problem (2). ■

Example 1. Let the following problem be given

$$(15)\qquad \begin{cases} -D_{0^+}^{\frac{3}{2}}u(t)=\frac{1}{(1+t^{\frac{1}{2}})^{\eta}}\cdot\frac{1}{1+t^2}(u^{\alpha_1}(t)+(D_{0^+}^{\frac{1}{4}}u(t))^{\alpha_1}), \quad t\in[0,+\infty),\\ u(0)=D_{0^+}^{\frac{1}{2}}u(\infty)=0,\end{cases}$$

where $a(t)=\frac{1}{(1+t^{\frac{1}{2}})^{\eta}}\cdot\frac{1}{1+t^2}$ and $g(x,y)=x^{\alpha_1}+y^{\alpha_1}$. If $\eta\geq\alpha_1>1$, then the hypotheses of Theorem 2 are satisfied and the boundary value problem (15) has at least one positive solution.

Theorem 3. *Assume that the conditions*

(b_1) *there exists functions* $p,q:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ *such that*

$$g((1+t^{\alpha-1})x,(1+t^{\alpha-\beta-1})y)\leq p(t)|x|+q(t)|y|$$

with $(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-\beta)})\int_0^{+\infty}a(s)[p(s)+q(s)]ds\leq 1, \forall\, x,y\in\mathbb{R}^+, x+y\in[0,R_2]$ *for some* $R_2>0$,

(b_2) $\liminf\limits_{x+y\to 0^+}\dfrac{g((1+t^{\alpha-1})x,(1+t^{\alpha-\beta-1})y)}{x+y}=+\infty$, *uniformly with respect to* $t\in[\frac{1}{\sigma},\sigma]$

hold. Then the boundary value problem (2) *has at least one positive solution.*

Proof. Let $\Omega_2=\{u\in Y:\|u\|_Y<R_2\}$. Then for $u\in P\cap\partial\Omega_2$ we have

$$\begin{aligned}\frac{Tu(t)}{1+t^{\alpha-1}} &= \int_0^{+\infty}\frac{G(t,s)}{1+t^{\alpha-1}}a(s)g(u(s),D_{0^+}^{\beta}u(s))ds\\ &= \int_0^{+\infty}\frac{G(t,s)}{1+s^{\alpha-1}}a(s)g\Big(\frac{(1+s^{\alpha-1})u(s)}{(1+s^{\alpha-1})},\frac{(1+s^{\alpha-\beta-1})D_{0^+}^{\beta}u(s)}{(1+s^{\alpha-\beta-1})}\Big)ds\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} a(s)\Big[p(s)\frac{u(s)}{1+s^{\alpha-1}}+q(s)\frac{D_{0^+}^{\beta}u(s)}{1+s^{\alpha-\beta-1}}\Big]ds\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} a(s)[p(s)\|u\|_Y+q(s)\|u\|_Y]ds\\
&\leq \Big(\frac{1}{\Gamma(\alpha)}\int_0^{+\infty} a(s)[p(s)+q(s)]ds\Big)\|u\|_Y.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\frac{D_{0^+}^{\beta}Tu(t)}{1+t^{\alpha-\beta-1}} &= \int_0^{+\infty}\frac{D_{0^+}^{\beta}G_t(t,s)}{1+t^{\alpha-\beta-1}}a(s)g(u(s),D_{0^+}^{\beta}u(s))ds\\
&= \int_0^{+\infty}\frac{D_{0^+}^{\beta}G_t(t,s)}{1+s^{\alpha-\beta-1}}a(s)g\Big(\frac{(1+s^{\alpha-1})u(s)}{(1+t^{\alpha-1})},\frac{(1+s^{\alpha-\beta-1})D_{0^+}^{\beta}u(s)}{(1+s^{\alpha-\beta-1})}\Big)ds\\
&\leq \frac{1}{\Gamma(\alpha-\beta)}\int_0^{+\infty} a(s)\Big[p(s)\frac{u(s)}{1+s^{\alpha-1}}+q(s)\frac{D_{0^+}^{\beta}u(s)}{1+s^{\alpha-\beta-1}}\Big]ds\\
&\leq \frac{1}{\Gamma(\alpha-\beta)}\int_0^{+\infty} a(s)[p(s)\|u\|_Y+q(s)\|u\|_Y]ds\\
&\leq \Big(\frac{1}{\Gamma(\alpha-\beta)}\int_0^{+\infty} a(s)[p(s)+q(s)]ds\Big)\|u\|_Y.
\end{aligned}
$$

Therefore, $\|Tu\|_Y \leq \|u\|_Y$. From (b_2) it follows that there exists $R_1 > 0$ such that $R_1 < R_2$ and $g((1+t^{\alpha-1})x,(1+t^{\alpha-\beta-1})y) > \xi(x+y)$ with $x+y \leq R_1$ for all $t \in [\frac{1}{\sigma},\sigma]$ and $\xi > 0$. Then for $u \in P\cap\partial\Omega_1$ with $\Omega_1 = \{u \in Y : \|u\|_Y < R_1\}$, we obtain

$$
\begin{aligned}
\frac{Tu(t)}{1+t^{\alpha-1}} &= \int_0^{+\infty}\frac{G(t,s)}{1+t^{\alpha-1}}a(s)g(u(s),D_{0^+}^{\beta}u(s))ds\\
&\geq \int_{\frac{1}{\sigma}}^{\sigma}\frac{G(t,s)}{1+t^{\alpha-1}}a(s)g\Big(\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}},\frac{(1+s^{\alpha-\beta-1})D_{0^+}^{\beta}u(s)}{1+s^{\alpha-\beta-1}}\Big)ds\\
&\geq \int_{\frac{1}{\sigma}}^{\sigma}\frac{G(t,s)}{1+t^{\alpha-1}}a(s)\xi\Big(\frac{u(s)}{1+s^{\alpha-1}}+\frac{D_{0^+}^{\beta}u(s)}{1+s^{\alpha-\beta-1}}\Big)ds\\
&\geq \xi\Big[\theta\int_{\frac{1}{\sigma}}^{\sigma}\theta_1 a(s)ds\Big]\|u\|_Y.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\frac{D_{0^+}^{\beta}Tu(t)}{1+t^{\alpha-\beta-1}} &= \int_0^{+\infty}\frac{D_{0^+}^{\beta}G_t(t,s)}{1+t^{\alpha-\beta-1}}a(s)g(u(s),D_{0^+}^{\beta}u(s))ds\\
&\geq \int_{\frac{1}{\sigma}}^{\sigma}\frac{D_{0^+}^{\beta}G_t(t,s)}{1+t^{\alpha-\beta-1}}a(s)g\Big(\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}},\frac{(1+s^{\alpha-\beta-1})D_{0^+}^{\beta}u(s)}{1+s^{\alpha-\beta-1}}\Big)ds
\end{aligned}
$$

$$\begin{aligned}&\geq \int_{\frac{1}{\sigma}}^{\sigma} \frac{D_{0^+}^{\beta} G_t(t,s)}{1+t^{\alpha-\beta-1}} a(s)\xi\Big(\frac{u(s)}{1+s^{\alpha-1}}+\frac{D_{0^+}^{\beta}u(s)}{1+s^{\alpha-\beta-1}}\Big)ds\\ &\geq \xi\Big[\theta\int_{\frac{1}{\sigma}}^{\sigma}\theta_2 a(s)ds\Big]\|u\|_Y.\end{aligned}$$

We choose ξ such that $\xi N \geq 1$, therefore $\|Tu\|_Y \geq \|u\|_Y$. By Theorem 1, the operator T has at least one fixed point, which is a positive solution of the boundary value problem (2). ■

Example 2. Let the following problem be given

$$(16)\qquad \begin{cases} -D_{0^+}^{\frac{3}{2}}u(t)=\frac{1}{3}\frac{e^{-t}}{1+t^2}(|u^{\alpha_2}(t)|+|D_{0^+}^{\frac{1}{4}}u(t)|^{\alpha_2}), & t\in[0,+\infty),\\ u(0)=D_{0^+}^{\frac{1}{2}}u(\infty)=0,\end{cases}$$

where $\alpha=\frac{3}{2},\beta=\frac{1}{4},a(t)=\frac{1}{3}\frac{e^{-t}}{1+t^2}$ and $g(x,y)=x^{\alpha_2}+y^{\alpha_2}$, for all $x,y\in\mathbb{R}^+$ with $0<\alpha_2<1$ and $\eta\geq\alpha_2$. Then the condition $(b2)$ of Theorem 3 is satisfied. By the condition $(b1)$, we obtain

$$\begin{aligned} g((1+t^{\frac{1}{2}})x,(1+t^{\frac{1}{4}})y) &= (1+t^{\frac{1}{2}})^{\alpha_2}x^{\alpha_2}+(1+t^{\frac{1}{4}})^{\alpha_2}y^{\alpha_2}\\ &\leq (1+t^{\frac{1}{2}})^{\alpha_2}|x|+(1+t^{\frac{1}{4}})^{\alpha_2}|y|.\end{aligned}$$

We put $p(t)=(1+t^{\frac{1}{2}})^{\alpha_2},q(t)=(1+t^{\frac{1}{4}})^{\alpha_2}$, with $\Gamma(\frac{3}{2})\approx 0,8862,\Gamma(\frac{1}{4})\approx 3,6256$, and we obtain

$$\Big(\frac{1}{\Gamma(\frac{3}{2})}+\frac{1}{\Gamma(\frac{1}{4})}\Big)\frac{1}{3}\int_0^{+\infty}\frac{e^{-t}}{1+t^2}[(1+t^{\frac{1}{2}})^{\alpha_2}+(1+t^{\frac{1}{4}})^{\alpha_2}]dt\ \leq\ 0,9361\leq 1.$$

Then, the boundary value problem (16) has at least one positive solution.

Theorem 4. *Assume that the conditions*

(c_1) $\limsup_{x+y\to0^+}\frac{1}{(1+t^{\alpha-1})^{\eta}}\frac{g((1+t^{\alpha-1})x,(1+t^{\alpha-\beta-1})y)}{x+y}<M^{-1}$, *uniformly with respect to* $t\in\mathbb{R}^+$,

(c_2) $\liminf\limits_{x+y\to+\infty}\frac{g(x,y)}{x+y}>N^{-1}$

hold. Then the boundary value problem (2) *has at least one positive solution.*

Proof. From (c_1), it follows that for $0<\varepsilon_1<M^{-1}$, there exists $R_1>0$ such that, $\frac{1}{(1+t^{\alpha-1})^{\eta}}g((1+t^{\alpha-1})x,(1+t^{\alpha-\beta-1})y)\leq(M^{-1}-\varepsilon_1)(x+y)$ with $0<x+y\leq R_1$, for all $t\in\mathbb{R}^+$. Let $\Omega_1=\{u\in Y,\|u\|_Y<R_1\}$. So for any $u\in P\cap\partial\Omega_1$, we have

$$\begin{aligned}
\frac{Tu(t)}{1+t^{\alpha-1}} &= \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) g(u(s), D_{0^+}^{\beta} u(s)) ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} a(s) \frac{(1+s^{\alpha-1})^{\eta}}{(1+s^{\alpha-1})^{\eta}} g\Big(\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}, \frac{(1+s^{\alpha-\beta-1})D_{0^+}^{\beta}u(s)}{1+s^{\alpha-\beta-1}}\Big) ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} a(s)(1+s^{\alpha-1})^{\eta}(M^{-1}-\varepsilon_1)\Big(\frac{u(s)}{1+s^{\alpha-1}} + \frac{D_{0^+}^{\beta}u(s)}{1+s^{\alpha-\beta-1}}\Big) ds \\
&\leq (M^{-1}-\varepsilon_1)\Big[\frac{1}{\Gamma(\alpha)} \int_0^{+\infty} a(s)(1+s^{\alpha-1})^{\eta} ds\Big] \|u\|_Y.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\frac{D_{0^+}^{\beta} Tu(t)}{1+t^{\alpha-\beta-1}} \\
&= \int_0^{+\infty} \frac{D_{0^+}^{\beta} G_t(t,s)}{1+t^{\alpha-\beta-1}} a(s) g(u(s), D_{0^+}^{\beta} u(s)) ds \\
&\leq \frac{1}{\Gamma(\alpha-\beta)} \int_0^{+\infty} a(s) \frac{(1+s^{\alpha-1})^{\eta}}{(1+s^{\alpha-1})^{\eta}} g\Big(\frac{(1+s^{\alpha-1})u(s)}{1+s^{\alpha-1}}, \frac{(1+s^{\alpha-\beta-1})D_{0^+}^{\beta}u(s)}{1+s^{\alpha-\beta-1}}\Big) ds \\
&\leq \frac{1}{\Gamma(\alpha-\beta)} \int_0^{+\infty} a(s)(1+s^{\alpha-1})^{\eta}(M^{-1}-\varepsilon_1)\Big(\frac{u(s)}{1+s^{\alpha-1}} + \frac{D_{0^+}^{\beta}u(s)}{1+s^{\alpha-\beta-1}}\Big) ds \\
&\leq (M^{-1}-\varepsilon_1)\Big[\frac{1}{\Gamma(\alpha-\beta)} \int_0^{+\infty} a(s)(1+s^{\alpha-1})^{\eta} ds\Big] \|u\|_Y.
\end{aligned}$$

Then

$$\begin{aligned}
\|Tu\|_Y &\leq (M^{-1}-\varepsilon_1) M \|u\|_Y \\
&\leq \|u\|_Y.
\end{aligned}$$

From (c_2), it follows that there exists $\overline{R_2} > 0$ and $\varepsilon_2 > 0$ such that $g(x,y) \geq (N^{-1}+\varepsilon_2)(x+y)$, for $x+y \geq \overline{R_2}$. Let $R_2 = \max\{2R_1, \frac{\overline{R_2}}{\theta}\}$ and $\Omega_2 = \{u \in Y, \|u\|_Y < R_2\}$. Then for $u \in P \cap \partial\Omega_2$, we have

$$\begin{aligned}
u(t) + D_{0^+}^{\beta} u(t) &\geq \frac{u(t)}{1+t^{\alpha-1}} + \frac{D_{0^+}^{\beta}u(t)}{1+t^{\alpha-\beta-1}} \\
&\geq \theta_1 \|u\|_X + \theta_2 \|D_{0^+}^{\beta} u\|_{X'} \\
&\geq \theta(\|u\|_X + \|D_{0^+}^{\beta} u\|_{X'}) \\
&\geq \theta R_2 \geq \overline{R_2}, \ \ \forall \ t \in [\frac{1}{\sigma}, \sigma]
\end{aligned} \tag{17}$$

with $\theta = \min(\theta_1, \theta_2)$. We obtain

$$\begin{aligned}
\frac{Tu(t)}{1+t^{\alpha-1}} &= \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) g(u(s), D_{0^+}^{\beta} u(s)) ds \\
&\geq \int_{\frac{1}{\sigma}}^{\sigma} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) g(u(s), D_{0^+}^{\beta} u(s)) ds \\
&\geq \int_{\frac{1}{\sigma}}^{\sigma} \frac{G(t,s)}{1+t^{\alpha-1}} a(s)(N^{-1}+\varepsilon_2)(u(s) + D_{0^+}^{\beta} u(s)) ds \\
&\geq \int_{\frac{1}{\sigma}}^{\sigma} \frac{G(t,s)}{1+t^{\alpha-1}} a(s)(N^{-1}+\varepsilon_2)\Big(\frac{u(s)}{1+s^{\alpha-1}} + \frac{D_{0^+}^{\beta} u(s)}{1+s^{\alpha-\beta-1}}\Big) ds \\
&\geq (N^{-1}+\varepsilon_2)\theta \|u\|_Y \int_{\frac{1}{\sigma}}^{\sigma} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) ds \\
&\geq \theta(N^{-1}+\varepsilon_2)\Big[\int_{\frac{1}{\sigma}}^{\sigma} \theta_1 a(s) ds\Big] \|u\|_Y,
\end{aligned}$$

and

$$\begin{aligned}
\frac{D_{0^+}^{\beta} Tu(t)}{1+t^{\alpha-\beta-1}} &= \int_0^{+\infty} \frac{D_{0^+}^{\beta} G_t(t,s)}{1+t^{\alpha-\beta-1}} a(s) g(u(s), D_{0^+}^{\beta} u(s)) ds \\
&\geq \int_{\frac{1}{\sigma}}^{\sigma} \frac{D_{0^+}^{\beta} G_t(t,s)}{1+t^{\alpha-\beta-1}} a(s) g(u(s), D_{0^+}^{\beta} u(s)) ds \\
&\geq \int_{\frac{1}{\sigma}}^{\sigma} \frac{D_{0^+}^{\beta} G_t(t,s)}{1+t^{\alpha-\beta-1}} a(s)(N^{-1}+\varepsilon_2)(u(s) + D_{0^+}^{\beta} u(s)) ds \\
&\geq \int_{\frac{1}{\sigma}}^{\sigma} \frac{D_{0^+}^{\beta} G_t(t,s)}{1+t^{\alpha-\beta-1}} a(s)(N^{-1}+\varepsilon_2)\Big(\frac{u(s)}{1+s^{\alpha-1}} + \frac{D_{0^+}^{\beta} u(s)}{1+s^{\alpha-\beta-1}}\Big) ds \\
&\geq (N^{-1}+\varepsilon_2)\theta \|u\|_Y \int_{\frac{1}{\sigma}}^{\sigma} \frac{D_{0^+}^{\beta} G_t(t,s)}{1+t^{\alpha-\beta-1}} a(s) ds \\
&\geq \theta\ (N^{-1}+\varepsilon_2)\Big[\int_{\frac{1}{\sigma}}^{\sigma} \theta_2 a(s) ds\Big] \|u\|_Y.
\end{aligned}$$

Then

$$\begin{aligned}
\|Tu\|_Y &\geq (N^{-1}+\varepsilon_2)\theta(\theta_1+\theta_2)\Big[\int_{\frac{1}{\sigma}}^{\sigma} a(s) ds\Big] \|u\|_Y \\
&\geq (N^{-1}+\varepsilon_2) N \|u\|_Y \geq \|u\|_Y.
\end{aligned}$$

Therefore, by Theorem 1, the operator T has at least one fixed point which is a positive solution of the boundary value problem (2). ■

By the same way, one can prove the following result.

Theorem 5. *Assume that*

(d_1) *there exist functions* $p, q : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *such that*

$$g((1+t^{\alpha-1})x, (1+t^{\alpha-\beta-1})y) \leq p(t)|x| + q(t)|y|$$

with $(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-\beta)}) \int_0^{+\infty} a(s)[p(s)+q(s)]ds \leq 1, \forall\, x, y \in \mathbb{R}^+$, $x + y \in [0, R_2]$ *for some* $R_2 > 0$,

(d_2) $\liminf\limits_{x+y\to 0^+} \frac{g((1+t^{\alpha-1})x,(1+t^{\alpha-\beta-1})y)}{x+y} > N^{-1}$, *uniformly with respect to* $t \in [\frac{1}{\sigma}, \sigma]$.

Then the boundary value problem (2) *has at least one positive solution.*

Example 3. Let the problem

$$\left\{ \begin{array}{l} -D^{\frac{3}{2}}_{0^+}u(t) = \frac{1}{8}e^{-t}\Big(u^{\alpha_3}(t) + (D^{\frac{1}{4}}_{0^+}u(t))^{\alpha_3}\Big), \quad t \in [0, +\infty), \\ u(0) = D^{\frac{1}{2}}_{0^+}u(\infty) = 0, \end{array} \right. \tag{18}$$

with $a(t) = \frac{1}{8}e^{-t}$ and $g(x, y) = x^{\alpha_3} + y^{\alpha_3}$ be given.

If $\eta \geq \alpha_3 > 1$, then the hypotheses of Theorem 4 are satisfied and the boundary value problem (18) has at one positive solution.

If $0 < \alpha_3 < 1$ with $\eta \geq \alpha_3$, then the hypotheses of Theorem 5 are satisfied and the boundary value problem (18) has at one positive solution.

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